EXPANSIONS OF DISTRIBUTIONS IN TERMS OF GENERALIZED HEAT POLYNOMIALS AND THEIR APPELL TRANSFORMS

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ABSTRACT. This paper is concerned with expansions of distributions in terms of

the generalized heat polynomials and of their Appell transforms. Two different

techniques are used to prove theorems concerning expansions of distributions. A

theorem which provides an orthogonal series expansion of generalized functions is

also established. It is shown that this theorem gives an inversion formula for a

certain generalized integral transformation.

1. INTRODUCTION.

Zemanian [1] has developed a procedure for expanding certain Schwartz distri-

butions (or generalized functions) in a series of the Fourier type. He has also

shown that every distribution of polynomial growth (or tempered distribution) can

be expanded in a series of Hermite functions. Some general results concerning the

orthogonal series expansion of generalized functions are also available in recent

literature including reference listed by Zemanian [1].

In a paper, Korevaar [2] has developed a general theory of Fourier transforms

and pansions based upon the ideas different from those of the Schwartz-Sobolev

approach [3], and from those of Ehrenpreis [4]. This theory did not include the

notion of convergence for pansions as there exists such notions for distributions.

In fact, the theory is essentially in algebraic in nature. It is also shown that

distributions which are finite order derivatives of certain functions growing no

faster than exp (αx^2), $\alpha < \frac{1}{2}$ are uniquely determined by their Hermite series.

However, no topology was introduced there, and hence there is no way of expressing

the generalized function as a functional in terms of the Hermite coefficients.

In two papers [5-6], Haimo studied the expansion of functions in terms of heat

polynomials and their Appell transforms. This work is an extension of some results

of Rosenbloom and Widder [7] on expansions in terms of heat polynomials and associ-

ated functions. Despite these works, no attention is given to the expansions of

distributions in terms of heat polynomials and their Appell transforms.

This paper is devoted to the study of expansions of distributions in terms of

the generalized heat polynomials and their Appell transforms. A theorem which pro-

vides an orthogonal series expansion of generalized functions is also proved. It

is shown that this theorem gives an inversion formula for a certain generalized

integral transformation.

2. PRELIMINARY RESULTS ON THE GENERALIZED HEAT POLYNOMIALS.

For real values of x and t, the generalized heat polynomials, $P_{n,\nu}(x,t)$ and its

Appell transform, $W_{n,\nu}(x,t)$ are defined by

$$P_{n,\nu}(x,t) = \sum_{r=0}^{n} 2^{2r} \binom{n}{r} \frac{\Gamma(\nu + \frac{1}{2})}{\Gamma(\nu + \frac{1}{2} + n - r)} x^{2(n-r)} t^r , \qquad (2.1)$$

$$W_{n,\nu}(x,t) = t^{-2n} G(x,t) P_{n,\nu}(x,-t), \qquad t > 0, \qquad (2.2)$$

where n = 0,1,2,... ; ν is a fixed positive number and G(x,t) is given by

$$G(x,t) = (2t)^{-(\nu + \frac{1}{2})} \exp(- \frac{x^2}{4t}) . \qquad (2.3)$$

When ν = 0, it follows from (2.1) that

$$P_{n,o}(x,t) = v_{2n}(x,t), \qquad (2.4)$$

$$P_{n,o}(x,-1) = H_{2n}(\frac{x}{2}), \qquad (2.5)$$

where $v_{2n}(x,t)$ is the ordinary heat polynomial of even order defined by Rosenbloom

and Widder [7, p. 222], and H_{2n} is the Hermite polynomial of even order given in

Erdélyi's book [8].

It can readily be verified that for $-\infty < x, t < \infty$, $P_{n,\nu}(x,t)$ satifies the

generalized heat equation

$$\Delta_x u(x,t) = \frac{\partial u}{\partial t}, \qquad\qquad (2.6)$$

where the operator $\Delta_x = \frac{\partial^2}{\partial x^2} + (\frac{2\nu}{x})\frac{\partial}{\partial x}$ with some fixed positive number ν.

It is noted that $W_{n,\nu}(x,t)$ is also a solution of (2.6). Further, $W_{n,\nu}(x,t)$

satifies the following operator formulas

$$W_{n,\nu}(x,t) = 2^{2n} \Delta_x^n G(x,t), \qquad\qquad (2.7)$$

with

$$\Delta_x^k W_{n,\nu}(x,t) = 2^{-2k} W_{n+k,\nu}(x,t). \qquad\qquad (2.8)$$

With the help of the following results reported by Erdélyi [8]

$$P_{n,\nu}(x,-t) = (-1)^n \, 2^{2n} \, n! \, t^n \, L_n^{(\nu-\frac{1}{2})}(\frac{x^2}{4t}), \; t > 0, \qquad\qquad (2.9)$$

$$\left| L_n^\alpha(x) \right| \le \Delta_{n,\alpha} \, \exp(\frac{x}{2}), \qquad\qquad (2.10)$$

where $L_n^\alpha(x)$ is the Laguerre polynomial of degree n and order $\alpha > -1$ and

