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ON AN ABSTRACT EVOLUTION EQUATION WITH A SPECTRAL OPERATOR OF SCALAR TYPE

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It is shown that the weak solutions of the evolution equation $y'(t) = Ay(t)$, $t \in [0,T)$ $(0 < T \leq \infty)$, where A is a spectral operator of scalar type in a complex Banach space X, defined by Ball (1977), are given by the formula $y(t) = e^{tA}f$, $t \in [0,T)$, with the exponentials understood in the sense of the operational calculus for such operators and the set of the initial values, f's, being $\bigcap_{0\leq t<T} D(e^{tA})$, that is, the largest possible such a set in X.

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1. Introduction. Consider the evolution equation

$$y'(t) = Ay(t), \quad t \in [0,T)\ (0 < T \leq \infty), \tag{1.1}$$

in a complex Banach space X with a *spectral operator* A of *scalar type* [2, 5].

Following [1], by a *weak solution* of (1.1) with a densely defined linear operator A in a Banach space X, we understand a vector function $y : [0,T) \to X$ that satisfies the following conditions:

(a) $y(\cdot)$ is *strongly continuous* on $[0,T)$;

(b) for any $g^* \in D(A^*)$,

$$\frac{d}{dt}\langle y(t), g^*\rangle = \langle y(t), A^*g^*\rangle, \quad 0 \leq t < T, \tag{1.2}$$

where $D(\cdot)$ is the domain of an operator, A^* is the operator adjoint to A, and $\langle \cdot,\cdot\rangle$ is the pairing between the space X and its dual X^*.

Note that the weak solutions thus defined are not expected to satisfy (1.1) in the *classical* plug-in sense, that is, when the requirements of $y(\cdot)$, being *strongly differentiable* and taking values exclusively in $D(A)$, are presupposed implicitly.

It is also readily seen that the notion of a *weak solution* of (1.1) is more general than that of the *classical* one.

When the operator A is *closed*, the *classical solutions* of (1.1) are precisely those of its *weak solutions* that are *strongly differentiable* (see [1] for details).

The purpose of the present paper is to stretch out [8, Theorem 3.1] which states that the *general weak solution* of (1.1) with a *normal operator* A in a complex Hilbert space is of the form

$$y(t) = e^{tA}f, \quad t \in [0,T),\ f \in \bigcap_{0\leq t<T} D(e^{tA}), \tag{1.3}$$

the exponentials being understood in the sense of the *operational calculus* for such operators [4, 9], to the more general case of a *spectral operator of scalar type* (*scalar operator*) in a complex Banach space.

Note for that matter that, in a Hilbert space, the *scalar operators* are the operators similar to *normal* ones [10].

The latter result suggests that the weak solutions of (1.1), with the set of their initial values $\bigcap_{0\le t<T} D(e^{tA})$ being the largest such a set, most inherently represent the exponential nature of the equation, more so than their classical fellows.

Observe that the same state of affairs is the case when A generates a C_0-semigroup of bounded linear operators $\{e^{tA} \mid t \ge 0\}$ in a Banach space [6], the *classical* and *weak solutions* being the orbits $e^{tA}f$ with the initial value sets $D(A)$ and $X = \bigcap_{0\le t<T} D(e^{tA})$, respectively, [1].

As is to be expected, the departure from a Hilbert space, immediately depriving us of its powerful inner product techniques, causes certain challenges to be faced in the following generalization endeavor of ours.

2. Preliminaries.

Hereafter, unless specifically stated otherwise, A is a *scalar operator* in a complex Banach space X with a norm $\|\cdot\|$ and $E_A(\cdot)$ is its *spectral measure* (*resolution of the identity*) [2, 5]. Borel sets of the complex plane that has as its values bounded projection operators on X and enjoys a number of distinctive properties [2, 5].

For such operators, there is an *operational calculus* for Borel measurable functions on the *spectrum* [2, 5].

