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CHARACTERIZATIONS OF PROJECTIVE AND k-PROJECTIVE SEMIMODULES

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To my late father

This paper deals with projective and k-projective semimodules. The results for projective semimodules are generalization of corresponding results for projective modules.

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1. Introduction. Throughout this paper, R denotes a semiring with identity 1, all semimodules M are left R-semimodules and in all cases are unitary semimodules, that is, $1 \cdot m = m$ for all $m \in M$ all left R-semimodule ${}_RM$.

We recall here (cf. [1, 2, 3, 4, 5]) the following facts:

(a) let $\alpha : M \to N$ be a homomorphism of semimodules. The subsemimodule $\operatorname{Im}\alpha$ of N is defined as follows: $\operatorname{Im}\alpha = \{n \in N : n + \alpha(m') = \alpha(m)$ for some m, $m' \in M\}$. The homomorphism α is said to be an isomorphism if α is injective and surjective; to be i-regular if $\alpha(M) = \operatorname{Im}\alpha$; to be k-regular if for $m, m' \in M$, $\alpha(m) = \alpha(m')$ implying that $m + k = m' + k'$ for some $k, k' \in \operatorname{Ker}\alpha$; and to be regular if it is both i-regular and k-regular;

(b) the sequence $K \xrightarrow{\alpha} M \xrightarrow{\beta} N$ is called an exact sequence if $\operatorname{Ker}\beta = \operatorname{Im}\alpha$, and proper exact if $\operatorname{Ker}\beta = \alpha(K)$;

(c) for any two R-semimodules N, M, $\operatorname{Hom}_R(N,M) := \{\alpha : N \to M \mid \alpha$ is an R-homomorphism of semimodules$\}$ is a semigroup under addition. If M, N, U, are R-semimodules and $\alpha : M \to N$ is a homomorphism, then $\operatorname{Hom}(I_U,\alpha) : \operatorname{Hom}_R(U,M) \to \operatorname{Hom}_R(U,N)$ is given by $\operatorname{Hom}(I_U,\alpha)\gamma = \alpha\gamma$ where I_U is the identity;

(d) P is a projective semimodule if and only if for each surjective R-homomorphism $\alpha : M \to N$, the induced homomorphism

$$\bar{\alpha} : \operatorname{Hom}(P,M) \longrightarrow \operatorname{Hom}(P,N) \tag{1.1}$$

is surjective;

(e) a left R-semimodule P is Mk-projective if and only if it is projective with respect to every surjective k-regular homomorphism $\varphi : M \to N$.

In Section 2, we study the structure of k-projective semimodules. Proposition 2.2 shows that for a semimodule P, the class of all semimodules M such that P is Mk-projective is closed under subtractive subsemimodules, factor semimodules, and gives a sufficient condition for the class to be closed undertaking homomorphic images. Example 2.3 sheds light upon one difference between the structure of projectivity in module theory and semimodule theory. In Section 3, we characterize projective and k-projective semimodules via the Hom functor. Theorems 3.5 and 3.7 assert that P is M-projective (Mk-projective) if and only if $\operatorname{Hom}_R(P,-)$ preserves the exactness of all proper exact sequences $M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$, with β k-regular (both α and β k-regular).

2. k-projective semimodules. We study the structure of k-projective semimodules via the Hom function. We show that the class of all semimodules M, such that P is Mk-projective, is closed under subtractive subsemimodules, factor semimodule and undertaking homomorphic image for a k-regular homomorphism.

For proving Proposition 2.2 we need the following proposition, which is modified from [5, Theorem 2.6].

Proposition 2.1. *Let R be a semiring,*

(i) *if $0 \to M \xrightarrow{\alpha} M' \xrightarrow{\beta} M''$ is a proper exact sequence of R-semimodules and α is k-regular, then for every R-semimodule E,*

$$0 \longrightarrow \operatorname{Hom}_R(E,M) \xrightarrow{\bar{\alpha}} \operatorname{Hom}_R(E,M') \xrightarrow{\bar{\beta}} \operatorname{Hom}(E,M'') \tag{2.1}$$

is a proper exact sequence of Abelian semigroups and $\bar{\alpha}$ is regular, where $\bar{\alpha}(\xi) = \alpha\xi$ for $\xi \in \operatorname{Hom}(E,M)$ and $\bar{\beta}(\gamma) = \beta\gamma$ for $\gamma \in \operatorname{Hom}(E,M')$;

(ii) *if $M \xrightarrow{\alpha} M' \xrightarrow{\beta} M'' \to 0$ is a proper exact sequence of R-semimodules and β is k-regular, then for every R-semimodule E,*

$$0 \longrightarrow \operatorname{Hom}(M'',E) \xrightarrow{\bar{\beta}} \operatorname{Hom}(M',E) \xrightarrow{\bar{\alpha}} \operatorname{Hom}(M,E) \tag{2.2}$$

is a proper exact sequence of Abelian semigroups and $\bar{\beta}$ is regular, where $\bar{\beta}(\xi) = \beta\xi$.