$$\Delta_{n,\alpha} = \left[\begin{array}{l} \frac{1}{n!}(\alpha+1)_n \, , \qquad\qquad \alpha \ge 0 \\[2mm] 2 - \frac{1}{n!}(\alpha+1)_n \, , \; -1 < \alpha < 0 \end{array} \right], \qquad\qquad (2.11ab)$$

with $(\alpha)_n = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)}$; it can easily be shown that for $t > 0$

$$\left| P_{n,\nu}(x,-t) \right| \le n! \, 2^{2n} \, \Delta_{n,\nu-\frac{1}{2}} \, t^n \, \exp(\frac{x^2}{8t}), \qquad\qquad (2.12)$$

$$\left| W_{n,\nu}(x,t) \right| \le 2^{2n-\nu-\frac{1}{2}} \, n! \, \Delta_{n,\nu-\frac{1}{2}} \, t^{-(n+\nu+\frac{1}{2})} \, \exp(-\frac{x^2}{8t}). \qquad\qquad (2.13)$$

The differential $d\mu(x)$ is defined in a paper of Haimo [6] as

$$d\mu(x) = 2^{(\frac{1}{2}-\nu)} \, [\Gamma(\nu + \frac{1}{2})]^{-1} \, x^{2\nu} dx. \qquad\qquad (2.14)$$

It is interesting to observe that $P_{n,\nu}(x,t)$ form a biorthogonal system in

$0 < x < \infty$. Indeed,

$$\int_0^\infty W_{n,\nu}(x,t) \, P_{m,\nu}(x,-t) \, d\mu(x) = \frac{\delta_{nm}}{b_n} \qquad t > 0, \qquad (2.15)$$

where

$$b_n = \Gamma(\nu + \tfrac{1}{2})/[2^{4n} \, n! \; \Gamma(\nu + n + \tfrac{1}{2})] \; . \qquad (2.16)$$

An important consequence of (2.15)-(2.16) is the bilinear generating function for the biorthogonal set $P_{n,\nu}$ and $W_{n,\nu}$ as

$$G(x,y; \; s + t) = \sum_{n=o}^\infty b_n \, W_{n,\nu}(y,s) \, P_{n,\nu}(x,t). \qquad (2.17)$$

The source solution of (2.6) is given by $G(x,t)$ in the form

$$G(x,y;t) = (2t)^{-(\nu+\frac{1}{2})} \exp[-(x^2 + y^2)/4t] \, \mathcal{J}\left(\tfrac{xy}{2t}\right), \qquad (2.18)$$

$$\mathcal{J}(x) = 2^{\nu-\frac{1}{2}} \, \Gamma(\nu + \tfrac{1}{2}) \; x^{\frac{1}{2}-\nu} \; I_{\nu-\frac{1}{2}}(x), \qquad (2.19)$$

and $I_r(x)$ is the modified Bessel function of imaginary argument of order r; and $G(x,0;t) = G(x,t)$.

3. THE TEST FUNCTION SPACE $U_{\sigma,\nu}$.

We denote the open interval $(0,\infty)$ by I. A complex valued infinitely differentiable function $\phi(x)$ defined over I belongs to the space $U_{\sigma,\nu}(I)$ if

$$\gamma_k(\phi) \triangleq \sup_{0<x<\infty} \left|\exp(\tfrac{x^2}{4\sigma}) \, \Delta_x^k \, \phi(x)\right| < \infty, \qquad (3.1)$$

for any fixed k, where k assumes the values $0,1,2,\ldots$; σ is a positive real number, $\nu > -\frac{1}{2}$ and Δ_x is the differential operator involved in section 2.

The topology in the space $U_{\sigma,\nu}$ is induced by the collection of seminorms $\{\gamma_k\}_{k=0}^{k=\infty}$. Since γ_0 is a norm, the collection of seminorms is separating as indicated by Zemanian [1, p8]. A sequence $\{\phi_r\}_1^\infty$ is said to converge to ϕ in $U_{\sigma,\nu}(I)$ if for each k, $\gamma_k(\phi_r - \phi)$ tends to zero as $r \to \infty$. A sequence $\{\phi_r\}_{r=1}^\infty$ with each $\phi_r(x)$ belonging to $U_{\sigma,\nu}(I)$ is a Cauchy sequence in $U_{\sigma,\nu}(I)$ if $\gamma_k(\phi_r - \phi_s) \to 0$ as $r,s \to \infty$ independently of each other for every fixed k, where $k = 0,1,2,\ldots$. It is noted here that $U_{\sigma,\nu}(I)$ is a locally convex, sequentially complete, Hausdorff topological linear space. Its dual space $U'_{\sigma,\nu}(I)$ is the space of generalized functions under consideration.

From the estimate (2.10), it follows that for x > 0, s > $\frac{\sigma}{2}$ and ν > $-\frac{1}{2}$,

$\mu(x)$ $P_{n,\nu}(x,-s)$ $\in U_{\sigma,\nu}(I)$. Using results (2.4), (2.5) and (2.8), it can also be

seen that $W_{n,\nu}(x,t)$ $\in U_{\sigma,\nu}(I)$ for x > 0 and 0 < t < σ.