If $F(\cdot)$ is a Borel measurable function on the spectrum of A, $\sigma(A)$, a new *scalar operator*

$$F(A) = \int_{\sigma(A)} F(\lambda)\, dE_A(\lambda) \tag{2.1}$$

is defined as follows:

$$\begin{aligned} F(A)f &:= \lim_{n\to\infty} F_n(A)f, \quad f \in D(F(A)), \\ D(F(A)) &:= \Big\{ f \in X \mid \lim_{n\to\infty} F_n(A)f \text{ exists} \Big\}, \end{aligned} \tag{2.2}$$

where

$$F_n(\cdot) := F(\cdot)\chi_{\{\lambda\in\sigma(A) \mid |F(\lambda)|\le n\}}(\cdot), \quad n = 1, 2, \ldots, \tag{2.3}$$

($\chi_\alpha(\cdot)$ is the *characteristic function* of a set α), and

$$F_n(A) := \int_{\sigma(A)} F_n(\lambda)\, dE_A(\lambda), \quad n = 1, 2, \ldots, \tag{2.4}$$

being the integrals of *bounded* Borel measurable functions on $\sigma(A)$, are *bounded scalar operators* on X defined in the same way as for *normal operators* (e.g., [4, 9]).

In particular,

$$A = \int_{\sigma(A)} \lambda\, dE_A(\lambda). \tag{2.5}$$

Note that, if $F(\cdot)$ is a function *analytic* on an *open* set U such that $E(U) = I$ (I is the *identity operator*) and $\{e_n\}_{n=1}^{\infty}$ is an arbitrary sequence of *bounded* Borel sets whose *closures* are contained in U and $E_A(\bigcup_{n=1}^{\infty} e_n) = I$, the operator $F(A)$ can also be defined as follows [5]:

$$\begin{aligned} D(F(A)) &:= \Big\{ f \in X \mid \lim_{n\to\infty} F(A|E_A(e_n)X)E_A(e_n)f \text{ exists} \Big\}, \\ F(A)f &:= \lim_{n\to\infty} F(A|E_A(e_n)X)E_A(e_n)f, \quad f \in D(F(A)), \end{aligned} \tag{2.6}$$

where $P|Y$ is the restriction of an operator P to a subspace Y.

The properties of the *spectral measure*, $E_A(\cdot)$, and the *operational calculus* for *scalar operators* underlying the entire argument to follow, are exhaustively delineated in [2, 5].

Here, we single out one of them, which is a real cornerstone for the statement of the next section: *the spectral measure is bounded*, that is, there is an $M > 0$ such that

$$||E_A(\delta)|| \le M \quad \text{for any Borel set } \delta. \tag{2.7}$$

Note that here the same notation as for the norm in X, $\|\cdot\|$, is used to designate the norm in the space of bounded linear operators on X, $\mathcal{L}(X)$. We do so henceforth for the operator norm as well as for the norm in the dual space X^*, such an economy of symbols being a rather common practice.

On account of compactness, the terms *spectral measure* and *operational calculus* for spectral operators will be abbreviated to s.m. and o.c., respectively.

3. A characterization of the domain of a scalar operator. As is well known [4, 9], for a *normal operator* A with a *spectral measure* $E_A(\cdot)$ in a complex Hilbert space H with an inner product $(\cdot,\cdot)$, the domain of the operator $F(A)$, $F(\cdot)$ being a complex-valued Borel measurable function on $\sigma(A)$, can be characterized in terms of positive measures:

$$f \in D(F(A)) \text{ if and only if } \int_{\sigma(A)} |F(\lambda)|^2 d(E(\lambda)f,f) < \infty. \tag{3.1}$$

Our purpose here is to obtain an analogue of such a description for *scalar operators*.

Before we proceed, we agree to use the notation $v(f,g^*,\cdot)$, $f \in X$ and $g^* \in X^*$, for the *total variation* of the complex-valued Borel measure $\langle E_A(\cdot)f,g^*\rangle$.

Proposition 3.1. *Let $F(\cdot)$ be a complex-valued Borel measurable function on the spectrum of a scalar operator A. Then $f \in D(F(A))$ if and only if*

(i) *for any $g^* \in X^*$,*

$$\int_{\sigma(A)} |F(\lambda)|\, dv(f,g^*,\lambda) < \infty; \tag{3.2}$$

(ii)

$$\sup_{\{g^*\in X^* \mid \|g^*\|=1\}} \int_{\{\lambda\in\sigma(A) \mid |F(\lambda)|>n\}} |F(\lambda)|\,dv(f,g^*,\lambda) \longrightarrow 0 \quad as\ n \longrightarrow \infty. \tag{3.3}$$

Proof

"Only if" part. Let $f \in D(F(A))$. Then, by the properties of the o.c. [5],

$$\int_{\sigma(A)} F(\lambda)\,d\langle E_A(\lambda)f,g^*\rangle = \langle F(A)f,g^*\rangle, \quad g^* \in X^*, \tag{3.4}$$

whence condition (i) follows immediately (e.g., [3]).

