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ON THE BACKWARD HEAT PROBLEM: EVALUATION OF THE NORM OF $\frac{\partial u}{\partial t}$

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ABSTRACT. We show in this paper that $\| \Delta u \| = \| u_t \|$ is bounded $\forall\ t \le T^{(o)} < T$ if one imposes on u (solution of the backward heat equation) the condition $\| u(x,t) \| \le M$. A Hölder type of inequality is also given if one supposes $\| u_t(x,T) \| \le K$.

KEY WORDS AND PHRASES: *Parabolic equations, Improperly posed problems.*
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1. INTRODUCTION.

A lot of authors have dealt the backward heat problem and considered equations of various kind. It is known that this problem is an improperly posed problem and the dependence of the solution as function of the initial data is an important aspect of it. The a priori inequalities (see Sigillito [1]) give immediately several informations. Among the methods of investigation, that of the logarithmic convexity is relatively simple when one is able to define - the difficulty is there - the functional which leads us to the required result (see, e.g. Knops [2]). But in the particular case of the hereafter equation, the functional is, for reasons of analogy, rather easy to determine.

2. EVALUATION OF $\|u(x,t)\|$ AND $\|u_t(x,t)\|$ WITH $u \in C_{\alpha\beta}$.

Let us consider the backward heat equation:

$$\Delta u + u_t = 0 \quad \text{in } D\times[0,T) \tag{2.1}$$

$$u = 0 \quad \text{on } \partial D\times[0,T) \tag{2.2}$$

and the initial data $u(x,0) = f(x)$ (2.3)

D is a bounded open domain in $\mathbb{R}^n$ with smooth boundary ∂D.

Let $C_{\alpha\beta}$ be the class of the functions $\theta(x,t)$, continuous in $\bar{D}\times[0,T]$, so that

α) for fixed $t \in (0,T)$, θ is twice continuously differentiable in the x-variable and for $x \in D$, θ is continuously differentiable in t $[t \in (0,T)]$;

β) $\int_D |\frac{\partial}{\partial t}\theta(x,T)|^2 dx \leq K^2$.

Then, if the solution u of (2.1-3) is subjected to belong to $C_{\alpha\beta}$, one has: $\forall\ t \leq T,\quad \|u_t\| \leq K^{t/T} \| u_t(x,0)\|^{1-t/T}$ (2.4)

where, by definition $\|v(x,t)\|^2 = \int_D v^2(x,t)\ dx$.

Let us set $g(x) = u_t(x,0)$; then, the evaluation of $\|u_t\|$ is found if an upper bound of $\|g(x)\|$ is known.

To prove (2.4), let us use the property of the logarithmic convexity applied to the functional (see [3], p. 11-12)

$$\Psi(t) = \int_D u_t^2(x,t)\ dx \tag{2.5}$$

One has: $\Psi'(t) = 2\int_D u_t u_{tt} dx = -2\int_D u_t \Delta u_t dx = -2\oint_{\partial D} u_t \cdot \frac{\partial}{\partial n} u_t\ d\sigma +$

$$+ 2\int_D |\mathrm{grad}\, u_t|^2\ dx = 2\int_D |\mathrm{grad}\, u_t|^2\ dx \tag{2.6}$$

since (2.2) implies that $\frac{\partial u}{\partial t} = 0$ on ∂D when $t\in(0,T)$. It follows that

$$\Psi''(t) = 4\int_D \text{grad}u_t.\text{grad}u_{tt}\, dx = 4\oint_{\partial D} u_{tt}\cdot\frac{\partial}{\partial n}u_t\, d\sigma - 4\int_D u_{tt}\Delta u_t\, dx = $$
$$= 4\int_D u_{tt}^2\, dx\ .$$

Using Schwarz's inequality one finds:

$$\Psi\Psi'' - \Psi'^2 \geq 0 \qquad \text{or} \qquad (\ln\Psi)'' \geq 0\ .$$

Thus, $\Psi \leq \Psi(T)^{t/T}.\Psi(0)^{1-t/T}$, that is to say

$$\|u_t\|^2 \leq K^{2t/T}\|g\|^{2(1-t/T)}\ .$$

On the other hand, one has:

$$\int_D uu_t\, dx = -\int_D u\Delta u\, dx = \int_D |\text{grad}u|^2 dx \geq \lambda_1\int_D u^2 dx$$

with λ_1 = first eigenvalue of the problem $\left.\begin{array}{l}\Delta\phi+\lambda\phi=0 \text{ in } D\\ \phi=0 \text{ on } \partial D\end{array}\right\}$ (2.7)

Then, for t=T

$$\lambda_1^2(\int_D u^2dx)^2 \leq \int_D u^2dx.\int_D u_t^2dx \quad , \quad \int_D u^2dx \leq \frac{1}{\lambda_1^2}\int_D u_t^2dx \leq (\frac{K}{\lambda_1})^2 = m^2$$

and consequently (see [3], p. 13)

$$\|u\| \leq (\frac{K}{\lambda_1})^{t/T}\|f\|^{1-t/T} \tag{2.8}$$

Now, let us substitute the condition β) of the class $C_{\alpha\beta}$ by

β') $\int_D |\theta(x,t)|^2\, dx \leq K'^2$

and let $C_{\alpha\beta'}$ be the class of functions which satisfy α) and β').

Let us show why one can find one bound of $\|u_t\|$ if $u\in C_{\alpha\beta'}$.