In order to make this paper self-contained to some extent, we now list a few

properties of the above spaces:

(i) $\mathcal{D}(I)$ denotes the space of infinitely differentiable functions defined over I

with a compact support. The dual space $\mathcal{D}'(I)$ is the space of Schwartz distributions

[9] on I. It can easily be shown that $\mathcal{D}(I)$ is a subspace of the space $U_{\sigma,\nu}(I)$ and

that the topology of $\mathcal{D}(I)$ is stronger than that induced on it by $U_{\sigma,\nu}(I)$.

(ii) The space $U_{\sigma,\nu}(I)$ is a dense subspace of $\mathcal{E}(I)$ which is the space of all

complex-valued smooth functions on I. The topology of $U_{\sigma,\nu}$ is stronger than the

topology induced on $U_{\sigma,\nu}$ by $\mathcal{E}(I)$. So $\mathcal{E}'(I)$ can be identified with a subspace of

$U'_{\sigma,\nu}$.

(iii) The space S' of tempered distributions is a subspace of $U'_{\sigma,\nu}$ for ν > $\frac{1}{2}$

[Pandey, 10].

(iv) For each f $\in$ $U'_{\sigma,\nu}$ there exists a non-negative integer r and a positive

constant C such that

$$|\langle f,\phi \rangle| \leq C \max_{0 \leq k \leq r} \gamma_k(\phi) , \qquad (3.2)$$

for every ϕ $\in$ $U_{\sigma,\nu}$, where r and C depend on f but not on ϕ [8,p19].

(v)   If f(x) is a locally integrable for x > 0 such that f(x) $\exp(-\frac{x^2}{4\sigma})$ is

absolutely integrable on 0 < x < ∞, then f(x) generates a regular generalized

function f $\in$ $U'_{\sigma,\nu}$ defined by the integral

$$\langle f,\phi \rangle = \int_0^\infty f(x) \overline{\phi(x)} \, dx , \qquad (3.3)$$

where $\phi(x)$ $\in$ $U_{\sigma,\nu}(I)$.

(vi) Let f(x) be a locally integrable function defined for x > 0 such that

$$f(x) = \begin{bmatrix} 0(x^\rho) & \rho + 1 > 0, & x \to 0 + \\ 0(e^{\delta x^2}) & 0 < \delta < 1/(4\sigma), & x \to \infty \end{bmatrix}$$

Then, clearly f $\in$ $U'_{\sigma,\nu}$. Let $\mathcal{A}$ denote Zemanian's test function space ([1], p252).

Then $\phi(x) = x^{\frac{1}{2}\alpha} e^{-x/2} L_n^{(\alpha)}(x) \in A$. But $\int_0^\infty f(x) \overline{\phi(x)} dx$ does not exist. Hence, $f(x)$

satisfying (2.4) does not belong to $A'([1], p258)$. Therefore, $U'_{\sigma,\nu} \not\subset A'$.

Since $D(I)$ is contained in both the spaces $U_{\sigma,\nu}$ and A, it follows that the

spaces $U'_{\sigma,\nu}$ and A' over-lap but $U'_{\sigma,\nu} \not\subset A'$.

4. EXPANSION OF DISTRIBUTIONS.

The expansion of $f \in U'_{\sigma,\nu}$ in terms of heat polynomials $P_{n,\nu}(x,-t)$ and their

Appell transforms $W_{n,\nu}(x,t)$ is given in the following theorem:

THEOREM 4.1. Let $f \in U'_{\sigma,\nu}$ where $\nu > \frac{1}{2}$ and $\sigma > 0$. Then, for $0 < \sigma/2 < t < \sigma$,

$$\langle f(x), \phi(x) \rangle = \lim_{s \to t-} \lim_{N \to \infty} \sum_{n=0}^{N} k_n a_n(t) \langle \mu(x) P_{n,\nu}(x,-s), \phi(x) \rangle, \quad (4.1)$$

where

$$a_n(t) \triangleq \langle f(y), W_{n,\nu}(y,t) \rangle, \quad (4.2)$$

$$K_n = \Gamma(\nu + \frac{1}{2})/[2^{4n} n! \ \Gamma(\nu + \frac{1}{2} + n)]. \quad (4.3)$$

We shall need the following lammas for the proof of this theorem.

LEMMA 4.1. For $f \in U'_{\sigma,\nu}$ where $\nu > 0$ and $\sigma > 0$, we define

$$a_n(t) \triangleq \langle f(y), W_{n,\nu}(y,t) \rangle \quad , \quad n = 0,1,2,\ldots , \quad (4.4)$$

Then for $0 < t < \sigma$,

$$|a_n(t)| \leq C(4/t)^n [(\nu + \frac{1}{2})_{n+r} + 2(n+r)!], \quad (4.5)$$

where C and r are independent of n.

