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GENERALIZED DERIVATION MODULO THE IDEAL OF ALL COMPACT OPERATORS

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We give some results concerning the orthogonality of the range and the kernel of a generalized derivation modulo the ideal of all compact operators.

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1. Introduction. Let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded operators acting on a complex Hilbert space $\mathcal{H}$. For A and B in $\mathcal{L}(\mathcal{H})$, let $\delta_{A,B}$ denote the operator on $\mathcal{L}(\mathcal{H})$ defined by $\delta_{A,B}(X) = AX - XB$. If $A = B$, then δ_A is called the inner derivation induced by A. In [1, Theorem 1.7], Anderson showed that if A is normal and commutes with T then, for all $X \in \mathcal{L}(\mathcal{H})$,

$$\left\| T - (AX - XA) \right\| \geq \|T\|. \tag{1.1}$$

In [4], we generalized this inequality, we showed that if the pair (A,B) has the Putnam-Fuglede's property (in particular if A and B are normal operators) and $AT = TB$, then for all $X \in \mathcal{L}(\mathcal{H})$,

$$\left\| T - (AX - XB) \right\| \geq \|T\|. \tag{1.2}$$

The related inequality (1.1) was obtained by Maher [3, Theorem 3.2] who showed that, if A is normal and $AT = TA$, where $T \in C_p$, then $\|T - (AX - XA)\|_p \geq \|T\|_p$ for all $X \in \mathcal{L}(\mathcal{H})$, where C_p is the von Neumann-Schatten class, $1 \leq p < \infty$, and $\|\cdot\|_p$ its norm. Here we show that Maher's result is also true in the case where C_p is replaced by $\mathcal{K}(\mathcal{H})$, the ideal of all compact operators with $\|\cdot\|_\infty$ its norm. Which allows to generalize these results, we prove that if the pair (A,B) has $(\mathrm{PF})_{\mathcal{K}(\mathcal{H})}$, the Putnam-Fuglede's property in $\mathcal{K}(\mathcal{H})$, and $AT = TB$, where $T \in \mathcal{K}(\mathcal{H})$, then $\|T - (AX - XB)\|_\infty \geq \|T\|_\infty$ for all $X \in \mathcal{L}(\mathcal{H})$.

2. Normal derivations. In this section, we investigate on the orthogonality of the range and the kernel of a normal derivation modulo the ideal of all compact operators. We recall that the pair (A,B) has the property $(\mathrm{PF})_{\mathcal{K}(\mathcal{H})}$ if $AT = TB$, where $T \in \mathcal{K}(\mathcal{H})$ implies $A^*T = TB^*$. Before proving this result we need the following lemmas.

Lemma 2.1. *Let $N, X \in \mathcal{L}(\mathcal{H})$, where N is a diagonal operator. If $\delta_N(X) + S \in \mathcal{K}(\mathcal{H})$, then $S \in \mathcal{K}(\mathcal{H})$ and $\|\delta_N(X) + S\|_\infty \geq \|S\|_\infty$.*

Proof. Let $\lambda_1,\lambda_2,\dots,\lambda_n$ be eigenvalues of the diagonal operator N. Then, the operator N can be written under the following matrix form:

$$\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}. \tag{2.1}$$

According to the following decomposition of $\mathcal{H}$:

$$\mathcal{H} = \bigoplus_{i=1}^{n} \ker(N-\lambda_j). \tag{2.2}$$

Let $|\delta_{ij}|$ and $|X_{ij}|$ be the matrix representations of S and X according to the above decomposition of $\mathcal{H}$. Then

$$NX - XN = |(\lambda_i - \lambda_j)X_{ij}|. \tag{2.3}$$

Since $S \in \{N\}'$ (the commutant of N), we get $S_{ij} = 0$ for $i \neq j$. Consequently

$$NX - XN + S = \begin{bmatrix} S_{11} & * & * & * \\ * & S_{22} & * & * \\ * & * & * & * \\ * & * & * & S_{nn} \end{bmatrix}. \tag{2.4}$$

Here $*$ stands for some entry.

