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SOME FIXED POINT THEOREMS FOR SET VALUED DIRECTIONAL CONTRACTION MAPPINGS

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ABSTRACT. Let S be a subset of a metric space X and let B(X) be the class of all nonempty bounded subsets of X with the Hausdorff pseudometric H. A mapping $F : S \to B(X)$ is a directional contraction iff there exists a real $\alpha \in [0,1)$ such that for each $x \in S$ and $y \in F(x)$, $H(F(x), F(z)) \leq \alpha d(x,z)$ for each $z \in [x,y] \cap S$, where $[x,y] = \{z \in X : d(x,z) + d(z,y) = d(x,y)\}$. In this paper, sufficient conditions are given under which such mappings have a fixed point.

KEY WORDS AND PHRASES: *Directional contraction, Hausdorff pseudometric.*

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1. Introduction.

In this paper, we prove a fixed point theorem for set valued directional contraction mappings (see definition below). The main result extends an earlier result of Assad and Kirk [1] and has some interesting consequences.

Throughout this paper, (X,d) represents a complete metric space and $B(X)$ is the class of all nonempty bounded subsets of X with the Hausdorff pseudometric H induced by d (see [3] p. 33), that is if $A,B \in B(X)$, then

$$H(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\}.$$

It follows immediately from the definition of H, that for any $A,B \in B(X)$,

$$d(x,B) \leq H(A,B) \text{ for any } x \in A, \tag{1.1}$$

$$d(x,B) \leq d(x,A) + H(A,B) \text{ for any } x \in X, \tag{1.2}$$

and given $\varepsilon > 0$ and $x \in A$, there exists a $y \in B$ such that

$$d(x,y) \leq H(A,B) + \varepsilon. \tag{1.3}$$

For $x,y \in X$, we will denote

$$[x,y] = \{z \in X : d(x,z) + d(z,y) = d(x,y)\},$$

and $(x,y] = [x,y] \setminus \{x\}$, $(x,y) = (x,y] \setminus \{y\}$. The following result is due to Caristi [2] and is used in the proof of the main result.

THEOREM (Caristi) Let $f : X \to X$ be a mapping. If there exists a lower semi-continuous ($\ell.s.c.$) mapping $\phi : X \to [0,\infty)$ such that for each $x \in S$,

$$d(x,f(x)) \leq \phi(x) - \phi(f(x)), \tag{1.4}$$

then f has a fixed point.

2. MAIN RESULTS.

Let S be a nonempty subset of X.

DEFINITION 1. A mapping $F : S \to B(X)$ is a directional contraction (d.c) iff there exists a real $\alpha \in [0,1)$ such that for each $x \in S$ and $y \in F(x)$,

$$H(F(z), F(x)) \leq \alpha d(z,x), \tag{2.1}$$

for all $z \in [x,y] \cap S$.

The real α in (2.1) will be called a contraction constant of F.

THEOREM 1. Let S be a closed subset of X and $F : S \to B(X)$ be a d.c mapping with contraction constant α. If F satisfies

a) for each $x \in S$, $y \in F(x) \setminus S$, there exists a $z \in (x,y) \cap S$ with $F(z) \subseteq S$, (2.2)

b) the mapping $g : S \to [0,\infty)$ defined by $g(x) = d(x,F(x))$ is $\ell.s.c.$, (2.3)

then F has a fixed point, that is $x \in F(x)$ for some $x \in S$.

We first prove the following lemma which simplifies the proof of Theorem 1.