To prove (ii), note first that, the positive Borel measure

$$\int_{\cdot} |F(\lambda)|\,dv(f,g^*,\lambda) \tag{3.5}$$

being the total variation of the complex-valued measure

$$\int_{\cdot} F(\lambda)\,d\langle E_A(\lambda)f,g^*\rangle, \tag{3.6}$$

where the dots can be replaced by an arbitrary Borel set we have the estimate [3]

$$\int_{\alpha} |F(\lambda)|\,dv(f,g^*,\lambda) \le 4\sup_{\beta\subseteq\alpha} \left| \int_{\beta} F(\lambda)\,d\langle E_A(\lambda)f,g^*\rangle \right|, \tag{3.7}$$

where α and β are Borel sets.

Henceforth, let $\delta_n := \{\lambda \in \sigma(A) \mid |F(\lambda)| > n\}$, $n = 1,2,\ldots$, and let β be a Borel set. By (3.7),

$$\begin{aligned}
&\sup_{\{g^*\in X^* \mid \|g^*\|=1\}} \int_{\delta_n} |F(\lambda)|\,dv(f,g^*,\lambda) \\
&\quad\le 4 \sup_{\{g^*\in X^* \mid \|g^*\|=1\}} \sup_{\beta\subseteq\delta_n} \left| \int_{\delta_n} F(\lambda)\chi_\beta(\lambda)d\langle E_A(\lambda)f,g^*\rangle \right| \\
&\qquad\qquad \text{by the properties of the o.c.} \\
&\quad= 4 \sup_{\{g^*\in X^* \mid \|g^*\|=1\}} \sup_{\beta\subseteq\delta_n} \left| \left\langle \int_{\delta_n} F(\lambda)\chi_\beta(\lambda)\,dE_A(\lambda)f,g^* \right\rangle \right| \\
&\qquad\qquad \text{by the properties of the o.c.} \\
&\qquad\qquad \text{and definitions (2.2), (2.3), and (2.4)} \\
&\quad= 4 \sup_{\{g^*\in X^* \mid \|g^*\|=1\}} \sup_{\beta\subseteq\delta_n} |\langle E_A(\beta)(F(A)f - F_n(A)f),g^*\rangle| \\
&\quad\le 4 \sup_{\beta\subseteq\delta_n} \|E_A(\beta)\| \|F(A)f - F_n(A)f\| \quad \text{by (3.7)} \\
&\quad\le 4M\|F(A)f - F_n(A)f\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{3.8}$$

"If" part. Let $f \in X$ be a vector satisfying conditions (i) and (ii). Then, for any natural m and n $(m < n)$, we have, as follows from the *Hahn-Banach theorem*,

$$\begin{aligned}
&\|F_n(A)f - F_m(A)f\| \\
&\quad = \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \left|\langle F_n(A)f - F_m(A)f, g^*\rangle\right| \quad \text{by (2.3), (2.4)} \\
&\quad = \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \Bigg|\left\langle \int_{\{\lambda\in\sigma(A)\mid |F(\lambda)|\le n\}} F(\lambda)\,dE_A(\lambda)f, g^*\right\rangle \\
&\qquad\qquad - \left\langle \int_{\{\lambda\in\sigma(A)\mid |F(\lambda)|\le m\}} F(\lambda)\,dE_A(\lambda)f, g^*\right\rangle\Bigg| \quad \text{by condition (i)} \\
&\quad = \sup_{\|g^*\|=1} \Bigg|\left\langle \int_{\sigma(A)} F(\lambda)\,dE_A(\lambda)f, g^*\right\rangle - \left\langle \int_{\{\lambda\in\sigma(A)\mid |F(\lambda)|> n\}} F(\lambda)dE_A(\lambda)f, g^*\right\rangle \\
&\qquad\qquad - \left(\left\langle \int_{\sigma(A)} F(\lambda)\,dE_A(\lambda)f, g^*\right\rangle - \left\langle \int_{\{\lambda\in\sigma(A)\mid |F(\lambda)|> m\}} F(\lambda)\,dE_A(\lambda)f, g^*\right\rangle\right)\Bigg| \\
&\quad = \sup_{\|g^*\|=1} \Bigg|\left\langle \int_{\{\lambda\in\sigma(A)\mid |F(\lambda)|> m\}} F(\lambda)\,dE_A(\lambda)f, g^*\right\rangle \\
&\qquad\qquad - \left\langle \int_{\{\lambda\in\sigma(A)\mid |F(\lambda)|> n\}} F(\lambda)\,dE_A(\lambda)f, g^*\right\rangle\Bigg| \\
&\quad \le \sup_{\|g^*\|=1} \left|\int_{\{\lambda\in\sigma(A)\mid |F(\lambda)|> m\}} F(\lambda)\,d\langle E_A(\lambda)f, g^*\rangle\right| \\
&\qquad + \sup_{\|g^*\|=1} \left|\int_{\{\lambda\in\sigma(A)\mid |F(\lambda)|> n\}} F(\lambda)\,d\langle E_A(\lambda)f, g^*\rangle\right| \\
&\quad \le \sup_{\|g^*\|=1} \int_{\{\lambda\in\sigma(A)\mid |F(\lambda)|> m\}} |F(\lambda)|\,dv(f, g^*, \lambda) \\
&\qquad + \sup_{\|g^*\|=1} \int_{\{\lambda\in\sigma(A)\mid |F(\lambda)|> n\}} |F(\lambda)|\,dv(f, g^*, \lambda) \longrightarrow 0 \quad \text{as } m, n \longrightarrow \infty, \text{ by (ii).}
\end{aligned} \tag{3.9}$$

