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RESEARCH NOTES

CONTROLLABILITY, BEZOUTIAN AND RELATIVE PRIMENESS

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ABSTRACT. Let f(x) and g(x) be two polynomials of degree n. Then it is well-known that the Bezoutian matrix B_{fg} associated with f(x) and g(x) is nonsingular if and only if f(x) and g(x) are relatively prime. We give an alternative proof of this result. The proof is based on a result on controllability derived in this note.

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KEY WORKS AND PHRASES. Bezoutian, Controllability, Nullity

1. INTRODUCTION.

Let $f(x) = x^n - a_n x^{n-1} - a_{n-1}x^{n-2} \dots - a_2 x - a_1$ and $g(x) = x^n - b_n x^{n-1} - b_{n-1}x^{n-2} \dots - b_2 x - b_1$ be two polynomials of degree n .

Then the Bezoutian bilinear form defined by f(x) and g(x) is given by

$$B(f,g) = \frac{f(x)g(y) - f(y)g(x)}{x-y} = \sum_{i,k=0}^{n-1} b_{ik}\, x^i y^k .$$

The symmetric matrix $B_{fg} = (b_{ik})$ is known as the Bezoutian matrix.

THEOREM 1. B_{fg} is nonsingular iff $f(x)$ and $g(x)$ are relatively prime.

The above result is classical and is well-known. Various proofs of this result are available in the literature (for references see the survey of Krien and Naimark [4] and the paper of Honseholder [3]).

In this note, we give a proof of this result using the idea of controllability.

Lemma 1 that follows forms the main tool of our proof. Besides its application to the proof of theorem 1, it is important in its own right and many find applications elsewhere.

2. TWO LEMMAS ON CONTROLLABILITY.

A pair of matrices (A,B), where A is n×n and B is n×m, is controllable if the n×nm matrix $C(A,B) = (B, AB, A^2B,\dots,A^{n-1}B)$ has rank n.

LEMMA 1. Let

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ & & & \ddots & & \\ & & & & & 1 \\ a_1 & a_2 & \cdots & & & a_n \end{pmatrix} \qquad (1)$$

be the companion matrix of $f(x)$ and let X, with x_n as its last row, be a solution of $XA = A^TX$.

Then X is nonsingular iff (A^T, x_n^T) is controllable.

PROOF. Let $x_1, x_2,\dots,x_n$ be n rows of X. Then the equation $XA = A^TX$ is equivalent to:

$$x_1A = a_1x_n$$
$$x_iA = x_{i-1} + a_ix_n, \qquad i = 2, 3,\dots,n.$$

The last equations can be written in the form:

$$x_{n-1} = x_n A - a_n x_n$$

$$x_{n-2} = x_{n-1}A - a_{n-1}x_n = (x_n A - a_n x_n)A - a_{n-1}x_n$$
$$= x_n A^2 - a_n x_n A - a_{n-1}x_n$$

$$x_{n-3} = x_{n-2}A - a_{n-2}x_n = (x_n A^2 - a_n x_n A - a_{n-1}x_n)A$$
$$- a_{n-2}x_n = x_n A^3 - a_n x_n A^2 - a_{n-1}x_n A - a_{n-2}x_n$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$x_1 = x_n A^{n-1} - a_n x_n A^{n-2} - a_{n-1}x_n A^{n-3} \cdots - a_2 x_n \quad .$$

Thus, we have, for any solution X of $XA = A^T X$,

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 1 & -a_n & -a_{n-1} & \cdots & -a_2 \\ 0 & 1 & -a_n & \cdots & -a_3 \\ 0 & 0 & 1 & -a_n \cdots & -a_4 \\ \vdots & & & & \\ 0 & 0 & \cdots\cdots & 1 & -a_n \\ 0 & 0 & \cdots\cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} x_n A^{n-1} \\ x_n A^{n-2} \\ x_n A^{n-3} \\ \vdots \\ x_n A \\ x_n \end{pmatrix}$$

whence, X is nonsingular if and only if (A^T, x_n^T) is controllable.

LEMMA 2 [2]. Let A be the same as in (1) and let H be given by

$$H = \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{m1} & & b_{mn} \end{pmatrix}$$

then, (A^T, H^T) is controllable if and only if the polynomials $f(x)$ and $P_k(x) = b_{k1} + b_{k2}x + \ldots + b_{kn}x^{n-1}$ $(k = 1,\ldots,m)$ have no common zero.

3. PROOF OF THEOREM 1.

It is shown in [1] that

$$B_{fg}A = A^T B_{fg} .$$

So, by Lemma 1, B_{fg} is nonsingular if and only if (A^T, h_n^T), where h_n is the last row of the Bezoutian matrix B_{fg}, is controllable.

It is an easy computation to see that

$$h_n = (a_1 - b_1, a_2 - b_2, \ldots, a_n - b_n).$$

Applying Lemma 2 to the pair $[A^T, h_n^T]$, we see that B_{fg} is nonsingular if and only if the polynomials $f(x)$ and $h(x) = (a_n - b_n)x^{n-1} + (a_n - b_n)x^{n-2} + \ldots + (a_2 - b_2)x + (a_1 - b_1)$ are relatively prime. But, $h(x) = g(x) - f(x)$, and $f(x)$ and $g(x)$ are relatively prime if and only if $f(x)$ and $h(x)$ are so.

REMARK. When B_{fg} is singular, $f(x)$ and $g(x)$ have a common zero and in this case, the degree of g.c.d. is equal to the nullity of the controllability matrix $(h_n, A^T h_n^T, (A^T)^2 h_n^T, \ldots, (A^T)^{n-1} h_n^T)$.

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