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APPROXIMATION ON THE SEMI-INFINITE INTERVAL

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ABSTRACT. The approximation of a function $f \in C[a,b]$ by Bernstein polynomials is well-known. It is based on the binomial distribution. O. Szasz has shown that there are analogous approximations on the interval $[0,\infty)$ based on the Poisson distribution. Recently R. Mohapatra has generalized Szasz' result to the case in which the approximating function is

$$\alpha e^{-ux} \sum_{k=N}^{\infty} \frac{(ux)^{k\alpha+\beta-1}}{\Gamma(k\alpha+\beta)} f\left(\frac{k\alpha}{u}\right)$$

The present note shows that these results are special cases of a Tauberian theorem for certain infinite series having positive coefficients.

KEYWORDS AND PHRASES. *Szasz operators, Borel summability, Tauberian theorems.*
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1. INTRODUCTION.

Let us denote the class of functions f such that $f \in C[0,\infty)$ and for which

$\lim_{t\to\infty} f(t)$ exists by $C_{L,\infty}$. The subclass for which $\lim_{t\to\infty} f(t) = 0$ we shall denote by C_∞.

It is known that if $f \in C_{L,\infty}$ then

$$\lim_{u\to\infty} \alpha e^{-xu} \sum_{k=N}^{\infty} \frac{(xu)^{k\alpha+\beta-1}}{\Gamma(k\alpha+\beta)} f\left(\frac{k\alpha}{u}\right) = f(x) \tag{1}$$

for each $x \in (0,\infty)$. Here $\alpha > 0$, β is a real number and N is a positive integer exceeding $-\beta/\alpha$. This result was proved in [1] and is a generalization of a result due to O. Szász [2] which was the special case $\alpha = \beta = 1$, $N = 0$.

The proof of (1) depends heavily on a result due to D. Borwein [3], namely that

$$\lim_{u\to\infty} \alpha e^{-u} \sum_{k=N}^{\infty} \frac{u^{k\alpha+\beta-1}}{\Gamma(k\alpha+\beta)} = 1 \tag{2}$$

and it is the purpose of the present note to show that the deduction of (1) from (2) is a special case of a general theorem about infinite series. This theorem is of the Tauberian type and the method of proof which we give is of rather wide applicability. Our result is

<u>THEOREM</u>. Suppose that $f \in C_{L,\infty}$. Let $a_k \geq 0$, let K be a constant and let $\{v_k\}$ be a strictly increasing sequence of positive numbers. Then

$$\lim_{u\to\infty} e^{-u} \sum_{k=0}^{\infty} a_k u^{v_k} = 1 \tag{3}$$

$$\text{implies} \quad \lim_{u\to\infty} e^{-xu} \sum_{k=0}^{\infty} a_k (xu)^{v_k} f\left(\frac{v_k+K}{u}\right) = f(x)$$

for each $x \in (0,\infty)$.

2. PROOF OF THE THEOREM

Since the result is trivially true if f is a constant function there is no loss of generality in supposing $f \in C_\infty$

instead of $f \in C_{L,\infty}$. As usual we will denote by $||f||$ the norm of f in the space C_∞, namely $||f|| = \sup_{[0,\infty)} |f(x)|$. Now for each $x \in (0,\infty)$

$$\overline{\lim_{u\to\infty}} \; e^{-xu} \sum_{k=0}^{\infty} a_k (xu)^{v_k} f\left(\frac{v_k+K}{u}\right)$$

defines a linear functional on C_∞ which we will denote by $\overline{\ell}_x$. And if $\overline{\lim}$ is replaced by $\underline{\lim}$ the corresponding linear functional will be denoted by $\underline{\ell}_x$.

First we consider $\overline{\ell}_x$. Since

$$\left|e^{-xu} \sum_{k=0}^{\infty} a_k (xu)^{v_k} f\left(\frac{v_k+K}{u}\right)\right| \le ||f|| \; e^{-xu} \sum_{k=0}^{\infty} a_k (xu)^{v_k}$$

we see, on letting $u\to\infty$, that $|\overline{\ell}_x(f)| \le ||f||$. Hence $\overline{\ell}_x$ is a bounded linear functional on C_∞ and so we will have

$$\overline{\ell}_x(f) = \int_0^\infty f(t)\, d\alpha_x(t)$$

for some function $\alpha_x \in BV[0,\infty)$, and we shall take α_x as having been normalized in the usual way. Now if we take $f(t) = e^{-\lambda t}$ $(\lambda > 0)$ it is a simple matter to see that $\overline{\ell}_x(e^{-\lambda t}) = e^{-\lambda x}$. In this calculation the hypothesis (3) is used in the form

$$\lim_{u\to\infty} e^{-xu} \sum_{k=0}^{\infty} a_k (xu)^{v_k} = 1 \qquad (x > 0).$$

Hence

$$\overline{\ell}_x(e^{-\lambda t}) \equiv \int_0^\infty e^{-\lambda t}\, d\alpha_x(t) = e^{-\lambda x} \qquad (\lambda > 0).$$

By a well known theorem [4] this determines the normalized function α_x uniquely and by inspection it is seen to be

$$\alpha_x(t) = \begin{cases} 0 & (0 \le t < x) \\ \frac{1}{2} & (t = x) \\ 1 & (x < t) \end{cases}$$

Hence for $f \in C_\infty$ we have

$$\overline{\ell}_x(f) = \int_0^\infty f(t)\, d\alpha_x(t) = f(x) \ .$$

Now all of the above analysis involving $\overline{\ell}_x$ could be repeated with $\underline{\ell}_x$ instead. The same function α_x would be obtained and so we have

$$\underline{\ell}_x(f) = \overline{\ell}_x(f) = f(x)$$

That is to say, if $x > 0$

then $\lim\limits_{u\to\infty} e^{-xu} \sum\limits_{k=0}^{\infty} a_k(xu)^{v_k} f(\frac{v_k+K}{u})$ exists

and equals $f(x)$. This concludes the proof of the theorem.

We conclude with two remarks. The above theorem is about point-wise convergence whereas in [1] and [2] the uniform convergence of a set of functions $P_u(x)$ to $f(x)$ at each point $x_o \in [0,\infty)$ was considered. For the definition of this type of convergence we refer the reader to either of these sources but, when $f \in C_{L,\infty}$, to go from pointwise convergence to this other type of convergence is, any way, a simple matter. Secondly, we mention that in [1] the result (1) was stated for $x \in [0,\infty)$ but except in the case $N\alpha+\beta = 1$ the point $x = 0$ should be omitted.

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