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SUFFICIENT CONDITIONS FOR STARLIKENESS ASSOCIATED WITH PARABOLIC REGION

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An analytic function $f(z) = z + a_{n+1}z^{n+1} + \cdots$, defined on the unit disk $\triangle = \{z : |z| < 1\}$, is in the class S_p if $zf'(z)/f(z)$ is in the parabolic region $\operatorname{Re} w > |w-1|$. This class is closely related to the class of uniformly convex functions. Sufficient conditions for function to be in S_p are obtained. In particular, we find condition on λ such that the function $f(z)$, satisfying $(1-\alpha)(f(z)/z)^{\mu} + \alpha f'(z)(f(z)/z)^{\mu-1} \prec 1+\lambda z$, is in S_p.

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1. Introduction. Let $\mathcal{A}_n$ be the family of analytic functions $f(z) = z + a_{n+1}z^{n+1} + \cdots$ in the unit disk $\triangle = \{z : |z| < 1\}$, and let $\mathcal{A}_1 = \mathcal{A}$. For $0 \le \alpha < 1$, let $S^*(\alpha)$ and $C(\alpha)$ denote the subclasses of $\mathcal{A}$ of starlike functions and convex functions of order α, respectively; for $\alpha = 0$, $S^*(0) = S^*$, the class of starlike functions in $\triangle$. The function $f \in \mathcal{A}$ is uniformly convex (starlike) if, for every circular arc γ contained in $\triangle$ with center $\zeta \in \triangle$, the image arc $f(\gamma)$ is convex (starlike with respect to $f(\zeta)$). The class of all uniformly convex functions denoted by UCV was introduced by Goodman [1] in 1991. Rønning [5] and Ma and Minda [2] independently proved that $f \in \mathrm{UCV}$ if and only if

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \left|\frac{zf''(z)}{f'(z)}\right|, \quad z \in \triangle. \tag{1.1}$$

Further, Rønning [5] defined the class S_p of functions $f \in \mathcal{A}$ for which

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \left|\frac{zf'(z)}{f(z)} - 1\right| \tag{1.2}$$

holds for all $z \in \triangle$. It can be observed that $f \in \mathrm{UCV}$ if and only if $zf' \in S_p$. Let $\Omega = \{w : |w-1| < \operatorname{Re} w\}$. It follows that $f \in \mathrm{UCV}$ or S_p are equivalent to saying that $1 + zf''(z)/f'(z)$ or $zf'(z)/f(z)$ are in Ω, respectively. Note that Ω is a parabolic region symmetric with respect to the real axis and (1/2,0) as its vertex. The function $k(z)$, with $k(0) = k'(0) - 1 = 0$ and

$$1 + \frac{zk''(z)}{k'(z)} = 1 + \frac{2}{\pi^2}\left[\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right]^2, \tag{1.3}$$

is an example of function in UCV.

Ponnusamy and Singh [4] obtained bounds on λ such that the Alexander transform of $f \in \mathcal{A}$, satisfying $f' \prec 1+\lambda z$, is uniformly convex. We extend their result in two directions. Specifically, we find condition on λ such that the function $f(z)$, satisfying

$$(1-\alpha)\left(\frac{f(z)}{z}\right)^{\mu}+\alpha f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} \prec 1+\lambda z, \\ \left|\alpha z f''(z)+f'(z)-1\right| < \lambda, \tag{1.4}$$

is in S_p.

Let $a > 1/2$ and let $R_a = \min\{|w-a| : |w-1| = \operatorname{Re} w\}$. A simple computation gives

$$R_a = \begin{cases} a-\dfrac{1}{2} & \text{if } \dfrac{1}{2} < a \le \dfrac{3}{2}, \\ \sqrt{2a-2} & \text{if } a \ge \dfrac{3}{2}. \end{cases} \tag{1.5}$$

Now, $D(a,R_a) = \{w : |w-a| < R_a\}$ is the largest disk centered at a which lies inside Ω. If we restrict the value of a by $3/4 < a < 3$, then the disk will contain the point 1.

Lemma 1.1 [6]. *Let $f \in \mathcal{A}$. If, for any a, $3/4 < a < 3$,*

$$\left|\frac{zf'(z)}{f(z)}-a\right| < R_a, \quad z \in U, \tag{1.6}$$

then $f \in S_p$.