Proof. (i) Since the sequence $0 \to M \xrightarrow{\alpha} M' \xrightarrow{\beta} M''$ is proper exact, then the sequence is exact with α being i-regular.

Using [5, Theorem 2.6], the sequence

$$0 \longrightarrow \operatorname{Hom}(E,M) \xrightarrow{\bar{\alpha}} \operatorname{Hom}(E,M') \xrightarrow{\bar{\beta}} \operatorname{Hom}(E,M') \tag{2.3}$$

is exact with $\bar{\alpha}$ being regular. This means that the sequence is proper exact. (ii) can be proved by the same argument. □

Proposition 2.2. *Let P be a left R-semimodule. If $0 \to M' \xrightarrow{\theta} M \xrightarrow{\eta} M'' \to 0$ is a proper exact sequence with θ being regular, η being k-regular, and P is Mk-projective, then P is k-projective relative to both M' and M''.*

Proof. Let $\Psi : M'' \to N$ be surjective k-regular homomorphism and $\alpha : P \to N$ be homomorphism. Since η is surjective k-regular, then $\Psi\eta$ is k-regular. Since P is Mk-projective, then there exists a homomorphism $\varphi : P \to M$ such that the following diagram commutative:

$$\begin{tikzcd} & & P \arrow[dll, dotted, "\varphi"'] \arrow[d, "\alpha"] & \\ M \arrow[r, "\eta"] & M'' \arrow[r, "\Psi"] & N \arrow[r] & 0 \end{tikzcd} \tag{2.4}$$

Therefore P is $M''k$-projective.

To prove that P is $M'k$-projective. Let $\Psi : M' \to N$ be a surjective k-regular homomorphism and set $K = \operatorname{Ker}\Psi$. Since Ψ is surjective k-regular homomorphism, then $M'/K \simeq N$. Define $\bar{\theta} : M'/K \to M/\theta(K)$ by the rule $\bar{\theta}(m'/K) = \theta(m')/\theta(K)$, and $\bar{\eta} : M/\theta(K) \to M''$ by the rule $\bar{\eta}(m/\theta(K)) = \eta(m)$. Clearly, both $\bar{\theta}$ and $\bar{\eta}$ are well defined homomorphisms. Consider the sequence $0 \to M'/K \xrightarrow{\bar{\theta}} M/h(K) \xrightarrow{\bar{\eta}} M'' \to 0$. Let $m/\theta(K) \in \operatorname{Ker}\bar{\eta}$, then $\eta(m) = 0$, hence $m \in \operatorname{Ker}\eta = \theta(M')$. Hence $\operatorname{Ker}\bar{\eta} = \bar{\theta}(M'/K)$. Clearly $\bar{\eta}$ is surjective, and $\bar{\theta}$ is injective. Since θ is i-regular, then $\bar{\theta}$ is i-regular. Now consider the following commutative diagram:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
0 \arrow[r] & M' \arrow[r, "\theta"] \arrow[d, "\pi_k"'] & M \arrow[r, "\eta"] \arrow[d, "\pi_{h(k)}"'] & M'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M'/K \arrow[r, "\bar{\theta}"] \arrow[d] & M/\theta(K) \arrow[r, "\bar{\eta}"] \arrow[d] & M'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{2.5}$$

Applying $\operatorname{Hom}_R(P,-)$ to this diagram we have the commutative diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] \\
0 \arrow[r] & \operatorname{Hom}(P,M') \arrow[r, "\theta*"] \arrow[d, "(\pi_k)_*"'] & \operatorname{Hom}(P,M) \arrow[r, "\eta*"] \arrow[d, "(\pi_{\theta(k)})*"'] & \operatorname{Hom}(P,M'') \arrow[d, "I_*"] \\
0 \arrow[r] & \operatorname{Hom}(P,M'/K) \arrow[r, "\bar{\theta}*"] & \operatorname{Hom}(P,M/\theta(K)) \arrow[r, "\bar{\eta}*"] \arrow[d] & \operatorname{Hom}(P,M'') \arrow[d] \\
 & & 0 & 0
\end{tikzcd} \tag{2.6}$$