LEMMA. Under the hypothesis of Theorem 1, for any $\beta, \alpha < \beta < 1$, there exists a mapping $A : S \to B(X)$ with the following properties

i) for each $x \in S$, $A(x) \neq \phi$ and $A(x) \subseteq F(x)$, (2.4)

ii) if $y \in A(x)$, then $d(x,y) \leq (1-\beta+\alpha)^{-1} d(x,F(x))$, (2.5)

iii) if $A(x) \cap S = \phi$ for some $x \in S$, then there exists a $y = y(x) \in A(x)$ and a $z = z(x,y) \in (x,y) \cap S$ such that

$$d(x,y) \leq d(x,F(x)) + (\beta-\alpha)d(x,z). \tag{2.6}$$

PROOF. Define a mapping $A : S \to B(X)$ by

$$A(x) = \{y \in F(x) : d(x,y) \leq (1-\beta+\alpha)^{-1} d(x,F(x))\}.$$

Since $(1-\beta+\alpha) < 1$, $A(x) \neq \phi$ for any $x \in S$ and satisfies (2.4) and (2.5). Suppose $A(x) \cap S = \phi$ for some $x \in S$. Choose a sequence $\{y_n\} \subseteq F(x)$ such that

$$d(x,y_n) \to d(x,F(x)). \tag{2.7}$$

Since the sequence $\{y_n\}$ is eventually in $A(x)$, we may assume that the sequence $\{y_n\} \subseteq A(x)$. It then follows by the supposition that for each $n \in I$ (positive integers), $y_n \in F(x) \setminus S$ and consequently by (2.2) for each $n \in I$, there exists a z_n satisfying

$$z_n \in (x\,,\,y_n) \cap S \text{ and } F(z_n) \subseteq S. \tag{2.8}$$

Now, since $d(x,z_n) \leq d(x,y_n)$, it follows by (2.7) that there is a subsequence $\{z_{n_k}\}$ of the sequence $\{z_n\}$ and a real $\lambda \geq 0$ such that

$$d(x,z_{n_k}) \to \lambda. \tag{2.9}$$

We claim that $\lambda > 0$. Suppose $\lambda = 0$. Then the sequence $\{z_{n_k}\} \to x$. Moreover, since $y_n \in F(x)$, it follows by the definition of F and (2.8) that

$$H(F(x),\ F(z_{n_k})) \le \alpha d(x,z_{n_k}) \to 0 \text{ as } k \to \infty. \tag{2.10}$$

Now, (2.10) implies that $F(x) \subseteq S$, for if y is an arbitrary element of F(x), then by (1.3) for each $k \in I$, there is a $w_k \in F(z_{n_k})$ such that $d(y,w_k) \le H(F(x),\ F(z_{n_k})) + \frac{1}{k} \to 0$ as $k \to \infty$. Since $\{w_k\} \subseteq S$ and S is closed, it follows that y and hence $F(x) \subseteq S$. However, this contradicts the supposition that $A(x) \cap S = \phi$. Thus $\lambda > 0$. Now choose an $\varepsilon > 0$ such that $\delta = (\beta - \alpha)\lambda - \varepsilon > 0$. Then by (2.9), $(\beta - \alpha)d(x,z_{n_k}) \ge \delta$ eventually and hence by (2.7) and the last inequality,

$$d(x,y_{n_k}) \le d(x,F(x)) + \delta \le d(x,F(x)) + (\beta - \alpha)d(x,z_{n_k})$$

eventually. Thus there exists a $y = y_{n_k}$ and the corresponding $z = z_{n_k}$ satisfying (2.8) such that (2.6) holds.

PROOF OF THEOREM 1. Define a mapping $f : S \to S$ as follows: for $x \in S$, let f(x) be any element of $A(x) \cap S$ if $A(x) \cap S \neq \phi$; and if $A(x) \cap S = \phi$, then by the lemma, there exist elements $y = y(x) \in A(x)$ and $z = z(x,y) \in (x,y) \cap S$ satisfying (2.6), let $f(x) = z$ in this case. Note that for any $x \in S$,