Also, we need the following result.

Lemma 1.2 [3]. *Let $h(z)$ be convex and $\gamma \neq 0$, $\operatorname{Re}\gamma \ge 0$. If $p(z) = a + p_n z^n + \cdots$, $n \ge 2$, is analytic in $\triangle$ and*

$$p(z)+\frac{zp'(z)}{\gamma} \prec h(z), \quad h(0) = p(0), \tag{1.7}$$

then

$$p(z) \prec \frac{\gamma}{n} z^{-\gamma/n}\int_0^z h(t)t^{\gamma/n-1}dt. \tag{1.8}$$

2. Main results. We begin with proving the following result.

Theorem 2.1. *Let $\mu > 0$, $\alpha \ge 0$, and $0 \le \beta < 1$. Let $f \in \mathcal{A}_n$ and*

$$0 < \lambda \le \frac{\alpha(\mu+\alpha n)(a-\beta-|1-a|)}{\mu[1+(a-\beta)\alpha+|(a-1)\alpha+1|]+\alpha n}. \tag{2.1}$$

Then, for $a > (1+\beta)/2$,

$$(1-\alpha)\left(\frac{f(z)}{z}\right)^{\mu}+\alpha f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} \prec 1+\lambda z \tag{2.2}$$

implies

$$\left|\frac{zf'(z)}{f(z)}-a\right| \le \frac{\lambda[\mu+\alpha n+\mu|(a-1)\alpha+1|]+\alpha|1-a|(\mu+\alpha n)}{\alpha(\mu+\alpha n-\lambda\mu)} \le a-\beta, \tag{2.3}$$

and $f \in S^*(\beta)$.

Proof. Define the functions $Q(z)$ and $w(z)$ by

$$Q(z) = \left(\frac{f(z)}{z}\right)^{\mu}, \quad w(z) = \frac{zf'(z)}{f(z)} - a, \quad z \in \triangle. \tag{2.4}$$

Then, $Q(z)$ and $w(z)$ are analytic in $\triangle$, and $w(0) = 1 - a$. Clearly,

$$(1-\alpha)Q(z) + \alpha[w(z)+a]Q(z) = (1-\alpha)\left(\frac{f(z)}{z}\right)^{\mu} + \alpha\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\mu} \prec 1 + \lambda z,$$
$$\frac{1}{\mu}\frac{zQ'(z)}{Q(z)} + 1 = w(z) + a. \tag{2.5}$$

This shows that

$$Q(z) + \frac{\alpha}{\mu} zQ'(z) \prec 1 + \lambda z, \tag{2.6}$$

and hence, by Lemma 1.2, we have

$$Q(z) \prec 1 + \frac{\lambda\mu}{\mu + \alpha n} z, \quad z \in \triangle. \tag{2.7}$$

Since

$$\lambda \le \frac{\alpha(\mu+\alpha n)(a-\beta-|1-a|)}{\mu[1+(a-\beta)\alpha + |(a-1)\alpha+1|] + \alpha n} \le \frac{\mu + \alpha n}{\mu} \tag{2.8}$$

and $a \ge (1+\beta)/2$, we see that $\mu + \alpha n - \lambda\mu > 0$.

Since

$$\begin{aligned}\frac{zf'(z)}{f(z)} - a &= w(z) \\ &= \frac{[(1-\alpha)Q(z) + \alpha Q(z)(w(z)+a) - 1] - (Q(z)-1)[(a-1)\alpha+1] + \alpha(1-a)}{\alpha Q(z)},\end{aligned} \tag{2.9}$$

we have

$$\begin{aligned}\left|\frac{zf'(z)}{f(z)} - a\right| &\le \frac{\lambda + (\lambda\mu/(\mu+\alpha n))|(a-1)\alpha+1| + \alpha|1-a|}{\alpha(1-\lambda\mu/(\mu+\alpha n))} \\ &\le \frac{\lambda[\mu+\alpha n+\mu|(a-1)\alpha+1|] + \alpha|1-a|(\mu+\alpha n)}{\alpha(\mu+\alpha n-\lambda\mu)} \\ &\le a - \beta\end{aligned} \tag{2.10}$$

provided condition (2.1) is satisfied. This shows that $\operatorname{Re} zf'(z)/f(z) > \beta$ and $f(z)$ is starlike of order β. □

Note that to prove (2.10) it is enough to assume that $0 < \lambda \le (\mu + \alpha n)/\mu$.