PROOF. By the boundedness property of generalized functions there exist a

positive constant C and a non-negative integer r such that

$$|a_n| = |\langle f(y), W_{n,\nu}(y,t) \rangle|$$

$$\leq C \max_{0 \leq k \leq r} \gamma_k(W_{n,\nu}(y,t))$$

$$= C \max_{0 \leq k \leq r} \sup_{0 < y < \infty} |2^{-2k} e^{y^2/(4\sigma)} W_{n+k,\nu}(y,t)| \ .$$

We arrive at the desired estimate using result (2.11).

LEMMA 4.2. Let $\phi(x) \in U_{\sigma,\nu}$. Then, for $0 < \sigma/2 < s < t < \sigma$,

$$\exp(y^2/4\sigma) |\Delta_y^k \int_0^\infty \sum_{n=N}^{\infty} k_n W_{n,\nu}(y,t) P_{n,\nu}(x,-s) \phi(x) d\mu(x)| \to 0$$

as $N \to \infty$ uniformly for all $y > 0$.

PROOF. Using the estimates (2.10) and (2.11) we can write

$$|J| \equiv \exp(y^2/4\sigma) \left| \Delta_y^k \int_0^\infty \sum_{n=N}^\infty k_n W_{n,\nu}(y,t) P_{n,\nu}(x,-s) \phi(x) d\mu(x) \right|$$

$$\leq \exp(y^2/4\sigma) \int_0^\infty \sum_{n=N}^\infty k_n \frac{2^{\frac{1}{2}-\nu-2k}}{\Gamma(\nu+\frac{1}{2})} |W_{n+k,\nu}(y,t)| |P_{n,\nu}(x,-s)| |\phi(x)| d\mu(x)$$

$$\leq \exp[-\frac{y^2}{4\sigma}(\frac{1}{t} - \frac{1}{\sigma})] \sum_{n=N}^\infty \frac{\Gamma(\nu + \frac{1}{2})}{\Gamma(\nu + \frac{1}{2} + n)} (n+k)! \, \Delta_{n+k,\nu-\frac{1}{2}} \Delta_{n,\nu-\frac{1}{2}} 2^{2k-\nu-\frac{1}{2}} \times$$

$$\times t^{-2k-\nu-\frac{1}{2}} (s/t)^n \int_0^\infty \exp(x^2/8s) |\phi(x)| d\mu(x) \ .$$

Since $\phi(x) \in U_{\sigma,\nu}$, there exists a positive constant C such that

$$|J| \leq C \sum_{n=N}^\infty \frac{\Gamma(\nu + 1 + k + n)}{(n + 1)} (s/t)^n$$

$$\leq C_1 \sum_{n=N}^\infty n^{\nu+k-\frac{1}{2}} (s/t)^n \ ,$$

where C_1 is another positive constant. Clearly the last term tends to zero as

$N \to \infty$ for $s < t$.

LEMMA 4.3. For $0 < \sigma/2 < s < t < \sigma$, let $G(x,y;t-s)$ be the function defined by

(2.14) and let $\phi(x) \in U_{\sigma,\nu}$. Then there exist $\epsilon > 0$ and a large positive constant

$A_1(n)$ such that for $y > A_1(n)$,

$$\exp(y^2/4\sigma) \left| \int_0^\infty \Delta_y^k \{G(x,y,t-s)\} \phi(x) d\mu(x) \right| < \frac{\epsilon}{2} , \qquad k = 0,1,2, \ldots .$$

PROOF. Proceeding as in the proof of Lemma 4.2, we write

$$\exp(y^2/4\sigma) \left| \int_0^\infty \Delta_y^k \{G(x,y,t-s)\} \phi(x) d\mu(x) \right|$$

$$\leq C_2 \exp[-y^2/4(\frac{1}{t} - \frac{1}{\sigma})] \sum_{n=o}^\infty \frac{\Gamma(\nu + \frac{1}{2} + k + n)}{\Gamma(n + 1)} (\frac{s}{t})^n$$

$$= C_3 \exp[-y^2/4(\frac{1}{t} - \frac{1}{\sigma})](1 - \frac{s}{t})^{-\nu-k-\frac{1}{2}}$$

for appropriate constants C_2 and C_3.

Hence for $t < \sigma$, the last expression tends to zero as $y \to \infty$.

LEMMA 4.4. Let A_1, ε, $G(x,y,t-s)$ and $\phi(x)$ be the same as in Lemma 4.3 and let $\nu > \frac{1}{2}$. Then for $t-s < \delta(n,\varepsilon)$,

$$\exp(y^2/4\sigma) \; \left| \; \int_0^\infty \Delta_y^k \{G(x,y,t-s)\} \; \phi(x) \; d\mu(x) - \phi_k(y) \right| < \varepsilon$$

where $\phi_k(y) = \Delta_y^k \phi(y)$, uniformly for all $y \in (0,A_1)$.