As $\delta_N(X) + S \in \mathcal{K}(\mathcal{H})$, so $S \in \mathcal{K}(\mathcal{H})$ and the result of Gohberg and Kreĭn [2] guarantee that $\|\delta_N(X)+S\|_\infty \geq \|S\|_\infty$. □

Lemma 2.2. *Let $N \in \mathcal{L}(\mathcal{H})$ be a normal operator and let $\mathcal{H}_1 = \mathrm{Vect}_{\lambda\in\mathbb{C}}\ker(N-\lambda)$. If $S \in \{N\}'$ and there exists $X \in \mathcal{L}(\mathcal{H})$ such that $\delta_N(X)+S \in \mathcal{K}(\mathcal{H})$, then $\mathcal{H}_1$ reduces S and the restriction $S|_{\mathcal{H}_1^\perp} = 0$.*

Proof. Since N is a normal operator, $\mathcal{H}_1$ reduces N and the restriction $N|_{\mathcal{H}_1}$is a diagonal operator, then the Putnam-Fuglede's theorem guarantees that $S^* \in \{N\}'$. Hence, $\mathcal{H}_1$ reduces S. Let

$$N = \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix}, \qquad S = \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}, \qquad X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \tag{2.5}$$

on $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, where $\mathcal{H}_2 = \mathcal{H}_1^\perp$. The hypothesis $\delta_N(X)+S \in \mathcal{K}(\mathcal{H})$ would imply that $\delta_{N_2}(X_{22}) + S_2 \in \mathcal{K}(\mathcal{H})$. The result of Anderson [1] (applied to the Calkin algebra $\mathcal{L}(\mathcal{H}_2)\setminus\mathcal{K}(\mathcal{H}_2)$) guarantees that $S_2 \in \mathcal{K}(\mathcal{H})$. Since the normal operator N_2 is without eigenvalues and the selfadjoint operator $S_2^*S_2$ is compact and belongs to the commutant of N_2, it results that $S_2^*S_2 = 0$ and thus $S_2 = 0$. □

Theorem 2.3. *Let $N \in \mathcal{L}(\mathcal{H})$ be a normal operator, $S \in \{N\}'$, and $X \in \mathcal{L}(\mathcal{H})$. If $\delta_N(X)+S \in \mathcal{K}(\mathcal{H})$, then $S \in \mathcal{K}(\mathcal{H})$ and*

$$||\delta_N(X)+S||_\infty \geq \|S\|_\infty. \tag{2.6}$$

Proof. Since $\delta_N(X)+S \in \mathcal{K}(\mathcal{H})$, it follows from Lemma 2.2 that

$$N = \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix}, \qquad S = \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} \tag{2.7}$$

on $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_1^\perp$, where $\mathcal{H}_1 = \mathrm{Vect}_{\lambda\in\mathbb{C}} \ker(N-\lambda)$. If

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \tag{2.8}$$

on $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_1^\perp$, then

$$\delta_N(X)+S = \begin{bmatrix} \delta_{N_1}(X_{11})+S_1 & * \\ * & * \end{bmatrix}. \tag{2.9}$$

Since $\delta_N(X)+S \in \mathcal{K}(\mathcal{H})$, it results that $\delta_{N_1}(X_{11})+S_1 \in \mathcal{K}(\mathcal{H})$. As N is a diagonal operator and $S_1 \in \{N_1\}'$, it follows from Lemma 2.1 that S_1 is compact and

$$||\delta_{N_1}(X_{11})+S_1||_\infty \geq ||S_1||_\infty. \tag{2.10}$$

Consequently, S is compact and

$$||\delta_N(X)+S||_\infty \geq ||\delta_{N_1}(X_{11})+S_1||_\infty \geq ||S_1||_\infty = \|S\|_\infty. \tag{2.11}$$

□

Corollary 2.4. *Let $N,M,S \in \mathcal{L}(\mathcal{H})$ such that N and M are normal operators and $NS = SM$. If $X \in \mathcal{L}(\mathcal{H})$ such that $\delta_{N,M}(X)+S \in \mathcal{K}(\mathcal{H})$, then $S \in \mathcal{K}(\mathcal{H})$ and*

$$||\delta_{N,M}(X)+S||_\infty \geq \|S\|_\infty. \tag{2.12}$$

Proof. Consider the operators L, T, and Y defined on $\mathcal{H} = \mathcal{H} \oplus \mathcal{H}$ by

$$L = \begin{bmatrix} N & 0 \\ 0 & M \end{bmatrix}, \qquad S = \begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix}, \qquad Y = \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}, \tag{2.13}$$

then L is normal, $T \in \{L\}'$ and

$$\delta_L(Y)+T = \begin{bmatrix} 0 & \delta_{N,M}(X)+S \\ 0 & 0 \end{bmatrix}. \tag{2.14}$$

