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MAZUR SPACES

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ABSTRACT. A Mazur space is a locally convex topological vector space X such that every $f \in X^s$ is continuous where X^s is the set of sequentially continuous linear functionals on X; X^s is studied when X is of the form C(H), H a topological space, and when X is the weak * dual of a locally convex space. This leads to a new classification of compact T_2 spaces H, those for which the weak * dual of C(H) is a Mazur space. An open question about Banach spaces with weak * sequentially compact dual ball is settled: the dual space need not be Mazur.

KEY WORDS AND PHRASES. *Sequentially continuous linear maps, classification of compact T_2 spaces.*

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1. INTRODUCTION.

By a ℓ.c. space we shall understand a locally convex Hausdorff topological vector space over the real numbers, and use the terminology and ideas of standard texts such as [9], [14]. For a ℓ.c. space X, its dual X' is the set of continuous linear functionals (= real valued functions) and X^S is the set of sequentially continuous linear functionals, so that $X' \subset X^S$. Let H be a completely regular Hausdorff space; C(H), (respectively, C*(H)) is the set of real, continuous (respectively, continuous and bounded) functions on H; βH, υH are, respectively, the Stone-Cech and real compactifications of H. (See [13], §8.6). As usual we identify C*(H) with C(βH), C(H) with C(νH).

REMARK 0.1. We shall refer several times to GB spaces. A GB space is a Banach space X such that weak * convergent sequences in X' are weakly convergent. An example is ℓ^∞. See [14], §14-7.

A Mazur space, [15], p. 51, is a ℓ.c. space such that $X^S = X'$. The study of such spaces leads naturally to a new class of compact T_2 spaces H called μ spaces; those for which C(H)' with its weak * topology is a Mazur space. These include the Eberlein compacts and lie at the opposite end of a spectrum from the G spaces, those for which C(H) is a GB space. See §4 for details.

The study of Mazur spaces had its origins in an unpublished result of S. Mazur, which is the last sentence of Th 1.1. The Theorem is known [8], [12]. We give a new and somewhat simpler proof.

In Section 2 a generic method is given for constructing non-Mazur spaces. Section 3 shows that the important class of weakly compactly generated Banach spaces have Mazur duals and lays the groundwork for proving that Eberlein compacts are μ spaces in §4. Whether the converse holds is an intriguing question. Heredity properties are studied in Section 5 and in Section 6 are considered relevant properties of Banach spaces whose duals have weak * sequentially compact unit discs.

1. POINTWISE C(H). Each point t of υH may be considered as a linear

functional on C(H), namely, to $f \in C(H) = C(\upsilon H)$ we assign the number $f(t)$. This functional is called the evaluation at t. For any set S in a vector space, $[S]$ denotes the span of S.

THEOREM 1.1. Let H be a completely regular T_2 space, $X = C(H)$ with the pointwise topology. Then $X^s = [\upsilon H]$ i.e. a linear functional is sequentially continuous iff it is a linear combination of evaluations at points of υH. In particular X is a Mazur space iff H is realcompact.

PROOF. First let $z \in \upsilon H$ and define $F(x) = x(z)$ for $x \in X$. Now for each $x \in X$ there exists $h \in H$ such that $x(h) = x(z)$; see for example [13], Problem 8.5.9. If $\{x_n\}$ is a sequence in X we apply this to $x = \Sigma\, |x_n - x_n(z)| \wedge 2^{-n}$ and obtain $h \in H$ such that $x_n(h) = x_n(z)$ for each n i.e. $F(x_n) = x_n(h)$ for each n. Hence F is sequentially continuous.

The converse will be proved after three Lemmas, the first two of which are due to A. K. Snyder.

LEMMA 1.2. Let μ be a finite positive regular Borel measure with infinite support S on a regular T_2 space H. Then there exist sequences $\{F_n\}$ of closed sets and $\{G_n\}$ of open sets with $\{G_n\}$ pairwise disjoint, $F_n \subset G_n$, and $\mu(F_n) > 0$ for each n.