Using Proposition 2.1, and since P is Mk-projective, then all rows and columns are proper exact sequence. We should show that $(\pi_K)_*$ is surjective. Let $\alpha \in \operatorname{Hom}(P,M'/K)$. Since $(\pi_{\theta(K)})_*$ is surjective, then there exists $\beta \in \operatorname{Hom}(P,M)$ such that $(\pi_{\theta(K)})_*(\beta) = \bar{\theta}_*(\alpha)$. Now $\bar{\eta}_*\bar{\theta}_*(\alpha) = \bar{\eta}_*((\pi_{\theta(K)})_*(\beta)) = I_*\eta_*(\beta) = 0$. Hence $\eta_*(\beta) \subset \operatorname{Ker} I_* = 0$. Hence $\beta = \theta_*(\gamma)$ where $\gamma \in \operatorname{Hom}(P,M')$. Thus $\bar{\theta}_*(\alpha) = (\pi_{\theta(K)})_*(\beta) = (\pi_{\theta(K)})_*\theta_*(\gamma) = \bar{\theta}_*(\pi_K)_*(\gamma)$. Again by Proposition 2.1, $\bar{\theta}_*$ is injective, hence $\alpha = (\pi_K)_*(\gamma)$. Thus $(\pi_K)_*$ is surjective. Therefore P is $M'k$-projective. $\square$

Let $\Omega(P)$ be the collection of all semimodules M such that P is Mk-projective. The above results show that this class is closed under subtractive subsemimodules and give us a sufficient condition to be closed undertaking a homomorphic image. Since for every subsemimodule K of M, the canonical surjection $\pi_K : M \to M/K$ is k-regular surjective, then the class $\Omega(P)$ is closed under factor semimodules.

We know that in module theory any projective module is a direct summand of a free module. However, for arbitary semirings this is not true.

Example 2.3. Let R be the field $\mathbb{Z}/\langle p\rangle$ for any prime integer. Let $S = \{\{\bar{\theta}\}, \mathbb{Z}/\langle p\rangle\}$ set $R' = \{(\bar{a}, I) : \bar{a} \in I \in S\}$, that is

$$R' = \{(\bar{0}, \{\bar{0}\}), (\bar{a}, \mathbb{Z}/\langle p\rangle),\ \bar{a} \in \mathbb{Z}/\langle p\rangle\}. \tag{2.7}$$

Define operations $\oplus$ and $\otimes$ on R' by setting

$$\begin{aligned} (\bar{a}, I) \oplus (\bar{b}, H) &= (\bar{a} + \bar{b}, I + H), \\ (\bar{a}, \bar{I}) \otimes (\bar{b}, H) &= (\bar{a}\bar{b}, IH). \end{aligned} \tag{2.8}$$