Then Theorem 2.3 would imply that T is compact and

$$||\delta_L(Y)+T||_\infty \geq \|T\|_\infty, \tag{2.15}$$

consequently, S is compact and

$$\left\|\delta_{N,M}(X)+S\right\|_\infty \geq \|S\|_\infty. \tag{2.16}$$

□

3. Generalized derivations. In this section, we generalize the above results to a large class of operators. We show that if the pair (A,B) has the property $(\mathrm{PF})_{\mathcal{K}(\mathcal{H})}$, and $AS = SB$ such that $\delta_{N,M}(X)+S \in \mathcal{K}(\mathcal{H})$, then $S \in \mathcal{K}(\mathcal{H})$ and

$$\left\|\delta_{A,B}(X)+S\right\|_\infty \geq \|S\|_\infty, \quad \forall x \in \mathcal{L}(\mathcal{H}). \tag{3.1}$$

Before proving this result, we need the following lemma.

Lemma 3.1. *Let $A,B \in \mathcal{L}(\mathcal{H})$. The following statements are equivalent:*
(1) *the pair (A,B) has the property* $(\mathrm{PF})_{\mathcal{K}(\mathcal{H})}$*;*
(2) *if $AT = TB$, where $T \in \mathcal{K}(\mathcal{H})$, then $\overline{R(T)}$ reduces A, $\ker(T)^\perp$ reduces B, and $A|_{\overline{R(T)}}$ and $B|_{\ker(T)^\perp}$ are normal operators.*

Proof. (1)⇒(2). Since $\mathcal{K}(\mathcal{H})$ is a bilateral ideal and $T \in \mathcal{K}(\mathcal{H})$, then $AT \in \mathcal{K}(\mathcal{H})$. Hence, as $AT = TB$ and (A,B) satisfies $(\mathrm{PF})_{\mathcal{K}(\mathcal{H})}$, $A^*T = TB^*$ and $\overline{R(T)}$, and $\ker(T)^\perp$ are reducing subspaces for A and B, respectively. Since $A(AT) = (AT)B$ implies $A^*(AT) = (AT)B^*$ by $(\mathrm{PF})_{\mathcal{K}(\mathcal{H})}$, and the identity $A^*T = TB^*$ implies that $A^*AT = AA^*T$, thus we see that $A|_{\overline{R(T)}}$ is normal. Clearly, (B^*,A^*) satisfies $(\mathrm{PF})_{\mathcal{K}(\mathcal{H})}$ and $B^*T^* = T^*A^*$. Therefore, it follows from the above argument that $B^*|_{\overline{R(T^*)}} = B|_{\ker(T)^\perp}$ is normal.

(2)⇒(1). Let $T \in \mathcal{K}(\mathcal{H})$ such that $AT = TB$. Taking the two decompositions of $\mathcal{H}$, $\mathcal{H}_1 = \mathcal{H} = \overline{R(T)} \oplus \overline{R(T)}^\perp$ and $\mathcal{H}_2 = \mathcal{H} = \ker(T)^\perp \oplus \ker T$. Then we can write A and B on $\mathcal{H}_1$ into $\mathcal{H}_2$, respectively,

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \qquad B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}, \tag{3.2}$$

where A_1 and B_1 are normal operators. Also we can write T and X on $\mathcal{H}_2$ into $\mathcal{H}_1$

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}. \tag{3.3}$$

It follows from $AT = TB$ that $A_1T_1 = T_1B_1$. Since $A_1$and B_1 are normal operators, then, by applying the Fuglede-Putnam's theorem, we obtain $A_1^*T_1 = T_1B_1^*$, that is, $A^*T = TB^*$. □