PROOF. Call a set G <u>considerable</u> if G is open, $S \cap G$ is infinite and $\mu(S \cap G) > 0$. We show first that <u>if G is considerable, there exists $s \in S \cap G$ which has a closed neighborhood U such that $G \quad U$ is also considerable</u>: Namely, choose $s \in S \cap G$ which has a neighborhood V such that $(S \cap G) \quad V$ is infinite. Now $\mu(\{s\}) < \mu(S \cap G)$ since $(S \cap G) \quad \{s\}$ is not the support of μ, so s has a closed neighborhood U with $U \subset V$, $\mu(U) < \mu(S \cap G)$.

Apply this with $G = H$ obtaining U_1 with considerable complement; again with $G = H \quad U_1$ yielding $s_2 \in S \quad U_1$ with a closed neighborhood U_2 disjoint from U_1 such that $(H \quad U_1) \quad U_2$ is considerable. Continuing, we obtain a disjoint sequence $\{U_n\}$ whose interiors $\{G_n\}$ have $\mu(G_n) \geq \mu(G_n \cap S) > 0$. Since μ is regular each G_n includes a closed F_n with $\mu(F_n) > 0$.

LEMMA 1.3. Let H be a completely regular T_2 space, $X = C^*(H) = C(\beta H)$ with the pointwise topology, and $f \in X^s$. Then there exist $z_1, z_2, \ldots, z_n \in \beta H$ such that $f(x) = \Sigma\, \alpha_i x(z_i)$ for all $x \in X$.

PROOF. Since uniform convergence implies pointwise convergence, f is $||\cdot||_\infty$ continuous on X and the Riesz theorem gives a measure μ on βH with $f(x) = \int x \, d\mu$. If the result is false, the support of μ is infinite and we may apply Lemma 1.2 to $|\mu|$ on βH. The Radon-Nikodym theorem implies that $d\mu = \alpha d|\mu|$ with $\alpha \in L_1(\beta H, \mu)$ and $|\alpha(h)| \equiv 1$. Let $C_n \in L_1$ be the characteristic function of F_n and choose $u_n \in C^*(H)$ with $||u_n - \bar{\alpha}C_n||_1 < t_n = |\mu|(F_n)$. We may assume also that $u_n = 0$ on $\beta H \quad G_n$ for it can be multiplied by a continuous function which is 1 on F_n, 0 off G_n. Now for any h, $u_n(h) = 0$ for all but one value of n, and so $\lambda_n u_n \to 0$ pointwise for any choice of scalars λ_n. Since this implies that $\lambda_n f(u_n) = f(\lambda_n u_n) \to 0$ it follows that $f(u_n) = 0$ for sufficiently large n. This is contradicted by noting that $|f(u_n) - t_n| = |\int (u_n - \bar{\alpha}C_n)\,d\mu| \leq ||u_n - \bar{\alpha}C_n||_1 < t_n$ for all n.

LEMMA 1.4. Let H be a completly regular T_2 space, $z \in \beta H \quad \upsilon H$, and $z_1, z_2, \ldots, z_n \in \beta H$. Then there exists $x \in C^*(H)$ with $x(z) = 0$, $x(t) \neq 0$ for $t \in H$ and $t = z_1, z_2, \ldots, z_n$.

PROOF. Let $f : \beta H \to R^+$ (the one point compactification of R) have $f(z) = \infty$, $f \in C(H)$. Let $g = 1/(|f| \vee 1)$. Then $g(z) = 0$, $g(h) \neq 0$ for $h \in H$. Let $u \in C^*(H)$, $0 \leq u \leq 1$, $u(z) = 0$, $u(z_i) = 1$ for $i = 1, 2, \ldots, n$. Finally let $x = g + u$.

To complete the proof of Theorem 1.1 in which now $X = C(H)$, let $g \in X^s [\upsilon H]$, $f = g|C^*(H)$. By Lemma 1.3, the first half of Theorem 1.1, and the fact that $C^*(H)$ is dense in X, we may assume that $f(x) = \alpha x(z) + \Sigma_{i=1}^{n} \alpha_i x(z_i)$ with $\alpha \neq 0$, $z \in \beta H \setminus \upsilon H$. Choose x as in Lemma 1.4 and set $u_k = 1/(1+kx^2)$ for $k = 1,2,\ldots$. Then $u_k \to 0$ pointwise on H but $f(u_k) = \alpha + \Sigma\, \alpha_i/[1+kx(z_i)^2] \to \alpha \neq 0$.