Clearly R' is semiring. Let $I^+(R')$ be the set of all additively idempotent elements of $R' : I^+(R') = \{(\bar{0}, \{\bar{0}\}), (\bar{0}, \mathbb{Z}/\langle p\rangle)\}$. We note that the function $\alpha : R' \to I^+(R')$, defined by $\alpha(\bar{0}, \{\bar{0}\}) = (\bar{0}, \{\bar{0}\})$ and $\alpha(\bar{a}, \mathbb{Z}/\langle p\rangle) = (\bar{0}, \mathbb{Z}/\langle p\rangle)$, is a surjective R'-homomorphism of left R'-semimodules. Furthermore, the restriction of α to $I^+(R')$ is the identity map. Therefore $I^+(R')$ is a retract of R'. Since R' is projective, as a left semimodule over itself, by [5, Corollary 15.13] we see that $I^+(R')$ is also projective. If $I^+(R')$ is a free semimodule, then $(\bar{0}, \mathbb{Z}/\langle p\rangle)$ is a basis, but $(\bar{a}, \mathbb{Z}/\langle p\rangle)(\bar{0}, \mathbb{Z}/\langle p\rangle) = (\bar{0}, \mathbb{Z}/\langle p\rangle)$ for every $\bar{a} \in \mathbb{Z}/\langle p\rangle$. Hence $I^+(R')$ is not free. Now, suppose that $I^+(R')$ is a direct summand of a free R'-semimodule, say F. Then $F \cong K \oplus I^+(R')$ for an R'-semimodule K. Let $\varphi : K \oplus I^+(R') \to F = \oplus_\alpha R'_\alpha$ be an isomorphism, where $R'_\alpha = R'$ for all α. Since $(0, (\bar{0}, \mathbb{Z}/\langle p\rangle))$ is an idempotent element where $0 \in K$, then $\varphi(0, (\bar{0}, \mathbb{Z}/\langle p\rangle))$ is an idempotent element in F. Since the only idempotent elements of R' are $(\bar{0}, \{\bar{0}\})$ and $(\bar{0}, \mathbb{Z}/\langle p\rangle)$, then $\varphi(0, (\bar{0}, \mathbb{Z}/\langle p\rangle)) = (x_\alpha)$, where only finite numbers of x_α are nonzeros and $x_\alpha = (\bar{0}, \mathbb{Z}/\langle p\rangle)$. Let (y_α) have components $y_\alpha = (\bar{1}, \mathbb{Z}/\langle p\rangle)$ for α, with $x_\alpha = (\bar{0}, \mathbb{Z}/\langle p\rangle)$, and otherwise $y_\alpha = (\bar{0}, \{\bar{0}\})$. Suppose $\varphi(k, (\bar{0}, \mathbb{Z}/\langle p\rangle)) = (y_a)$, where $0 \neq k \in K$. Clearly, $p(y_\alpha) = (x_a)$, $p\varphi(k, (\bar{0}, \mathbb{Z}/\langle p\rangle)) = \varphi(0, (\bar{0}, \mathbb{Z}/\langle p\rangle))$. Hence $p(k, (\bar{0}, \mathbb{Z}/\langle p\rangle)) = (0, (\bar{0}, \mathbb{Z}/\langle p\rangle))$. Hence $pk = 0$. Therefore $p(k, (\bar{0}, \{\bar{0}\})) = 0$. Hence $(k, (\bar{0}, \{\bar{0}\}))$ has an additive inverse. Thus, $\varphi(k, (\bar{0}, \{\bar{0}\}))$ also has an additive inverse. Now, every element of F is of the form (u_α), $u_\alpha \in R'$. Since every nonzero element of R' has no additive inverse, then every nonzero element of F has no additive inverse. Thus we have a contradiction. Therefore, $I^+(R')$ is not a direct summand of a free R'-semimodule.

3. Characterizations of projective and k-projective semimodules. We characterize projective and k-projective semimodules via the Hom functor.

We state and prove the following lemma and corollaries which are needed in the proof of Theorem 3.5.

Lemma 3.1. *let R be a semiring and let $M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$ be a sequence of R-semimodules. Then, the sequence is exact if there exists a commutative diagram*

$$\begin{tikzcd}
0 \arrow[dr] & 0 & \\
& N \arrow[u] \arrow[dr, "\eta"] & \\
M' \arrow[r, "\alpha"] \arrow[dr, "\Psi"'] & M \arrow[u, "\varphi"] \arrow[r, "\beta"] & M'' \\
& K \arrow[u, "\theta"] \arrow[dr] & \\
& 0 \arrow[u] & 0
\end{tikzcd} \tag{3.1}$$

of R-semimodule in which the nonhorizontal *sequences are all exact.*

Proof. Let $x \in \operatorname{Ker}\beta$, then $\eta\varphi(x) = \beta(x) = 0$, hence $\varphi(x) \in \operatorname{Ker}\eta$. Since $\operatorname{Ker}\eta = 0$, then $x \in \operatorname{Ker}\varphi = \operatorname{Im}\theta$. Hence $x + \theta(k_l) = \theta(k_2)$, therefore $x + \theta(\Psi(m_l)) = \theta(\Psi(m_2))$. Since $\theta\Psi = \alpha$, then $x + \alpha(m_1) = \alpha(m_2)$. Thus $x \in \operatorname{Im}\alpha$. Conversely, let $x \in \operatorname{Im}\alpha$, then $x + \alpha(m_1) = \alpha(m_2)$ for some $m_1, m_2 \in M$. Again since $\theta\Psi = \alpha$, then $x + \theta\Psi(m_1) = \theta\Psi(m_2)$, hence $x \in \operatorname{Im}\theta$. But $\operatorname{Im}\theta = \operatorname{Ker}\varphi$, hence $\beta(x) = \eta\varphi(x) = 0$. Thus $\operatorname{Im}\alpha = \operatorname{Ker}\beta$. □

Since every proper exact sequence is exact sequence. Then we have the following corollary.