Thus, $\{F_n(A)f\}_{n=1}^{\infty}$ is a *Cauchy sequence* converging in the Banach space X, which implies that f belongs to $D(F(A))$. □

4. The principal statement.

The following lemma consists of three easy to prove statements, which become handy when engaging *dual space* techniques.

Lemma 4.1. (i) *For any Borel set δ, $E_A(\delta)^*$ is a bounded projection operator in the dual space X^*.*

(ii) *For any bounded Borel set δ,*

$$E_A^*(\delta)X^* \subseteq D(A^*). \tag{4.1}$$

(iii) *For any Borel set δ,*

$$E_A^*(\delta)A^* \subset A^*E_A^*(\delta), \tag{4.2}$$

where $P \subset Q$ means that an operator Q is an extension of an operator P.

Proof. (i) Immediately follows from the properties of *conjugates*.

(ii) Let δ be a *bounded* Borel set. For any $g^* \in X^*$, consider the following linear functional:

$$D(A) \ni f \longmapsto \langle Af, E_A^*(\delta)g^* \rangle. \tag{4.3}$$

We have

$$\langle Af, E_A^*(\delta)g^* \rangle = \langle E_A(\delta)Af, g^* \rangle, \quad f \in D(A),\ g^* \in X^*. \tag{4.4}$$

By the properties of s.m., $E_A(\delta)A \subset AE_A(\delta)$ and $E_A(\delta)X \subseteq D(A)$. By the *closed graph theorem*, the closed linear operator $AE_A(\delta)$ defined on the entire space X is *bounded* and so is $E_A(\delta)A$ (note that the operator $AE_A(\delta)$ is the *closure* of $E_A(\delta)A$). Whence the boundedness of functional (4.3) follows immediately.

Therefore, $E_A^*(\delta)g^* \in D(A^*)$ and

$$\langle Af, E_A^*(\delta)g^* \rangle = \langle f, A^*E_A^*(\delta)g^* \rangle. \tag{4.5}$$

(iii) By the properties of s.m., $E_A(\delta)A \subset AE_A(\delta)$, which immediately implies that

$$E_A^*(\delta)A^* \subset (AE_A(\delta))^* \subset (E_A(\delta)A)^* = E_A(\delta) \text{ is } \textit{bounded} = A^*E_A(\delta)^*. \tag{4.6}$$

□

Theorem 4.2. *A vector function $y : [0,T) \mapsto X$ is a weak solution of (1.1) on the interval $[0,T)$ $(0 < T \le +\infty)$ if and only if there is a vector $f \in \bigcap_{0 \le t < T} D(e^{tA})$ such that*

$$y(t) = e^{tA}f, \quad t \in [0,T). \tag{4.7}$$

Proof

"Only if" part. Let $y(\cdot)$ be a weak solution of (1.1) on the interval $[0,T)$ and $\Delta_n := \{\lambda \in \sigma(A) \mid |\lambda| \le n\}$, $n = 1,2,\ldots$.

Consider the following sequence of vector functions:

$$y_n(t) = E_A(\Delta_n)y(t), \quad t \in [0,T),\ n = 1,2,\ldots. \tag{4.8}$$

The *strong continuity* of the functions $y_n(\cdot)$'s on $[0,T)$ follows from that of $y(\cdot)$ the boundedness of the projections $E_A(\Delta_n)$'s.