EXAMPLE 1.5. Using Theorem 1.1 we can give a very easy example to show that <u>Mazur is not inherited by closed subspaces</u>. (See [10] for an example involving distribution spaces.) Let H be a Banach space; as a metrizable space, H is realcompact (at least for spaces of non-measurable cardinal; for example, all known spaces.) Then H' with its weak * topology is a closed subspace of $C(H)$, (in addition it is sequentially complete.) Now $C(H)$ is Mazur by Theorem 1.1 but H' is not if, for example, $H = \ell^\infty$. See Remark 0.1. It follows that <u>there is no extension theorem for sequentially continuous linear functionals</u>.

REMARK 1.6. The Nachbin-Shirota theorem has been extended in [1], Prop. 5.2: <u>$C(H)$ with the compact open topology is Mazur iff H is realcompact and iff $C(H)$ is Mazur in its weak topology</u>.

2. <u>SAME CONVERGENT SEQUENCES</u>. In this section we give a generic method for constructing non-Mazur spaces. Among the applications are an improvement of Example 1.5 and a simplified treatment of a result of J. Isbell.

THEOREM 2.1. A ℓ.c. space (X,T) is a non-Mazur space if X has another ℓ.c. Topology T_1 such that $(X,T_1)' \not\subset (X,T)'$ and T, T_1 have the same convergent sequences.

PROOF. A slightly stronger result is true. Suppose that (X,T) is Mazur and that every T convergent sequence is T_1 convergent. Then $(X,T_1)' \subset (X,T_1)^s \subset (X,T)^s = (X,T)'$. //

We give three applications. First let B be a non-reflexive GB space, Remark 0.1, and $X = (B', \text{weak } *)$. Then X is not Mazur.

EXAMPLE 2.2. D. J. H. Garling (see [14], Prob. 14-2-107) showed that $\tau(\ell^1, c_o)$ (the Mackey topology) and $||\cdot||_1$ have the same convergent sequences. Hence $(\ell, \tau(\ell^1, c_o))$ is not a Mazur space. Thus a space with the Mackey topology need not be Mazur.

EXAMPLE 2.3. Let B be a Banach space and S a dense barrelled proper subspace of (B', norm). Then $\sigma(B,S)$ and w, the weak topology of B, have the same convergent sequences. See [14], §15-1, 15-2, Problems 12-2-107, 9-3-104, Theorem 9-3-4. (Apply these to the natural embedding of B in B'.) By Theorem 2.1 $[B, \sigma(B,S)]$ is not Mazur.

EXAMPLE 2.4. A non-Mazur closed subspace of C(H) (pointwise) with H a compact metric space. Compare Example 1.5. Let B be a Banach space and H a weak * compact set in B' such that S, the span of H, is a barrelled dense proper subspace of (B', norm). (See Example 2.6). Then [C(H), pointwise] is Mazur by Theorem 1.1. Let $X = B|H \subset C(H)$, considering $B \subset B''$. Then X is not a Mazur space by Example 2.3 since its (pointwise) topology is $\sigma(X,S) = \sigma(B,S)$. That X is closed in C(H) will now be proved under weaker hypotheses.

LEMMA 2.5. Let B be a ℓ.c. space, $H \subset B'$, $X = B|H$ (as in Example 2.4). Then X is a pointwise closed subspace of C(H).

PROOF. Note that H has the topology $\sigma(B',B)$. Let x^α be a net in X, $F \in C(H)$, $x^\alpha(h) \to F(h)$ for $h \in H$. Extend F to S, the span of H in B', by $F(\Sigma\, t_i h^i) = \Sigma t_i F(h^i)$. This extension is well-defined, hence linear, since if $s = \Sigma\, t_i h^i = \Sigma\, u_i h^i$ we have $\Sigma\, t_i F(h^i) = \lim \Sigma\, t_i x^\alpha(h^i) = \lim x^\alpha(s) = \lim \Sigma\, u_i x^\alpha(h^i) = \Sigma\, u_i F(h^i)$. Now F is a linear functional on S which is $\sigma(S,B)$ continuous, thus $F \in B$.

EXAMPLE 2.6. In [7], p. 223, J. Isbell gives a non-Mazur subspace X of C(H) in which H is the Cantor set, not considering whether X is closed.