$$H(F(x),\ F(f(x)) \le \alpha d(x,f(x)). \tag{2.11}$$

This is obvious if $A(x) \cap S = \phi$ and if $A(x) \cap S \neq \phi$, then since $f(x) \in F(x)$ and $f(x) \in [x,f(x)] \cap S$, therefore the definition of F implies (2.11). Set $\phi(x) = (1 - \beta)^{-1}g(x)$. Then ϕ is $\ell.s.c.$ on S. We show that f satisfies (1.4). Let $x \in S$. We consider cases (i) when $A(x) \cap S \neq \phi$ and case (ii) when $A(x) \cap S = \phi$. In case (i), $f(x) \in A(x)$ and hence by (2.5), $d(x,f(x)) \le (1 - \beta + \alpha)^{-1}d(x,F(x))$. This implies that $\alpha(1 - \beta)^{-1}d(x,f(x)) \le \phi(x) - d(x,f(x))$. Therefore, by (1.1), (2.11) and the last inequality

$$\phi(f(x)) = (1 - \beta)^{-1}g(f(x)) \le (1 - \beta)^{-1}H(F(x),\ F(f(x))) \le \phi(x) - d(x,f(x)).$$

Thus (1.4) holds in this case. In case (ii), there is a $y = y(x) \in F(x)$ such that $f(x) \in (x,y)$ and satisfies (2.6). Thus by (2.6),

$d(f(x),F(x)) \leq d(f(x),y) = d(x,y) - d(x,f(x)) \leq d(x,F(x)) - (1-\beta+\alpha)d(x,f(x))$.

It now follows by (1.2) and (2.11) and the above inequality that

$(1-\beta)\phi(f(x)) = g(f(x)) \leq d(f(x),F(x)) + H(F(x),F(f(x))) \leq d(x,F(x)) - (1-\beta)d(x,f(x))$,

that is

$$d(x,f(x)) \leq \phi(x) - \phi(f(x)).$$

Thus f satisfies (1.4) and consequently by Caristi's theorem $f(x) = x$ for some $x \in S$. This implies that $x \in F(x)$ for otherwise $f(x) \notin A(x) \cap S$ and hence by the definition of f, $A(x) \cap S = \phi$. Thus $f(x) \in (x,y(x))$ for some $y(x) \in A(x)$. This contradicts $x \neq f(x)$. Consequently, $x \in F(x)$.

Recall, that a metric space is called convex iff for each $x,y \in X$, $x \neq y$ there exists a $z \in (x,y)$. It is easy to show (see [4]) that if S is a closed subset of a complete, convex metric space and $x \in S$ and $y \notin S$, then there exists a $z \in [x,y) \cap \partial S$ where ∂S denotes the boundary of S. As a result of this, the following is an immediate consequence of Theorem 1.

COROLLARY 1. Let X be convex and S a closed subset of X. Let $F : S \to B(X)$ be a d.c mapping such that $f(\partial S) \subseteq S$. If $g(x) = d(x,F(x))$ is $\ell.s.c.$ on S, then F has a fixed point.

The following special case of Corollary 1 extends to B(X) an earlier result of Assad and Kirk [1].

COROLLARY 2. Let X be convex and S a closed subset of X. Suppose $F : X \to B(X)$ satisfies the condition: there exists an $\alpha \in [0,1)$ such that for all $x,y \in S$,

$$H(F(x), F(y)) \leq \alpha d(x,y). \tag{2.12}$$

If $F(\partial S) \subseteq S$, then F has a fixed point.

PROOF. Since a mapping F satisfying (2.12) is a d.c mapping, it suffices to show that the mapping g on S defined by $g(x) = d(x,F(x))$ is continuous. To prove this, let $\{x_n\}$ be a sequence in S such that $\{x_n\} \to x \in S$. It follows that for each $n \in I$,

$$g(x) = d(x,F(x)) \leq d(x,x_n) + d(x_n,F(x)) \leq d(x,x_n) + g(x_n) + H(F(x_n), F(x)).$$

That is, $g(x) \leq g(x_n) + (1+\alpha)d(x_n,x)$. Similarly, it follows that for each $n \in I$, $g(x_n) \leq g(x) + (1+\alpha)d(x_n,x)$. Thus $|g(x_n) - g(x)| \to 0$ as $n \to \infty$.

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