Further, for any natural n and each $g^* \in X^*$,

$$\begin{aligned}
&\frac{d}{dt}\langle y_n(t), g^*\rangle \\
&\quad = \frac{d}{dt}\langle E_A(\Delta_n)y(t), g^*\rangle = \frac{d}{dt}\langle y(t), E_A^*(\Delta_n)g^*\rangle \\
&\qquad\qquad \text{since by Lemma 4.1 } E_A^*(\Delta_n)g^* \in D(A^*) \\
&\qquad\qquad \text{and } y(\cdot) \text{ is a \textit{weak solution} of (1.1)} \\
&\quad = \langle y(t), A^*E_A^*(\Delta_n)g^*\rangle \quad \text{by Lemma 4.1,} \\
&\qquad\qquad A^*E_A^*(\Delta_n) = A^*\left[E_A^*(\Delta_n)\right]^2 = E_A^*(\Delta_n)A^*E_A^*(\Delta_n) \\
&\quad = \langle y(t), E_A^*(\Delta_n)A^*E_A^*(\Delta_n)g^*\rangle = \langle E_A(\Delta_n)y(t), A^*E_A^*(\Delta_n)g^*\rangle \\
&\qquad\qquad \text{by the properties of s.m., } \Delta_n \text{ being bounded,} \\
&\qquad\qquad AE_A(\Delta_n) \in \mathscr{L}(X) \text{ and is the \textit{closure} of } E_A(\Delta_n)A, \\
&\qquad\qquad \text{hence, } A^*E_A^*(\Delta_n) = \left(E_A(\Delta_n)A\right)^* = \left(AE_A(\Delta_n)\right)^* \\
&\quad = \langle y_n(t), \left(AE_A(\Delta_n)\right)^* g^*\rangle, \quad t \in [0,T).
\end{aligned} \tag{4.9}$$

Thus, for any natural n, $y_n(\cdot)$ is a weak solution of the equation

$$y'(t) = AE_A(\Delta_n)y(t), \quad 0 \le t < T, \tag{4.10}$$

which, since the operator $AE_A(\Delta_n)$ is bounded, implies [1] that

$$y_n(t) = e^{tAE_A(\Delta_n)}y_n(0) = e^{tAE_A(\Delta_n)}E_A(\Delta_n)f, \quad 0 \le t < T, \tag{4.11}$$

where $f := y(0)$.

Since $A|E_A(\Delta_n) \subset AE_A(\Delta_n)$, $n = 1,2,\dots$, $e^{tA|E_A(\Delta_n)X} \subset e^{tAE_A(\Delta_n)}$, $0 \le t < T$, $n = 1,2,\dots$ (all the operators are bounded).

Hence, for $0 \le t < T$ and $n = 1,2,\dots$,

$$e^{tA|E_A(\Delta_n)X}E_A(\Delta_n)f = e^{tAE_A(\Delta_n)}E_A(\Delta_n)f = E_A(\Delta_n)y(t). \tag{4.12}$$

Since $\{\Delta_n\}_{n=1}^{\infty}$ is an increasing sequence of bounded Borel sets such that $\bigcup_{n=1}^{\infty}\Delta_n = \mathbb{C}$, $\lim_{n\to\infty}E_A(\Delta_n)y(t) = y(t)$, $0 \le t < T$.

Whence, by definition (2.6), we infer that $f \in \bigcap_{0\le t<T}D(e^{tA})$ and $y(t) = e^{tA}f$, $0 \le t < T$.

"If" part. Consider an arbitrary segment $[a,b] \subset [0,T)$ $(0 \le a < b < T)$.

Let $\delta_n := \{\lambda \in \sigma(A) \mid \operatorname{Re}\lambda \le \ln n/b\}$, $n = 1,2,\dots$ and

$$A_n := AE_A(\delta_n), \quad n = 1,2,\dots. \tag{4.13}$$

Since, by the properties of s.m., $\sigma(A_n) \subseteq \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \le \ln n/b\}$, $n = 1,2,\dots$, the operator A_n generates the C_0-semigroup of linear bounded operators, which consists of its exponentials $\{e^{tA_n} \mid t \ge 0\}$ [7].

Then [1], for any $f \in X$ and $g^* \in X^*$,

$$\langle e^{tA_n} f, g^* \rangle - \langle f, g \rangle = \int_0^t \langle e^{sA_n} f, A_n^* g^* \rangle \, ds, \quad 0 \le t < T. \tag{4.14}$$

We show that, for any $f \in \bigcap_{0 \le t < T} D(e^{tA})$, the sequence of vector functions $e^{\cdot A_n} f$ converges to $e^{\cdot A} f$ *uniformly* on $[a,b]$.