His ultimate aim was to find such a subspace of countable dimension.) We shall show that his example has the properties we require. (Our proof in 2.4 of the non-Mazur character of the subspace is different from Isbell's). Let H be the set of all sequences of 0's and 1's. So $H \subset \ell^\infty$ and, on H, $\sigma(\ell^\infty, \ell^1)$ coincides with the product topology of 2^ω so that H is the Cantor set. Our task is completed by Example 2.4 and the observation that the span of H is dense and barrelled: see [14], Example 15-1-13.

3. WEAK * DUALS. In many situations involving a ℓ.c. space X, the object of greatest interest is the Space X' with the weak * topology. To mention only three examples: interesting forms of the closed graph theorem have been given involving X such that (X',weak *) is sequentially complete: see [14], Problem 15-3-110; Banach spaces in which the dual disc is weak * sequentially compact have been studied as members of a variety in [4]; and Grothendieck's famous discovery that ℓ^∞ is a GB space deals with the weak * dual of this space.

In this and the next 3 sections we pursue the study of these duals. For any ℓ.c. space X, let $sX = (X', \text{weak } *)^s$, the set of weak * sequentially continuous linear functionals on X'. Taking X to be a Banach space we have $X \subset sX \subset X''$. It is clear that $sX = X''$ iff X is a GB space. We shall call X a μB space iff $sX = X$ i.e. (X', weak *) is a Mazur space. Obviously a Banach space is μB and GB iff it is reflexive.

We remark that a GB space X satisfies $[X', \tau(X',X)]^s = X''$ but the latter condition is not sufficient [Take $X = c_o$ and apply 2.2.]

It is convenient to work in more generality to show the role of completeness. Let a ℓ.c. space X be called a $\mu\ell c$ space if $sX = X$ i.e. (X', weak *) is Mazur.

THEOREM 3.1. A $\mu\ell c$ space is complete in its strong topology.

PROOF. This is by [14], Cor. 8-6-6. A special case is that a barrelled $\mu\ell c$ space is complete. This is improved by:

THEOREM 3.2. A sequentially barrelled $\mu\ell c$ space is complete.

PROOF. Let F be an aw* continuous linear functional on X'. (For this and the rest of the proof see [14], §12-2.). Then $A = F^{\perp}$ is weak * sequentially closed for if $f_n \in A$, $f_n \to f$, the hypothesis yields that $\{f_n\}$ is equicontinuous so that $A \cap \{f_n\}$ is weak * closed in A; hence $f \in A$. Thus F is weak * sequentially continuous and so, by hypothesis it is continuous. The result follows by Grothendieck's completeness theorem.

In the converse direction we give a quite general criterion. A topological space T is called N sequential if for $A \subset T$, $t \in \bar{A}$, A contains a sequence converging to t.

THEOREM 3.3. Let X be a ℓ.c. complete space such that each equicontinuous set in X' with the weak * topology is N sequential. Then X is a $\mu\ell c$ space.

PROOF. Let $F \in sX$ and let E be an equicontinuous set in X' with the weak * topology. Let $A \subset E$, $f \in \bar{A}$. Then there exists $a_n \in A$ with $a_n \to f$, hence $F(f) = \lim f(a_n) \in \overline{F[A]}$. Thus $F[\bar{A}] \subset \overline{F[A]}$ and so $F|E$ is continuous, i.e. F is aw* continuous. By Grothendieck's theorem, F is continuous.

COROLLARY 3.4. A separable complete ℓ.c. space is a $\mu\ell c$ space.

COROLLARY 3.5. Every closed subspace Y of a weakly compactly generated Banach space X is a μB space.

PROOF. Let $i : Y \to X$ be inclusion. Then $i' : D_X \to D_Y$ (The unit discs) is onto. Now D_X is an Eberlein compact by [3], Corollary 5.2.3 and so D_Y is also by [2]. The result follows from 3.3 since Eberlein compacts are N sequential [14], §14-1.

REMARK 3.6. Some of these results can be deduced from Prop. 4.1 of [1] whose language we use without explanation. If X is separable $m(X) \supset \tau(X,X')$ since each equicontinuous set is metrizable. So X complete implies $m(X)$ complete, equivalently X is $\mu\ell c$. This is 3.4. Also $\nu(X)$ is the smallest sequentially barrelled topology so, if (X,T) is sequentially barrelled; $\nu(X) \subset T$ and so T is complete, yielding 3.2.