PROOF. If $\phi(x) \in U_{\sigma,\nu}$, then proceeding as in [10, p846], it follows that for $\nu > \frac{1}{2}$,

$$\phi^{(k)}(x) = 0(e^{-cx}), \; c > 0, \quad x \to \infty$$

$$\phi^{(k)}(0+) \quad \text{exist finitely,}$$

for each $k = 0,1,2,\ldots$. Using these orders of $\phi^{(k)}(x)$ and the identity $\Delta_x G(x,y,t) = \Delta_y G(x,y,t)$ and integrating by parts, we obtain

$$\int_0^\infty \Delta_y \{G(x,y,t-s)\} \; \phi(x) \; d\mu(x) = \int_0^\infty G(x,y,t-s) \; \Delta_x \; \phi(x) \; d\mu(x).$$

Consequently

$$\int_0^\infty \Delta_y^k \{G(x,y,t-s)\} \; d\mu(x) - \phi_k(y) = \int_0^\infty G(x,y,t-s)[\phi_k(x) - \phi_k(y)] \; d\mu(x) \, ,$$

where we have made use of the fact that

$$\int_0^\infty G(x,y,t-s) \; d\mu(x) = 1.$$

We next use the standard technique [10] and remember that $\phi(x)$ is not an element of $\mathcal{D}(I)$ although it does belong to $U_{\sigma,\nu}$ to show

$$\exp(y^2/4\sigma) \; \left| \; \int^\infty G(x,y,t-s) \; \{\phi_k(x) - \phi_k(y)\} \; d\mu(x) \right| < \varepsilon$$

uniformly for all y satisfying $0 < y \le A_1$.

We are now prepared to prove the main expansion theorem 4.1.

PROOF OF THEOREM 4.1. Let t be a fixed number such that $0 < \frac{\sigma}{2} < s < t < \sigma$ Then using the estimates (2.10) and (4.5), we obtain

$$\sum_{n=o}^{\infty} K_n |a_n| |\langle \mu(x) \ P_{n,\nu}(x,-s), \ \phi(x) \rangle|$$

$$\leq C \ \Gamma(\nu + \tfrac{1}{2}) \sum_{n=o}^{\infty} \frac{(\nu + \tfrac{1}{2})_{n+r} + 2(n+r)!}{\Gamma(\nu + \tfrac{1}{2} + r)} \ \Delta_{n,\nu -\tfrac{1}{2}} \ (\tfrac{s}{t})^n$$

$$= C \sum_{n=o}^{\infty} \frac{\Gamma(\nu + \tfrac{1}{2} + r + n)}{n!} (\tfrac{s}{t})^n + 2C \sum_{n=o}^{\infty} \frac{(n+r)!}{n!} (\tfrac{s}{t})^n, \qquad (\nu > \tfrac{1}{2})$$

$$\leq C_1 ,$$

where C_1 is a certain positive constant. This proves the existence of the limit

$N \to \infty$, when $s < t$.

Using the linearity of $U_{\sigma,\nu}$ and $U'_{\sigma,\nu}$ we can write

$$\sum_{n=o}^{N} a_n \ k_n \ \langle \mu(x) \ P_{n,\nu}(x,-s), \ \phi(x) \rangle$$

$$= \sum_{n=o}^{N} \langle f(y), \ W_{n,\nu}(y,t) \rangle \ \langle \mu(x) \ k_n \ P_{n,\nu}(x,-s), \ \phi(x) \rangle$$

$$= \sum_{n=o}^{N} \langle f(y), \ W_{n,\nu}(y,t) \rangle \ \langle \mu(x) \ k_n \ P_{n,\nu}(x,-s), \ \phi(x) \rangle$$

$$= \langle f(y), \ \langle \sum_{n=o}^{N} \mu(x) \ k_n \ W_{n,\nu}(y,t) \ P_{n,\nu}(x,-s), \ \phi(x) \rangle \rangle$$

$$= \langle f(y), \ \langle \mu(x) \ G_N(x,y,t,s), \ \phi(x) \rangle \rangle$$

where $$G_N(x,y,t,s) = \sum_{n=o}^{N} k_n \ W_{n,\nu}(y,t) \ P_{n,\nu}(x,-s) .$$

The corresponding infinite series equals $G(x,y,t-s)$. We observe that

$\mu(x) \ G(x,y,t-s) \in U'_{\sigma,\nu}$ and $\mu(x) \ G_N(x,y,t,s) \in U'_{\sigma,\nu}$ for $s > \frac{\sigma}{2}$. Furthermore, as a

function of y, $\langle \mu(x) \ G(x,y,t-s), \ \phi(x) \rangle$ is an element of $U_{\sigma,\nu}$. From Lemma 4.2,

we know that

$$\langle \mu(x) \ G_N(x,y,t,s), \ \phi(x) \rangle \to \langle \mu(x) \ G(x,y,t-s), \ \phi(x) \rangle$$

in $U_{\sigma,\nu}$ as $N \to \infty$.