Corollary 2.2. *If* $f(z) = z + a_{n+1}z^{n+1} + \cdots$ *is analytic in* $\triangle$ *and if*

$$\left| f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} - 1 \right| < \lambda, \quad z \in \triangle, \tag{2.11}$$

then, for $a > 1/2$*, we have*

$$\left| \frac{zf'(z)}{f(z)} - a \right| \le \frac{\lambda[\mu(a+1)+n] + (\mu+n)|1-a|}{\mu+n-\lambda\mu} \le a \tag{2.12}$$

provided $\mu > 0$ *and*

$$0 < \lambda \le \frac{(\mu+n)(a-|1-a|)}{\mu(1+2a)+n}. \tag{2.13}$$

When $\mu = 1$, Corollary 2.2 reduces to the result by Ponnusamy and Singh [4].

Theorem 2.3. *Let* λ *be defined by*

$$\lambda = \begin{cases} \dfrac{\alpha(\mu+\alpha n)(4a-3)}{\mu[2+(2a-1)\alpha+2|(a-1)\alpha+1|]+2\alpha n} & \left(\dfrac{3}{4} < a \le 1\right), \\ \dfrac{\alpha(\mu+\alpha n)}{\mu[\alpha(4a-1)+2]+2\alpha n} & \left(1 \le a \le \dfrac{3}{2}\right), \\ \dfrac{\alpha(\mu+\alpha n)(1-a+\sqrt{2a-2})}{\alpha n+\mu[2+\alpha(a-1)+\alpha\sqrt{2a-2}]} & \left(\dfrac{3}{2} \le a < 3\right). \end{cases} \tag{2.14}$$

If $f \in \mathcal{A}_n$ *satisfies*

$$(1-\alpha)\left(\frac{f(z)}{z}\right)^{\mu} + \alpha f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} \prec 1+\lambda z, \tag{2.15}$$

then $f \in S_p$*.*

It should be noted that if $3/4 < a \le 3/2$, then the condition on λ in Theorem 2.3 reduces to the condition in Theorem 2.1. Hence, with the same condition as in Theorem 2.1 (with $\beta = 1/2$), we get a stronger conclusion that $f \in S_p$.

Proof. We first verify that λ defined in Theorem 2.3 satisfies the condition $0 < \lambda \le (\mu+\alpha n)/\mu$. This condition is equivalent to

$$0 \le \begin{cases} \mu[\alpha(2a-1)+1] & \left(1 \le a \le \dfrac{3}{2}\right), \\ 2\mu+\alpha n & \left(\dfrac{3}{4} \le a \le 1,\ (a-1)\alpha+1 \ge 0\right), \\ 2\mu\alpha(1-a)+\alpha n & \left(\dfrac{3}{4} \le a \le 1,\ (a-1)\alpha+1 \le 0\right), \\ \alpha n+2\mu[\alpha(a-1)+1] & \left(\dfrac{3}{2} \le a < 3\right). \end{cases} \tag{2.16}$$

The above inequality is obviously correct. Let

$$R_a = \frac{\lambda[\mu+\alpha n+\mu|(a-1)\alpha+1|]+\alpha|1-a|(\mu+\alpha n)}{\alpha(\mu+\alpha n-\lambda\mu)}. \tag{2.17}$$

Then, a computation shows that

$$R_a = \begin{cases} a - \dfrac{1}{2} & \left(\dfrac{3}{4} < a \le \dfrac{3}{2}\right), \\ \sqrt{2a-2} & \left(\dfrac{3}{2} \le a < 3\right). \end{cases} \tag{2.18}$$

Then, from the proof of Theorem 2.1, we have

$$\left| \frac{zf'(z)}{f(z)} - a \right| \le R_a. \tag{2.19}$$

Using Lemma 1.1, we have the desired result. □

This result for $\mu = 1$ and $\alpha = 1$ is obtained in Ponnusamy and Singh [4].