Theorem 3.2. *Let $A,B \in \mathcal{L}(\mathcal{H})$ satisfying* $(\mathrm{PF})_{\mathcal{K}(\mathcal{H})}$ *and $AS = SB$. If $X \in \mathcal{L}(\mathcal{H})$ such that $\delta_{A,B}(X)+S \in \mathcal{K}(\mathcal{H})$, then $S \in \mathcal{K}(\mathcal{H})$ and*

$$\left\|\delta_{A,B}(X)+S\right\|_\infty \geq \|S\|_\infty. \tag{3.4}$$

Proof. Since the pair (A,B) satisfies the property $(\mathrm{PF})_{\mathcal{K}(\mathcal{H})}$, it follows by Lemma 3.1 that $\overline{R(S)}$ reduces A, $\ker(S)^{\perp}$ reduces B, and $A|_{\overline{R(S)}}$ and $B|_{\ker(S)^{\perp}}$ are normal operators. Let $\mathcal{H}_1 = \overline{R(S)} \oplus \overline{R(S)}^{\perp}$ and $\mathcal{H}_2 = \ker(S)^{\perp} \oplus \ker S$. Then

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \qquad B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix},$$
$$S = \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}. \tag{3.5}$$

It follows from

$$AS - SB = \begin{bmatrix} A_1S_1 - S_1B_1 & 0 \\ 0 & 0 \end{bmatrix} = 0 \tag{3.6}$$

that $A_1S_1 = S_1B_1$ and we have

$$||S - (AX - XB)||_{\infty} = \left|\left| \begin{bmatrix} S_1 - (A_1X_1 - X_1B_1) & * \\ * & * \end{bmatrix} \right|\right|_{\infty}. \tag{3.7}$$

Since A_1 and B_1 are two normal operators, then it results from Corollary 2.4 that S_1 is compact and

$$||S_1 - (A_1X_1 - X_1B_1)||_{\infty} \geq ||S_1||_{\infty}, \tag{3.8}$$

so

$$||S - (AX - XB)||_{\infty} \geq ||S_1 - (A_1X_1 - X_1B_1)||_{\infty} \geq ||S_1||_{\infty} = \|S\|_{\infty}. \tag{3.9}$$

□

Corollary 3.3. *Let $A,B \in \mathcal{L}(\mathcal{H})$ satisfying $(\mathrm{PF})_{\mathcal{K}(\mathcal{H})}$ and $AS = SB$. If $X \in \mathcal{L}(\mathcal{H})$ such that $\delta_{A,B}(X) + S \in \mathcal{K}(\mathcal{H})$, then $S \in \mathcal{K}(\mathcal{H})$ and*

$$\|S + AX - XB\|_{\infty} \geq \|S\|_{\infty} \tag{3.10}$$

in each of the following cases:

(1) *if $A,B \in \mathcal{L}(\mathcal{H})$ such that $\|Ax\| \geq \|x\| \geq \|Bx\|$ for all $x \in \mathcal{H}$;*

(2) *if A is invertible and B such that $\|A^{-1}\|\|B\| \leq 1$.*

Proof. (1) The result of Tong [5, Lemma 1] guarantees that the above condition implies that for all $T \in \ker(\delta_{A,B} \mid \mathcal{K}(\mathcal{H}))$, $\overline{R(T)}$ reduces A, $\ker(T)^{\perp}$ reduces B, and $A|_{\overline{R(T)}}$ and $B|_{\ker(T)^{\perp}}$ are unitary operators. Hence, it results from Lemma 3.1 that the pair (A,B) has the property $(\mathrm{PF})_{\mathcal{K}(\mathcal{H})}$ and the result holds by Theorem 3.2.

Inequality (3.10) holds in particular if $A = B$ is isometric; in other words, $\|Ax\| = \|x\|$ for all $x \in \mathcal{H}$.

(2) In this case, it suffices to take $A_1 = \|B\|^{-1}A$ and $B_1 = \|B\|^{-1}B$, then $\|A_1x\| \geq \|x\| \geq \|B_1x\|$ and the result holds by (1) for all $x \in \mathcal{H}$. □

References

[1] J. H. Anderson, *On normal derivations*, Proc. Amer. Math. Soc. **38** (1973), 135–140.
[2] I. C. Gohberg and M. G. Kreĭn, *Introduction to the Theory of Linear Nonselfadjoint Operators*, Translations of Mathematical Monographs, vol. 18, American Mathematical Society, Rhode Island, 1969.
[3] P. J. Maher, *Commutator approximants*, Proc. Amer. Math. Soc. **115** (1992), no. 4, 995–1000.
[4] S. Mecheri, *On minimizing* $\|S-(AX-XB)\|_p^p$, Serdica Math. J. **26** (2000), no. 2, 119–126.
[5] Y. S. Tong, *Kernels of generalized derivations*, Acta Sci. Math. (Szeged) **54** (1990), no. 1-2, 159–169.

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