Hence $$\lim_{N\to\infty} \sum_{n=o}^{N} a_n \ k_n \ \langle \mu(x) \ P_{n,\nu}(x,-s), \ \phi(x) \rangle$$

$$= \langle f(y), \ \langle \mu(x) \ G(x,y,t-s), \ \phi(x) \rangle \rangle .$$

To complete the proof we have to show that

$$\lim_{s \to t-} \langle \mu(x) \ G(x,y,t-s), \ \phi(x) \rangle = \phi(y) \qquad \text{in } U_{\sigma,\nu} \ .$$

In other words, we need to show that for all k and ε, there exists $\delta(n,\varepsilon)$ such

that, for $0 < t-s < \delta(n,\varepsilon)$,

$$\exp(y^2/4\sigma) \left| \int_0^\infty \Delta_y^k \{G(x,y,t-s)\} \ \phi(x) \ d\mu(x) - \phi_k(y) \right| < \varepsilon \ .$$

Since $\phi(y) \in U_{\sigma,\nu}$, we have for $y > A(n)$,

$$\exp(y^2/4\sigma) \ | \ \phi_k(y) | < \frac{\varepsilon}{2} \ .$$

From Lemma 4.3 we know that there exists $A_1 > A$ such that for $y > A_1(n)$ and

$t < t_0$ ($A_1(n)$ independent of t),

$$\exp(y^2/4\sigma) \ | \int_0^\infty \Delta_y^k \{G(x,y,t-s)\} \ \phi(x) \ d\mu(x) | < \frac{\varepsilon}{2} \ .$$

An application of Lemma 4.4 completes the proof of the theorem.

5.   THE TEST FUNCTION SPACE $V_{\sigma,\nu}(I)$ AND ITS DUAL.

Suppose $\Omega_{x,t}$ denotes the differential operator given by

$$\Omega_{x,t} \equiv 4t[\frac{d^2}{dx^2} - \frac{x^2}{16t^2} + \nu(\nu-1)\frac{1}{x^2}] \ , \qquad (5.1)$$

where $x \in I = (0,\infty)$ and t is a fixed real number.

A complex-valued smooth function $\phi(x)$ belong to the space $V_{\sigma,\nu}(I)$ if

$$\beta_k(\phi) \triangleq [\int_0^\infty | \ \Omega_{x,t}^k \ \phi(x)|^2 \ dx]^{\frac{1}{2}} < \infty, \qquad (5.2)$$

where   k = 0,1,2, ... ; and for each n,k

$$(\Omega^k\phi,\psi_n) = (\phi,\Omega^k\psi_n), \qquad (5.3)$$

Then $V_{\sigma,\nu}(I)$ is a linear space, and $\{\beta_k\}_{k=0}^\infty$ is a multinorm on $V_{\sigma,\nu}$. The topology

over $V_{\sigma,\nu}(I)$ is a subspace of $L_2(I)$ when we identify each function in $V_{\sigma,\nu}(I)$ with

the corresponding equivalence class in $L_2(I)$. Also convergence in $V_{\sigma,\nu}(I)$ implies

convergence in $L_2(I)$. The space $V_{\sigma,\nu}(I)$ is complete and therefore is Frechet .

The dual space of $V_{\sigma,\nu}(I)$ is denoted by $V'_{\sigma,\nu}(I)$. It can also be shown that $V'_{\sigma,\nu}(I)$

is sequentially complete.

We note that

$$\Omega_{x,t} \, \psi_n(x,t) = -\lambda_n \psi_n(x,t) \, , \tag{5.4}$$

where

$$\psi_n(x,t) = \{\frac{2^{\frac{1}{2}-\nu}}{\Gamma(\nu+\frac{1}{2})}\}^{\frac{1}{2}} \, x^{\nu}[G(x,t)]^{\frac{1}{2}} \, P_{n,\nu}(x,-t) \tag{5.5}$$

with

$$\lambda_n = 4n + 2\nu + 1 \, , \tag{5.6}$$

Therefore $\psi_n(x,t)$, as a function of x, is a member of $V_{\sigma,\nu}(I)$ for fixed t.

Following are important properties of these spaces: (i) for fixed t, $\Omega_{x,t}$ is a

continuous linear mapping of $V_{\sigma,\nu}(I)$ into itself. In view of the self adjoint

nature of the operator $\Omega_{x,t}$ [see (5.1)] we define the generalized differential

operator $\Omega_{x,t}$ on $V'_{\sigma,\nu}$ by

$$(\Omega_{x,t} \, f(x), \, \phi(x)) \triangleq (f(x), \, \Omega_{x,t} \, \phi(x)), \tag{5.7}$$

for fixed t, where $f \in V'_{\sigma,\nu}$ and $\phi \in V_{\sigma,\nu}$.

(ii) $\mathcal{D}(I)$ is a subspace of $V_{\sigma,\nu}(I)$, and the convergence in $\mathcal{D}(I)$ implies convergence

in $V_{\sigma,\nu}(I)$. Consequently, restriction of any $f \in V'_{\sigma,\nu}(I)$ to $\mathcal{D}(I)$ is in $\mathcal{D}'(I)$.

Moreover, convergence in $V'_{\sigma,\nu}(I)$ implies convergence in $\mathcal{D}'(I)$.