Thus, for $f \in \bigcap_{0 \le t < T} D(e^{tA})$,

$$\begin{aligned}
&\sup_{a \le t \le b} \|e^{tA} f - e^{tA_n} f\| \quad \text{as follows from the } \textit{Hahn-Banach theorem}, \\
&\sup_{a \le t \le b} \sup_{\{g^* \in X^* \mid \|g^*\| = 1\}} \left| \langle e^{tA} f - e^{tA_n} f, g^* \rangle \right|, \\
&\qquad\qquad \text{by the properties of the o.c.} \\
&= \sup_{a \le t \le b} \sup_{\{g^* \in X^* \mid \|g^*\| = 1\}} \left| \int_{\sigma(A)} \left[e^{t\lambda} - e^{t\lambda \chi_{\delta_n}(\lambda)} \right] d\langle E_A(f, g^*) \rangle \right| \\
&= \sup_{a \le t \le b} \sup_{\{g^* \in X^* \mid \|g^*\| = 1\}} \int_{\{\lambda \in \sigma(A) \mid \operatorname{Re}\lambda > \ln n / b\}} \left| e^{t\lambda} - 1 \right| dv(f, g^*, \lambda) \\
&\qquad\qquad \text{since, under the restrictions on } t \text{ and } \lambda, \ t \operatorname{Re}\lambda \ge 0 \\
&\le \sup_{a \le t \le b} \sup_{\{g^* \in X^* \mid \|g^*\| = 1\}} \int_{\{\lambda \in \sigma(A) \mid \operatorname{Re}\lambda > \ln n / b\}} 2 e^{t \operatorname{Re}\lambda} dv(f, g^*, \lambda) \\
&\le 2 \sup_{\{g^* \in X^* \mid \|g^*\| = 1\}} \int_{\{\lambda \in \sigma(A) \mid \operatorname{Re}\lambda > \ln n / b\}} e^{b \operatorname{Re}\lambda} dv(f, g^*, \lambda) \\
&= 2 \sup_{\{g^* \in X^* \mid \|g^*\| = 1\}} \int_{\{\lambda \in \sigma(A) \mid |e^{b\lambda}| > n\}} \left| e^{b\lambda} \right| dv(f, g^*, \lambda) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty,
\end{aligned} \tag{4.15}$$

by Proposition 3.1, since $f \in D(e^{bA})$, in particular.

Because $[a,b] \subset [0,T)$ is an arbitrary segment, the latter implies that the function $e^{\cdot A} f$ is *strongly continuous* on $[0,T)$ for any $f \in \bigcap_{0 \le t < T} D(e^{tA})$.

Furthermore, for any $g^* \in D(A^*)$,

$$\begin{aligned}
\|A^* g^* - A_n^* g^*\| &= \|A^* g^* - (A E_A(\delta_n))^* g^*\| = \|A^* g^* - E_A^*(\delta_n) A^* g^*\| \\
&= \|E_A(\{\lambda \in \sigma(A) \mid \operatorname{Re}\lambda > \ln n / b\}) A^* g^*\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty,
\end{aligned} \tag{4.16}$$

$\{\lambda \in \sigma(A) \mid \operatorname{Re}\lambda > \ln n / b\}$ being a decreasing sequence of Borel sets with empty intersection.

It is not difficult to make sure now that, for any $0 \le t < T$, $f \in \bigcap_{0 \le t < T} D(e^{tA})$, and $g^* \in D(A^*)$,

$$\sup_{0 \le s \le t} \left| \langle e^{sA_n} f, A_n^* g^* \rangle - \langle e^{sA} f, A^* g^* \rangle \right| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{4.17}$$

Passing to the limit in (4.14) as $n \to \infty$, for any $f \in \bigcap_{0 \le t < T} D(e^{tA})$ and $g^* \in D(A^*)$, we obtain:

$$\langle e^{tA} f, g^* \rangle - \langle f, g \rangle = \int_0^t \langle e^{sA} f, Ag^* \rangle \, ds, \quad 0 \le t < T. \tag{4.18}$$

Whence

$$\frac{d}{dt} \langle e^{tA} f, g^* \rangle = \langle e^{tA} f, Ag^* \rangle, \quad 0 \le t < T. \tag{4.19}$$

□

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