3. EVALUATION OF $\|u_t(x,t)\|$ WITH $u \in C_{\alpha\beta'}$.

LEMMA: if $u\in C_{\alpha\beta'}$, $\|u_t(x,t)\| \leq M_\tau$ for $t\leq T-\tau$, $0<\tau\leq T$ (3.1)

PROOF: We remark first that $\Phi(t) =: \int_D u^2(x,t)dx$ is an increasing function since $\Phi' = 2\int_D uu_t dx = 2\int_D |\text{grad}u|^2 dx > 0$. Thus,

$$\Phi(t) \leq K'^2 \quad \text{for } t\leq T \tag{3.2}$$

Let h(t) be a function of class C^2 and let us define

$$I = \int_{t=a}^{t=b} \int_D h(t)u_t^2(x,t)\, dxdt = -\int_a^b [\int_D u_t \Delta u\, dx] h(t)dt =$$
$$= -\int_a^b [\oint_{\partial D} u_t \cdot \frac{\partial u}{\partial n} d\sigma - \int_D \text{gradu.gradu}_t\, dx] h(t)dt =$$
$$= \frac{1}{2}\int_D [\int_a^b h\frac{\partial}{\partial t}|\text{gradu}|^2 dt] dx = \frac{1}{2}\int_D (h|\text{gradu}|^2 \Big|_{t=a}^{t=b} - \int_a^b |\text{gradu}|^2 h' dt) dx$$
$$= \frac{1}{2}[\int_D h|\text{gradu}|^2 dx \Big|_a^b] - \frac{1}{2}\int_a^b h'(\int_D |\text{gradu}|^2 dx) dt = \frac{1}{2}[\ldots] -$$
$$- \frac{1}{2}\int_a^b h' \int_D \frac{1}{2}\frac{\partial}{\partial t}u^2\, dxdt = \frac{1}{2}[\ldots] - \frac{1}{4}\int_D h'u^2 dx \Big|_{t=a}^{t=b} + \frac{1}{4}\int_D\int_a^b h''u^2 dxdt$$

$$\left.\begin{aligned} I = \frac{1}{2}[h(b)\int_D |\text{gradu}(x,b)|^2 dx - h(a)\int_D |\text{gradu}(x,a)|^2 dx] - \\ - \frac{1}{4}[h'(b)\int_D u^2(x,b)dx - h'(a)\int_D u^2(x,a)dx] + \\ + \frac{1}{4}\int_D\int_a^b h''(t)u^2(x,t)dxdt \quad . \end{aligned}\right\} \quad (3.3)$$

Then, $\forall\ T_1 < T$

$$\int_0^{T_1}\int_D u_t^2\, dxdt \le \int_0^{T_1}\int_D u_t^2\, dxdt + \int_{T_1}^T \int_D \frac{T-t}{T-T_1} u_t^2\, dxdt \ ,$$

and using (3.3) for each term of the sum, we obtain

$$\int_0^{T_1}\int_D u_t^2\, dxdt \le \frac{1}{2}[\int_D |\text{gradu}(x,T_1)|^2 dx - \int_D |\text{gradu}(x,0)|^2 dx] +$$
$$+ \frac{1}{2}[-\int_D |\text{gradu}(x,T_1)|^2 dx] + \frac{1}{4}\cdot\frac{1}{T-T_1}[\int_D u^2(x,T_1)dx - \int_D u^2(x,0)dx]$$
$$\le \frac{1}{4(T-T_1)}\int_D u^2(x,T_1)dx \le \frac{1}{4(T-T_1)}K'^2 =: M_1^2 \quad \text{by (3.2)}.$$

Clearly, $\int_0^s\int_D u_t^2\, dxdt$ is an increasing function of s. Thus,

$$\int_0^s\int_D u_t^2\, dxdt \le M_1^2 \quad \forall\ s \le T_1 < T \qquad (3.4)$$

Also,

$$M_1^2 \ge \int_0^{T_1}\int_D u_t^2\, dxdt \ge \int_{T_0}^{T_1}\int_D u_t^2\, dxdt = \int_{T_0}^{T_1} \Psi(t)dt \qquad (0 \le T_0 < T_1)$$

But, Ψ and Ψ' are positive; this leads to $M_1^2 \ge \Psi(T_0)(T_1-T_0)$,

$$\Psi(T_0) \le K'^2/4(T-T_1)(T_1-T_0) \ .$$

Setting finally $T_0 = T-\tau$, $T_1 = T-\tau/2$, one arrives to

$$\Psi(t) \le \Psi(T-\tau) \le K'^2/\tau^2 =: M_\tau^2 \quad , \quad t \le T-\tau \qquad \text{q.e.d.}$$

So, we can evaluate $\|\Delta u\|$ with a bound of $\|u(x,T)\|$.

Let us write $T-\tau = T'$ and $\|u_t(x,T')\| \le M'$. Then, if u_{tt} exists, one finds similarly $\int_D u_{tt}^2\, dx \le M''^2$, $\forall\ t \le T''$, since

$$\Delta(u_t) + (u_t)_t = 0 \quad \text{in } D\times[0,T) \quad \text{and} \quad (u_t) = 0 \quad \text{on } \partial D\times[0,T) \ .$$

And generally:

$$\int_D \left(\frac{\partial^k u}{\partial t^k}\right)^2 dx \leq [M^{(k)}]^2 \quad , \quad t\leq T^{(k)}<T,\ k\in \mathbb{N} \tag{3.5}$$

REMARK: A result analogous to (3.1) can be obtained if we consider the serie $u(x,t) = \sum_{n=1}^{\infty} a_n u_n(x) e^{\lambda_n t}$ where u_n are the eigenfunctions of (2.7), λ_n the eigenvalues and a_n the Fourier coefficients of f(x). But, because of the necessity of the uniform convergence, the developments are long and rather complicated.

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