EXAMPLE 3.7. The converse of Corollary 3.5 is false, i.e. a μB space need not be weakly compactly generated, or even a subspace of such a space. Our example assumes the continuum hypothesis. Let $X = \ell^1(I)$ with $|I| = c$. This is a μB space by [1], p. 29, Remark. To prove the assertion, it suffices by the argument of 3.5 to show that the disc $D \subset X'$ is not weak * sequentially compact, hence not N sequential. Take $I = [0,1]$, $X' = \ell^\infty[I]$. Let $f_n(h) = 1$ if $2k/n \leq h < (2k+1)/n$ for some $k = 0,1,\ldots,[\tfrac{1}{2}(n-1)]$; -1 otherwise. If $\{f_n\}$ has a subsequence $g_n \to g$ weak * (hence pointwise) then $\int_J g_n \to 0$ for every interval $J \subset I$. Hence $g = 0$ almost everywhere. But $|g(h)| = 1$ for all h.

4. <u>μ AND G SPACES</u>. A G space (μ space) is a compact T_2 space H such that $C(H)$ is a GB space, (a μB space.) G spaces have been extensively studied. See [14] §14-7. Only a finite space is both μ and G.

THEOREM 4.1. Each Eberlein compact H is a μ space.

PROOF. This is by 3.5 since $C(H)$ is weakly compactly generated; [3] Prop 4.2.1.

This includes all compact metric spaces, a result which also follows from 3.4; and the one point compactification of a discrete space.

We conjecture that the converse is false; that μ spaces exist which are not Eberlein compacts. This is made even more plausible in the next section.

5. <u>HEREDITY</u>. The four properties GB, G, μB, μ obey the following table

whose entries will be discussed below:

	GB	G	μB	μ
Q	yes	no	no	yes
C	no	yes	yes	?

The row headed Q signifies that a quotient of a GB (respectively, μ) space is a GB (respectively, μ) space; but a quotient of a G (respectively, μB) space need not be a G (respectively, μB) space. The row headed C signifies that a closed subspace of a G (respectively, μB) space is a G (respectively, μB) space but a closed subspace of a GB space need not be a GB space. It is thus most natural to conjecture that a closed subspace of a μ space need not be a μ space. An example to verify this would also settle the conjecture of §4 since a closed subspace of an Eberlein compact is also an Eberlein compact.

The properties given for GB are well known. The natural map form $\beta\omega \to \omega^+$ (ω = integers) is a quotient map from a G space onto a non-G space. Since a quotient of a GB space is a GB space it follows that a closed subspace of a G space is a G space. The natural map from $\ell^1(I) \to \ell^\infty$ is a quotient map from a μB space to a non-μB space. Inheritance of μ by quotients will follow when we show (Corollary 5.2) that μB is inherited by closed subspaces.

We need the concept of a Tauberian map $u : X \to Y$ i.e. a map with the property that $F \in X''$, $u''F \in Y$ implies $F \in X$. All we need is that the inclusion map from a closed subspace is Tauberian; see [14], Th. 11-4-5.

LEMMA 5.1. Let Y be a μB space, X a Banach space and suppose there exists a Tauberian $u : X \to Y$. Then X is μB.

PROOF. If $F \in sX$, then $u''F \in sY = Y$ hence $F \in X$.

COROLLARY 5.2. μB is inherited by closed subspaces.

There is also a 3-space theorem:

COLLARY 5.3. If X has a reflexive supspace S with X/S μB then X is μB.

PROOF. If $F \in sX$, $q''F \in s(X/S) = X/S$. Since q is Tauberian, the result follows.

6. SEQUENTIAL COMPACTNESS. Let us call a Banach space X an SCB space if the unit disc in X' is weak * sequentially compact. Every closed subspace of a weakly compactly generated space is an SCB space as shown in §3. Some interest attaches to the study of SCB spaces due to the proof in [5] that every non-SCB space has a separable quotient. It was conjectured by Faires [4] that every SCB space is a μB space. That this is false is shown by means of Example 6.1 due to W. Schachermayer which is published here with his kind permission. (The slightly stronger condition of 3.3 is sufficient.) By 3.5 it follows that an SCB space need not be (a closed subspace of) a weakly compactly generated space.