Corollary 3.2. *Let R be a semiring and let $M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$ be a sequence of R-semimodules. Then the sequence is proper exact if there exists a commutative diagram*

$$\begin{tikzcd}
0 \arrow[dr] & 0 & \\
& N \arrow[u] \arrow[dr, "\eta"] & \\
M' \arrow[r, "\alpha"] \arrow[dr, "\Psi"'] & M \arrow[u, "\varphi"] \arrow[r, "\beta"] & M'' \\
& K \arrow[u, "\theta"] \arrow[dr] & \\
& 0 \arrow[u] & 0
\end{tikzcd} \tag{3.2}$$

of R-semimodules in which the nonhorizontal *sequences are all proper exact.*

Proof. Since every proper exact sequence is exact sequence, then using Lemma 3.1, we have $\operatorname{Ker}\beta = \operatorname{Im}\alpha$, hence $\alpha(M') \subset \operatorname{Ker}\beta$. Now let $x \in \operatorname{Ker}\beta$, then $\beta(x) = \eta\varphi(x) = 0$, hence $\varphi(x) \in \operatorname{Ker}\eta$. But $\operatorname{Ker}\eta = 0$, therefore $x \in \operatorname{Ker}\varphi = \theta(K)$. Hence $x = \theta(k) = \theta(\Psi(m')) = \alpha(m')$. Thus $\alpha(M') = \operatorname{Ker}\beta$. $\square$

Corollary 3.3. *Let R be a semiring and let $M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$ be a sequence of R-semimodules with β being k-regular. Then the sequence is proper exact if and only if there exists a commutative diagram*

$$\begin{tikzcd}
0 \arrow[dr] & 0 & \\
& N \arrow[u] \arrow[dr, "\eta"] & \\
M' \arrow[r, "\alpha"] \arrow[dr, "\Psi"'] & M \arrow[u, "\varphi"] \arrow[r, "\beta"] & M'' \\
& K \arrow[u, "\theta"] \arrow[dr] & \\
& 0 \arrow[u] & 0
\end{tikzcd} \tag{3.3}$$

of R-semimodules in which the nonhorizontal *sequences are all proper exact.*

Proof. Let $M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$ be a proper exact sequence with β being k-regular. Consider the following diagram:

$$\begin{tikzcd}
0 \arrow[dr] & 0 & \\
& M/\operatorname{Ker}\beta \arrow[u] \arrow[dr, "\eta"] & \\
M' \arrow[r, "\alpha"] \arrow[dr, "\Psi"'] & M \arrow[u, "\varphi"] \arrow[r, "\beta"] & M'' \\
& \operatorname{Ker}\beta \arrow[u, "i"] \arrow[dr] & \\
& 0 \arrow[u] & 0
\end{tikzcd} \tag{3.4}$$

where $\Psi(m') = \alpha(m')$ for all $m' \in M'$, $i(x) = x$ for all $x \in \operatorname{Ker}\beta$, $\varphi(m) = m/\operatorname{Ker}\beta$ for all $m \in M$, and $\eta(m/\operatorname{Ker}\beta) = \beta(m)$ for all $m/\operatorname{Ker}\beta \in M/\operatorname{Ker}\beta$. Let $m_1/\operatorname{Ker}\beta = m_2/\operatorname{Ker}\beta$, then $m_1 + k_1 = m_2 + k_2$, $k_1, k_2 \in \operatorname{Ker}\beta$. Hence $\beta(m_1) = \beta(m_2)$, therefore η is well defined. Now if $\beta(m_1) = \beta(m_2)$, and since β is k-regular, we have $m_1 + k_1 = m_2 + k_2$, $k_1, k_2 \in \operatorname{Ker}\beta$.

Hence $m_1/\operatorname{Ker}\beta = m_2/\operatorname{Ker}\beta$, therefore η is injective. Clearly the sequence $0 \to \operatorname{Ker}\beta \xrightarrow{i} M \xrightarrow{\varphi} M/\operatorname{Ker}\beta \to 0$ is proper exact sequence. Thus diagram (3.4) is a commutative diagram in which the nonhorizontal sequences are all proper exact.

