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ON ELASTIC WAVES IN A MEDIUM WITH RANDOMLY DISTRIBUTED CYLINDERS

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ABSTRACT. A study is made of the problem of propagation of elastic waves in a medium with a random distribution of cylinders of another material. Neglecting 'back scattering', the scattered field is expanded in a series of 'orders of scattering'. With a further assumption that the n (n > 2) point correlation function of the positions of the cylinders could be factored into two point correlation functions, the average field in the composite medium is found to be resummable, yielding the average velocity of propagation and damping due to scattering. The calculations are presented to the order of $(ka)^2$ for the scalar case of axial shear waves in the composite material. Several limiting cases of interest are recovered.

KEY WORDS AND PHRASES. *Elastic waves, elastic matrix, randomly distributed cylinders, fibers, multiple scattering, correlation functions, forward scattering series, average wave, and specific damping capacity.*

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1. INTRODUCTION.

In a series of papers [1-4], Bose and Mal studied the problem of propagation of elastic waves in a medium consisting of randomly distributed cylinders and spheres in an elastic matrix. The focus in these papers was to extract the behaviour of the average or the coherent wave which propagates in the composite medium. These authors solved the problem of elastic waves for a sparse distribution of cylinders [4]. For dense systems [1-3], they used the methods devised by Fikioris and Waterman [6], and Mathur and Yeh [7] where these papers deal with the multiple scattering of acoustic and electromagnetic waves. The formalism in these papers consisted of first solving the scattering problem by a large number N of cylinders or spheres. The positions of the scatterers are then made random and the configurational average of the field depends on a hierarchy of equations which was broken by invoking Lax's quasicrystalline approximation [8]. This led to the average wave number as a function of the properties of the phases, the concentrations of the scatterers and the correlation in their positions.

In this formalism the existence of the average wave is apriori assumed. However, Twersky [9] demonstrated its existence for sparse distributions by expanding the 'compact form' of solution of the fixed scatterer problem into an 'expanded form' of 'orders of scattering' and then taking the average of the field. The wave number was in agreement with that obtained by Foldy [10] heuristically. This procedure of expansion and resummation of scattering series is also apparently related to the diagram method of Lloyd [11] and Lloyd and Berry [12], who considered the acoustic and electromagnetic cases. The same procedure is adopted here to study the problem of axial shear waves in a distribution of aligned circular cylinders, and the present calculation is carried out to the order $(ka)^2$. This approximately brings out the dispersion and attenuation characteristics due to scattering. The method can also be adopted for vector elastic wave propagation problems. The 'quasicrystalline approximation' as such can then be dispensed with. Two crucial approximations have been made. First, 'back scattering' is neglected in the expanded form of the solution. Second, third and higher order correlations are taken in the form of factors of pair correlation functions.

2. SCATTERING BY ARBITRARY CONFIGURATION OF N CYLINDERS.

The formulation of this problem is similar to that of Bose and Mal [1]. In order to avoid duplication, we summarize the formulation and refer to [1]. We assume that N cylinders of equal radius a are embedded in an infinite matrix material; the shear modulus and density of the two materials are μ', ρ' and μ, ρ respectively. With a suitable frame of reference normal to the cylinders, the polar coordinate of the centers of the cylinders O_i $(i = 1,2,\ldots,N)$ are $(r_i. \theta_i)$. We take the polar coordinates of a point $P(r, \theta)$ referred to O_i as the origin as (R_i, ϕ_i), and that of another cylinder O_j as (r_{ij}, θ_{ij}). If a monochromatic plane shear wave, $\exp[i(kx-\omega t)]$ is incident upon the cylinders, then suppressing the time factor $\exp(-i\omega t)$, we can represent the displacement in the matrix material as

$$w = e^{ikx} + \sum_{i=1}^{N}\sum_{m=-\infty}^{\infty} A_{im} H_m(kR_i) e^{im\phi_i} , \tag{2.1}$$

$$\text{where } k = \omega/\beta \text{ and } \beta = (\mu/\rho)^{\frac{1}{2}} , \tag{2.2ab}$$

and $H_m(z)$ is the Hankel function of the first kind. Apparently, the first term in (2.1) represents the incident wave and the second term in (2.1) denotes the wave scattered by the N cylinders.