(iii) $V_{\sigma,\nu}(I) \subset \mathcal{E}(I)$ and since $\mathcal{D}(I)$ is dense in $\mathcal{E}(I)$, $V_{\sigma,\nu}(I)$ is also dense in

$\mathcal{E}(I)$. Furthermore, if $\{\phi_m\}_{m=1}^{\infty}$ converges in $V_{\sigma,\nu}(I)$ to the limit ϕ, then $\{\phi_m\}$ also

converges in $\mathcal{E}(I)$ to the some limit ϕ. The space $\mathcal{E}'(I)$ is a subspace of $V'_{\sigma,\nu}(I)$.

(iv) We imbed $L_2(I)$ (and therefore, $V_{\sigma,\nu}(I)$ since $V_{\sigma,\nu}(I) \subset L_2(I)$ into $V'_{\sigma,\nu}(I)$ by

defining the number that $f \in L_2(I)$ assigns to any $\phi \in V_{\sigma,\nu}(I)$ as

$$(f,\phi) \triangleq \int_0^{\infty} f(x) \, \overline{\phi(x)} \, dx. \tag{5.8}$$

Then f is linear and continuous on $V_{\sigma,\nu}$. This imbedding of $L_2(I)$ into $V'_{\sigma,\nu}(I)$ is

one to one.

(v) If $f(x) = \Omega_{x,t}^k \, g(x)$, for fixed t and $g \in L_2(I)$ and some k, then $f \in V'_{\sigma,\nu}(I)$.

    Instead of working with the number $\langle f,\phi \rangle$ that $f \in V'_{\sigma,\nu}$ assigns to $\phi \in V_{\sigma,\nu}$

it is more convenient to work with the number that f assigns to the complex conjugate

of ϕ. We write

$$(f, \phi) = \langle f, \bar{\phi} \rangle \ . \tag{5.9}$$

This is consistent with the inner product notation of $L_2(I)$. The multiplication by a complex number a is given by

$$(af, \phi) = \langle af, \bar{\phi} \rangle = \langle f, a\bar{\phi} \rangle = a(f, \phi). \tag{5.10}$$

The following two lemmas are useful in the subsequent analysis.

LEMMA 5.1.   If $\phi(x) \in V_{\sigma, \nu}(I)$ then for $o < t \leq \sigma$,

$$\phi(x) = \sum_{n=0}^{\infty} (\phi(x), \psi_n(x, t)) \ \psi_n(x, t) \ , \tag{5.11}$$

where the series converges in $V_{\sigma, \nu}(I)$.

PROOF.   By (5.2), $\Omega_{x, t}^k \ \phi(x)$ is in $L_2(I)$ for each nonnegative integer k.   Hence by [5, pp. 470-471] we may expand $\Omega_{x, t}^k \ \phi(x)$ into a series of orthonormal functions $\psi_n(x, t)$.   Thus for fixed t, $o < t \leq \sigma$,

$$\Omega_{x, t}^k [\phi(x)] = \sum_{n=0}^{\infty} (\Omega_{x, t}^k \ \phi(x), \ \psi_n(x, t) \ \psi_n(x, t)$$

$$= \sum_n (\phi(x), \ \Omega_{x, t}^k \ \psi_n(x, t)) \ \psi_n(x, t) \quad \text{by}$$

$$= \sum_n (\phi(x), \ (-\lambda_n)^k \ \psi_n(x, t))$$

$$= \sum_n (\phi(x), \ \psi_n(x, t)) (-\lambda_n)^k \ \psi_n(x, t)$$

$$= \sum_{n=0}^{\infty} (\phi, \psi_n) \ \Omega_{x, t}^k \ \psi_n \ . \tag{5.12}$$

Consequently, for each k,

$$\beta_k [\phi(x) - \sum_{n=0}^{N} (\phi, \ \psi_n) \ \psi_n] \to 0 \ , \quad \text{as } N \to \infty \tag{5.13}$$

This proves the lemma.

Using (5.3), (5.12) and the fact that the inner product is continuous with repect to each of its arguments, for any two members ϕ and χ of $V_{\sigma, \nu}$, we obtain

$$(\Omega_{x, t} \ \phi(x), \ \chi(x)) = (\phi(x), \ \Omega_{x, t} \ \chi(x)) \ , \tag{5.14}$$

Therefore the operator $\Omega_{x, t}$ is a self adjoint operator.

LEMMA 5.2.   Let a_n denote complex numbers.   Then $\sum_{n=0}^{\infty} a_n \psi_n(x, t)$ for $o < t \leq \sigma$, converges in $V_{\sigma, \nu}(I)$ if and only if $\sum_{n=0}^{\infty} |\lambda_n|^{2k} \ |a_n|^2$ converges for every non-negative integer k.

PROOF.   The proof is similar to that of Zemaniar [1, p 255] and hence it is omitted.

6. <u>ORTHOGONAL SERIES EXPANSION OF GENERALIZED FUNCTIONS</u>.

The following theorem provides an orthonormal series expansion of generalized functions belonging to $V'_{\sigma,\nu}$ which in turn yields an inversion formula for a certain generalized integral transformation.