Conversely, see Corollary 3.2. □

Corollary 3.4. *Let R be a semiring and let $M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$ be a sequence of R-semimodules with β being k-regular. Then the sequence is exact if and only if there exists a commutative diagram*

$$
\begin{tikzcd}
0 \arrow[dr] & 0 & \\
& N \arrow[u] \arrow[dr, "\eta"] & \\
M' \arrow[r, "\alpha"] \arrow[dr, "\Psi"'] & M \arrow[u, "\varphi"] \arrow[r, "\beta"] & M'' \\
& K \arrow[u, "\theta"'] \arrow[dr] & \\
& 0 \arrow[u] & 0
\end{tikzcd}
\tag{3.5}
$$

of R-semimodules in which the nonhorizontal *sequences are all exact.*

Proof. Let $M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$ be an exact sequence with β being k-regular. Consider the following diagram:

$$
\begin{tikzcd}
0 \arrow[dr] & 0 & \\
& M/\operatorname{Im}\alpha \arrow[u] \arrow[dr, "\eta"] & \\
M' \arrow[r, "\alpha"] \arrow[dr, "\Psi"'] & M \arrow[u, "\varphi"] \arrow[r, "\beta"] & M'' \\
& \operatorname{Im}\alpha \arrow[u, "i"'] \arrow[dr] & \\
& 0 \arrow[u] & 0
\end{tikzcd}
\tag{3.6}
$$

where $\Psi(m') = \alpha(m') \in \operatorname{Im}\alpha$ for all $m' \in M'$, $i(x) = x$ for all $x \in \operatorname{Im}\alpha$, $\varphi(m) = m/\operatorname{Im}\alpha$ for all $m \in M$, and $\eta(m/\operatorname{Im}\alpha) = \beta(m)$ for all $m/\operatorname{Im}\alpha \in M/\operatorname{Im}\alpha$. Let $m_1/\operatorname{Im}\alpha = m_2/\operatorname{Im}\alpha$, then $m_1 + t_1 = m_2 + t_2$, $t_1, t_2 \in \operatorname{Im}\alpha = \operatorname{Ker}\beta$. Hence $\beta(m_1) = \beta(m_2)$, therefore η is well defined. Now if $\beta(m_1) = \beta(m_2)$, then since β is k-regular we have $m_1 + k_1 = m_2 + k_2$, $k_1, k_2 \in \operatorname{Ker}\beta = \operatorname{Im}\alpha$. Hence $m_1/\operatorname{Im}\alpha = m_2/\operatorname{Im}\alpha$, therefore η is injective. Clearly, the sequence $0 \to \operatorname{Im}\alpha \xrightarrow{i} M \xrightarrow{\varphi} M/\operatorname{Im}\alpha \to 0$ is exact. Thus diagram (3.6) is a commutative diagram in which the nonhorizontal sequences are all exact.

Conversely, see Lemma 3.1. ☐

Theorem 3.5. *The following statements about left R-semimodule P are equivalent:*

(i) *P is projective;*

(ii) *for every proper exact sequence of left R-semimodules $M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$ with β being k-regular the sequence*

$$\operatorname{Hom}(P,M') \xrightarrow{\bar{\alpha}} \operatorname{Hom}(P,M) \xrightarrow{\bar{\beta}} \operatorname{Hom}(P,M'') \tag{3.7}$$

is proper exact.

Proof. (ii)⇒(i). Let $\alpha : N \to M$ be surjective homomorphism. Since $N \xrightarrow{\alpha} M \to 0$ is proper exact sequence with $M \to 0$ being regular, then by (ii) $\operatorname{Hom}(P,N) \to \operatorname{Hom}(P,M) \to 0$ is proper exact. Therefore P is projective.

(i)⇒(ii). Let P be a projective semimodule. Suppose that $M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$ is proper exact with β being k-regular. Consider the sequence $0 \to \operatorname{Ker}\beta \xrightarrow{i} M \xrightarrow{\pi} M/\operatorname{Ker}\beta \to 0$, where $\pi(m) = m/\operatorname{Ker}\beta$ is the canonical surjection. Clearly i is injective. Since P is projective, and using [1, Theorem 10], the sequence

$$0 \longrightarrow \operatorname{Hom}_R(P,\operatorname{Ker}\beta) \xrightarrow{\bar{i}} \operatorname{Hom}(P,M) \xrightarrow{\bar{\pi}} \operatorname{Hom}(P,M/\operatorname{Ker}\beta) \longrightarrow 0 \tag{3.8}$$