By analogy with §4 a compact T_2 space H is called an SC space if $C(H)$ is an SCB space. Each Eberlein compact is an SC space but not conversely (Example 6.1). Every SC space is sequentially compact $[H \subset D(X')]$ but not conversely: M Talagrand presented an example at the 1979 Kent State conference of a first countable space H such that $D \subset C(H)'$ includes a copy of βN. (This shows that even an N sequential space need not be an Eberlein compact.)

EXAMPLE 6.1. (W. Schachermayer). Let $H = [0,\Omega]$ where Ω is the first uncountable ordinal. (See [13], §14.5.) H is not a G space since it has convergent sequences, and not an Eberlein compact since Ω is not a sequential limit, or because it is not a μ space as will be shown. Let $X = C(H)$ and define $F \in X''$ by $F(\mu) = \mu(\{\Omega\})$. Then if $F \in X$, we would have (letting $\bar{h}$

be the point mass at $h \in H$, $h \neq \Omega$), $0 = F(\bar{h}) = \int F d\bar{h} = F(h)$ and so $F = 0$ which is false. Thus $F \notin X$ and when we show $F \in sX$ we will know that H is not a μ-space.

LEMMA 6.1.1. Let $\{\mu_n\} \subset X'$. Then there exists $h \in H$, $h \neq \Omega$, such that, for all n, $\mu_n(e) = 0$ whenever $e \subset (h,\Omega)$.

PROOF. Let $\mu \in X'$. Since $\mu(\{i\}) \neq 0$ for only countably many i there exists h such that $\mu(\{i\}) = 0$ for $i \in (h,\Omega)$. For any compact subset e of (h,Ω), $H \quad e$ is open, thus includes some $(b,\Omega]$; hence $e \subset [0,b]$ is countable and $\mu(e) = 0$. By regularity this is true for all Borel sets. Doing this for each μ_n we obtain h_n and may take $h < \Omega$, an upper bound for all h_n.

Note that H is a compact T_2 space which supports no measure.

LEMMA 6.1.2. Let $\{\mu_n\} \subset X'$. There exists $x \in X$ such $F(\mu_n) = \int x d\mu_n$ for all n.

PROOF. Choose h as in 6.1.1 and let X be the characteristic function of the open and closed set $[h+1,\Omega]$. Then $\int x d\mu_n = \mu_n([h+1,\Omega)) + \mu_n(\{\Omega\}) = \mu_n(\{\Omega\}) = F(\mu_n)$.

It follows that $F \in sX$ as claimed. To prove that H is SC let $\{\mu_n\} \subset D$, the unit disc in X'. Choose h as in 6.1.1. Let $N = \{x \in X : x(i) = 0$ for $0 \leq i \leq h$ and $i = \Omega\}$, $N^{\perp} = \{\mu \in D : \mu(x) = 0$ for $x \in N\}$. Since each $\mu_n \in N^{\perp}$ the result will follow when it is shown that $N^{\perp}$ is weak * metrizable. It is sufficient, since it is compact, to show that it has a smaller metric. For this it is sufficient to find a sequence $\{x_n\} \subset X$ which is total over $N^{\perp}$. Let $x_o(\Omega) = 1$, $x_o = 0$ on $[0,h]$. For each isolated point $b \leq h$ let x^b be the characteristic function of b. For each non-isolated $b \leq h$, $[0,b)$ is countable, say $\{c_k\}$, let $x_n^b(i) = 1$ if $i = b$ and 0 if $i > b$ or if $i = c_1, c_2, \ldots, c_n$. Since $[0,h]$ is countable we have named in all a countable set of functions $\{x_0, x^b, x_n^b\}$. It is total over $N^\perp$ for let $0 \neq \mu \in N^\perp$. If $\mu(\{\Omega\}) \neq 0$, then, since μ is supported on $[0,h] \cup \{\Omega\}$, $\mu(x_0) = \int x_0 d\mu = \int_{[0,a]} x_0 d\mu + \mu(\{\Omega\}) = \mu(\{\Omega\}) \neq 0$. If $\mu(\{\Omega\}) = 0$ then $\mu(e) \neq 0$ for some $e \subset [0,h]$; hence $\mu(\{b\}) \neq 0$ for some $b \leq h$. If b is isolated, $\mu(x^b) = \mu(\{b\}) \neq 0$; if not, $x_n^b \to f$, the characteristic function of $\{b\}$, pointwise boundedly and so by the bounded convergence theorem, $\mu(x_n^b) = \int x_n^b \, d\mu \to \mu(\{b\}) \neq 0$ hence some $\mu(x_n^b) \neq 0$.