Theorem 2.4. *Suppose $\alpha \in \mathbb{C}$, $a > 1/2$, and $\lambda \in \mathbb{R}$ satisfy*

$$0 < \lambda \le \frac{|1+n\alpha|(\mu+n)(a-|1-a|)}{\mu(1+2a)+n}. \tag{2.20}$$

If

$$\left(\frac{f(z)}{z}\right)^{\mu-1}\left\{\alpha(\mu-1)\frac{z[f'(z)]^2}{f(z)} + \alpha z f''(z) + (1+(1-\mu)\alpha)f'(z)\right\} \prec 1+\lambda z, \quad z \in \triangle, \tag{2.21}$$

then

$$\left| \frac{zf'(z)}{f(z)} - a \right| \le \frac{\lambda_1[n+\mu(a+1)] + (\mu+n)|1-a|}{\mu(1-\lambda_1)+n}, \tag{2.22}$$

where $\lambda_1 = \lambda/|1+n\alpha|$.

Proof. Let $p(z) = f'(z)(f(z)/z)^{\mu-1}$, $z \in \triangle$. Then, $p(z)$ is analytic in $\triangle$ and

$$zp'(z) = \left(\frac{f(z)}{z}\right)^{\mu-1}\left\{zf''(z) + (\mu-1)\left(\frac{zf'(z)}{f(z)} - 1\right)f'(z)\right\}. \tag{2.23}$$

This shows that

$$p(z) + \alpha z p'(z) \prec 1+\lambda z, \tag{2.24}$$

and hence, by Lemma 1.2,

$$p(z) \prec 1 + \frac{\lambda}{1+n\alpha} z \prec 1 + \frac{\lambda}{|1+n\alpha|} z. \tag{2.25}$$

The result now follows from Theorem 2.1 where $\lambda_1 = \lambda/|1+n\alpha|$. □

Corollary 2.5. *Suppose that $\alpha \in \mathbb{C}$, $a > 1/2$, and $\lambda \in \mathbb{R}$ satisfy*

$$0 < \lambda \le \frac{|1+n\alpha|(1+n)(a-|1-a|)}{1+2a+n}. \tag{2.26}$$

If $f \in \mathcal{A}$ satisfies

$$|\alpha z f''(z) + f'(z) - 1| < \lambda, \quad z \in \triangle, \tag{2.27}$$

then

$$\left| \frac{zf'(z)}{f(z)} - a \right| \le \frac{\lambda_1[n+a+1] + (1+n)|1-a|}{1-\lambda_1+n}, \tag{2.28}$$

where $\lambda_1 = \lambda/|1+n\alpha|$.

The result follows from Theorem 2.4 when $a = n = 1$ and is obtained in [4].

Theorem 2.6. *Let λ be defined by*

$$\lambda = \begin{cases} \dfrac{|1+n\alpha|(1+n)(4a-3)}{4a+2n+1} & \left(\dfrac{3}{4} < a \le 1\right), \\ \dfrac{|1+n\alpha|(1+n)}{4a+2n+1} & \left(1 < a \le \dfrac{3}{2}\right), \\ \dfrac{|1+n\alpha|(1+n)(\sqrt{2a-2}+1-a)}{n+1+a+\sqrt{2a-2}} & \left(\dfrac{3}{2} \le a \le 3\right). \end{cases} \tag{2.29}$$

If $|\alpha z f''(z) + f'(z) - 1| < \lambda$, then $f \in S_p$.

Proof. From the definition of λ, it is clear that

$$\frac{\lambda_1[n+a+1]+(1+n)|1-a|}{1-\lambda_1+n} = \begin{cases} a-\dfrac{1}{2} & \left(\dfrac{3}{4} < a \le \dfrac{3}{2}\right), \\ \sqrt{2a-2} & \left(\dfrac{3}{2} \le a < 3\right), \end{cases} \tag{2.30}$$

where $\lambda_1 = \lambda/|1+n\alpha|$. Since

$$0 < \lambda \le \frac{|1+n\alpha|(1+n)(a-|1-a|)}{2a+n+1}, \tag{2.31}$$

the result follows from Corollary 2.5. □

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