is proper exact sequence. Define $\Psi : M' \to \operatorname{Ker}\beta$ and $\eta : M/\operatorname{Ker}\beta \to M''$, where $\Psi(m') = \alpha(m')$ and $\eta(m/\operatorname{Ker}\beta) = \beta(m)$. Let $\eta(m/\operatorname{Ker}\beta) = \eta(m'/\operatorname{Ker}\beta)$, then $\beta(m) = \beta(m')$. Since β is k-regular, then $m + k = m' + k'$, where $k, k' \in \operatorname{Ker}\beta$. Hence $m/\operatorname{Ker}\beta = m'/\operatorname{Ker}\beta$. Therefore η is injective. Now consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[drr] & & 0 & & \\
& & \operatorname{Hom}_R(P,M/\operatorname{Ker}\beta) \arrow[u] \arrow[drr, "\bar{\eta}"] & & \\
\operatorname{Hom}_R(P,M') \arrow[rr, "\bar{\alpha}"] \arrow[drr, "\bar{\Psi}"'] & & \operatorname{Hom}_R(P,M) \arrow[u, "\bar{\pi}"] \arrow[rr, "\bar{\beta}"] & & \operatorname{Hom}_R(P,M'') \\
& & \operatorname{Hom}_R(P,\operatorname{Ker}\beta) \arrow[u, "\bar{i}"] \arrow[drr] & & \\
& & 0 \arrow[u] & & 0
\end{tikzcd} \tag{3.9}$$

where $\bar{\Psi}(\xi) = \Psi\xi$ and $\bar{\eta}(\gamma) = \eta\gamma$, $\xi \in \operatorname{Hom}_R(P,M')$ and $\gamma \in \operatorname{Hom}_R(P,M/\operatorname{Ker}\beta)$. Now let $\xi, \gamma \in \operatorname{Hom}_R(P,M/\operatorname{Ker}\beta)$ such that $\bar{\eta}(\xi) = \bar{\eta}(\gamma)$. Since η is injective, then $\xi = \gamma$. Let $\xi \in \operatorname{Hom}(P,\operatorname{Ker}\beta)$. Since P is projective, then there exist $\theta : P \to M'$ such that the following diagram is commutative:

$$\begin{tikzcd} & P \arrow[dl, dotted, "\theta"'] \arrow[d, "\xi"] & \\ M' \arrow[r, "\Psi"'] & \operatorname{Ker}\beta \arrow[r] & 0 \end{tikzcd} \tag{3.10}$$

Thus $\bar{\Psi}$ is surjective. Thus the nonhorizontal sequences are all proper exact. Using Corollary 3.2, the sequence

$$\operatorname{Hom}_R(P,M') \xrightarrow{\bar{\alpha}} \operatorname{Hom}_R(P,M) \xrightarrow{\bar{\beta}} \operatorname{Hom}_R(P,M'') \tag{3.11}$$

is proper exact. □

Corollary 3.6. *The following statement about left R-semimodule P are equivalent:*

(i) *P is projective;*

(ii) *for every proper exact sequence of left R-semimodules $M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$ with β being regular, the sequence*

$$\operatorname{Hom}(P,M') \xrightarrow{\bar{\alpha}} \operatorname{Hom}(P,M) \xrightarrow{\bar{\beta}} \operatorname{Hom}(P,M'') \tag{3.12}$$

is proper exact.

Proof. It is a consequence of Theorem 3.5. □

Theorem 3.7. *The following statements about left R-semimodule P are equivalent:*

(i) *P is k-projective;*

(ii) *for every proper exact sequence of left R-semimodules $M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$ with both α and β being k-regular, the sequence*

$$\operatorname{Hom}(P,M') \xrightarrow{\bar{\alpha}} \operatorname{Hom}(P,M) \xrightarrow{\bar{\beta}} \operatorname{Hom}(P,M'') \tag{3.13}$$

is proper exact.

Proof. (ii)⇒(i). Let $\alpha : N \to M \to 0$ be k-surjective homomorphism. Since $N \to M \to 0$ is proper exact sequence with α being k-regular, then by (ii) $\operatorname{Hom}(P,N) \to \operatorname{Hom}(P,M) \to 0$ is proper exact. Therefore P is k-projective.

(i)⇒(ii). It is similar to the proof of Theorem 3.5 and using [2, Theorem 8]. □

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