7. FURTHER RESULTS

It is clear that if (X,T) is Mazur and T_1 is a smaller compatible topology, (X,T_1) is Mazur. This is not necessarily true if T_1 is larger, for $\tau(\ell^1, c_0)$ is not Mazur by 2.2 while $\sigma(\ell^1, c_0)$ is by 3.4. Also "compatible" cannot be omitted since $\tau(\ell^1, c_0)$ is not Mazur while the norm on ℓ^1 is.

The following result is due to J. H. Webb.

THEOREM 7.1. Let X be a ℓ.c. space with a Schauder basis such that $(X', \text{weak}\ *)$ is sequentially complete. Then X is a Mazur space.

Let $f \in X^s$. For $x \in X$, $x = \Sigma t_i b^i$, $f(x) = \Sigma t_i f(b^i) = \lim f_n(x)$ where $f_n = \Sigma_{i=1}^n f(b^i) t_i \in X'$. Thus $f \in X'$.

THEOREM 7.2. This result generalizes Theorem 2 of [6] and leads similarly to the result that for a space X with Schauder basis, weak * sequential completeness of X' leads to its strong sequential completeness. Neither hypothesis can be omitted in 7.1 as shown by [14], Prob. 10-3-301 (in which X is barrelled!), and $\tau(\ell^1, c_o)$, respectively.

EXAMPLE 7.3. $[\ell^\infty, \tau(\ell^\infty, \ell^1)]$ is a Mazur space. This follows from 7.1. The Schauder basis is given in [14], Prob. 9-5-107. This is in contrast with $[\ell^1, \tau(\ell^1, c_o)]$, Example 2.2. The difference lies only in the completeness part.

THEOREM 7.4. Equivalent conditions for a ℓ.c. space (X,T) are:

i. Every sequentially continuous linear map $u : X \to Y$, Y any ℓ.c. space, is continuous.

ii. X is C-sequential

iii. $T = T^+$.

PROOF. Webb's topology T^+ is the largest ℓ.c. topology with the same convergent sequences as T; X is called C-sequential if every absolutely convex sequential neighborhood U of 0 ($x_n \in U$ eventually whenever $x_n \to 0$) is a neighborhood of 0. That i => iii follows from consideration of $i : (X,T) \to (X,T^+)$. The rest is [14], §8.4, Probs. 128, 201.

The conditions of 7.4 do not imply that X has the Mackey topology; indeed there may exist two such comparable compatible topologies, for example, with $X = c_0$; let W (respectively, n) be the weak (respectively, norm) topologies. Now $w^+ \neq n$; further $T^{++} = T^+$ for every T so w^+ is C-sequential. Finally w^+ n by the next result.

THEOREM 7.5. Let (X,T) be a ℓ.c. space. Then T^+ is compatible with T iff (X,T), is Mazur.

PROOF. $\to$: $(X,T)^S = (X,T^+)' = (X,T)'$. $\leftarrow$: $(X,T^+)' = (X,T)^S = (X,T)' \subset (X,T^+)'$.

COROLLARY 7.6. A Mazur space which has the Mackey topology must be C-sequential, but need not be bornological.

PROOF. This follows from 7.3, 7.4 and 7.5. It has the interesting application that every sequentially continuous linear map from $[\ell^\infty, \tau(\ell^\infty, \ell^1)]$ to an arbitrary ℓ.c. space is continuous (by 7.3) even though this space is not bornological. Along the same lines one might conjecture that if (X,T) is C-sequential then $\tau(X,X')$ is also; but this is false: take $X = \ell$, $T = \sigma(\ell^1, c_0)^+$, then $\tau(\ell^1, c_0)$ is not even Mazur.

QUESTIONS. Must a μ space be a Eberlein compact? Must a closed subspace of a μ space be a μ space?

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