THEOREM 6.1.   Let F(n) be the generalized integral transformation of $f \in V'_{\sigma,\nu}$ defined for fixed t by

$$F(n) \triangleq (f(x), \psi_n(x,t)) \equiv T[f] \tag{6.1}$$

Then

$$f(x) = \sum_{n=0}^{\infty} F(n) \psi_n(x,t), \tag{6.2}$$

where the series converges in $V'_{\sigma,\nu}(I)$.

PROOF.   By Lemma 5.1, for any $\phi(x) \in V_{\sigma,\nu}(I)$, we obtain

$$(f,\phi) = (f, \sum(\phi(x), \psi_n(x,t)) \psi_n(x,t))$$

$$= \sum_{n=0}^{\infty} (f, \psi_n) \overline{(\phi, \psi_n)}$$

$$= \sum_{n=0}^{\infty} (f, \psi_n)(\psi_n,\phi) = \sum_{n=0}^{\infty} F_n(\psi_n(x,t), \phi(x,t)).$$

This proves that the series on the right of (6.1) converges in $V'_{\sigma,\nu}(I)$.

THEOREM 6.2 (UNIQUENESS).   Let f and g be elements of $V'_{\sigma,\nu}$ and let their transforms F(n) and G(n) satisfy F(n) = G(n) for all n, then f = g in the sense of equality in $V'_{\sigma,\nu}$.

PROOF.   We have

$$f - g = \sum (f - g, \psi_n)\psi_n = \sum [(f, \psi_n) - (g, \psi_n)] \psi_n = 0.$$

7. <u>CHARACTERIZATION THEOREMS</u>.   The following theorem gives a characterization of the transform F(n) of $f \in V'_{\sigma,\nu}$.   Its proof being similar to that of Theorem 9.6-1, p 261 in [1] and hence it is omitted.

THEOREM 7.1.   Let b_n denote complex numbers.   Then for fixed, t, $o < t \leq \sigma'$,

$$\sum_{n=0}^{\infty} b_n \psi_n(x,t), \tag{7.1}$$

converges in $V'_{\sigma,\nu}(I)$ if and only if there exists a nonnegative integer q such that the series

$$\sum_{\substack{\lambda_n \neq o}} |\lambda_n|^{-2q} |b_n|^2$$

converges. Furthermore, if f denotes the sum in $V'_{\sigma,\nu}$ of (7.1) then $b_n = (f, \psi_n)$.

The next theorem is analogous to that of Theorem 9.6-2, p 262 [1] and gives a characterization of elements of $V'_{\sigma,\nu}(I)$.

THEOREM 7.2. A necessary and sufficient condition for f to be a member of $V'_{\sigma,\nu}(I)$ is that there be some nonnegative integer q and a $g \in L_2(I)$ such that for fixed t,

$$f(x) = \Omega^q_{x,t} \ g(x) + \sum_{\substack{\lambda_n = o}} c_n \ \psi_n(x,t), \ o < t \leq \sigma \ , \tag{7.2}$$

where c_n are certain complex constants and $\sum_{\substack{\lambda_n = o}}$ denotes a summation on those n for which $\lambda_n = 0$; these are finite in number.

8. AN OPERATIONAL CALCULUS.

Since $\Omega_{x,t}$ is a continuous linear mapping of $V'_{\sigma,\nu}$ for every $f \in V'_{\sigma,\nu}$ we can write

$$\Omega^k_{x,t} \ f(x) = \sum_{n=o}^{\infty} (f(x), \psi_n(x,t)) \Omega^k_{x,t} \ \psi_n(x,t)$$

$$= \sum_{n=o}^{\infty} (f(x), \psi_n(x,t))(-\lambda_n)^k \ \psi_n(x,t) \ . \tag{8.1}$$

Using this fact we can solve the differential equation

$$P(\Omega_{x,t}) \ u(x) = g(x) \ , \tag{8.2}$$

where P is a polynomial and the given g and unknown u are required to be in $V'_{\sigma,\nu}$. Applying the transformation T defined by (6.1) we obtain

$$P(\lambda_n) \ U(n) = G(n), \quad U = T \ u, \quad G = T \ g.$$

If $P(\lambda_n) \neq 0$ for every n, we can divide $P(\lambda_n)$ and apply T^{-1} to obtain

$$u = \sum_{n=o}^{\infty} \frac{G(n)}{P(\lambda_n)} \ \psi_n \ . \tag{8.3}$$

By Theorems 6.2 and 7.1, this solution exists, and is unique in $V'_{\sigma,\nu}$. If $P(\lambda_n) = 0$ for some λ_n, say, for λ_{n_k} ($k = 1, \ldots, m$), then a solution exists in $V'_{\sigma,\nu}$ if and only if $G(\lambda_{n_k}) = 0$ for $k = 1, 2, \ldots, m$. In this case solution (8.3) is no longer unique and we may add to it any complementary solution

$$u_c = \sum_{k=1}^{m} a_k \ \psi_{n_k} \ , \quad \text{where } a_k \text{ are arbitrary constants,} \tag{8.4}$$

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