The displacement inside the ith cylinder can be similarly represented as

$$w_i = \sum_{m=-\infty}^{\infty} B_{im} J_m(k' R_i) e^{im\phi_i} , \tag{2.3}$$

$$\text{where} \qquad k' = \omega'/\beta' \qquad \text{and} \qquad \beta' = (\mu'/\rho')^{\frac{1}{2}} . \tag{2.4ab}$$

Invoking the conditions of continuity on the displacement and shear stress on the surfaces of the cylinders, we obtain the scattering coefficients using calculations similar to [1] in the form

$$A_{in} = iC_n F_{in} \tag{2.5}$$

$$B_{in} = D_n F_{in} \tag{2.6}$$

where

$$iC_n = \frac{\mu J_n(k'a)(\partial/\partial a)J_n(ka) - \mu' J_n(ka)(\partial/\partial a)J_n(k'a)}{\mu' H_n(ka)(\partial/\partial a)J_n(k'a) - \mu' J_n(k'a)(\partial/\partial a)H_n(ka)} , \tag{2.7}$$

$$D_n = \frac{2i}{\pi a}\mu\Big/[\mu J_n(k'a)\frac{\partial}{\partial a}H_n(ka) - \mu' H_n(ka)\frac{\partial}{\partial a}J_n(k'a)] \ . \qquad (2.8)$$

C_n and D_n are apparently even in n. The constants F_{in} satisfy the infinite system of linear equations:

$$F_{jn} = i^n e^{ikr_j\cos\theta_j} + i \sum_{i=1}^{N}{}' \sum_{m=-\infty}^{\infty} C_m F_{im} H_{m-n}(kr_{ij}) \exp[i(m-n)\theta_{ij}] \qquad (2.9)$$

The prime in the summation means $i \neq j$.

The solution of equation (2.9) is the point of difficulty in the multiple scattering theory. Following Twersky [9] we may iterate (2.9) and obtain an expanded form of solution. For brevity we introduce the following notations:

$$x_j = r_j \cos\theta_j \ , \qquad (2.10)$$

$$v_{ijmn} = i\, C_m H_{m-n}(kr_{ij})\exp[i(m-n)\theta_{ij}]\, i^{(m-n)} \ , \qquad (2.11)$$

and to fix our idea we take j=1. The expanded solution can then be written as

$$i^{-n}F_{1n} = e^{ikx_1} + \sum_i \sum_m v_{i1mn}(e^{ikx_i} + \sum_p e^{ikx_1} v_{1ipm} + \sum_p \sum_q e^{ikx_i} v_{i1qp} v_{1ipm} + \ldots)$$
$$+ \sum_i \sum_j{}' \sum_m \sum_p v_{i1mn} v_{jipm}(e^{ikx_j} + \sum_q e^{ikx_i} v_{jiqp} + \sum_q e^{ikx_1} v_{1jqp} + \ldots)$$
$$+ \sum_i \sum_j{}' \sum_k{}'' \sum_m \sum_p \sum_q v_{i1mn}\, v_{jipm}\, v_{kjqp}(e^{ikx_k} + \ldots) + \ldots \ , \qquad (2.12)$$

where the summations over i,j,k... run from 2 to N and those over m,p,q,... from $-\infty$ to ∞. The single prime means $j \neq i$, the double means $k \neq j$ and $j \neq i$ etc. As an approximation to (2.12) we shall retain the leading terms in the series within the different summations and neglect the rest. We note that in this approximation the suffixes ...,k,j,i,1 appear in that order and hence do not involve back scattering (Twersky [9]). In the sequal we shall assume this approximation and accordingly we take

$$i^{-n}F_{ln} = e^{ikx_l} + \sum_i \sum_m e^{ikx_i} v_{ilmn} + \sum_i \sum_j{}' \sum_m \sum_p e^{ikx_j} v_{jipm} v_{ilmn} + \sum_i \sum_j{}' \sum_k{}' \sum_m \sum_p \sum_q e^{ikx_k} v_{kjqp} v_{jipm} v_{ilmn} + \dots \quad (2.13)$$

The solution of the problem is thus formally and commpletely obtained.

3. RANDOM DISTRIBUTION OF A LARGE NUMBER OF CYLINDERS.

If there are a large number of N cylinders which are aligned but are otherwise distributed at random, we can utilize the above formulation provided the position vector $\underset{\sim}{r}_i$ of 0_i is considered as a random variable. We shall however assume that they always remain confined to a large region S. The random variable $(\underset{\sim}{r}_1, \underset{\sim}{r}_2, \dots, \underset{\sim}{r}_N)$ shall have a probability density which we denote by $p_N(\underset{\sim}{r}_1, \underset{\sim}{r}_2, \dots, \underset{\sim}{r}_N)$. Then, due to indistinguishability of the cylinders the density function is symmetric in its arguments. Furthermore, we can write

$$\begin{aligned} p_N(\underset{\sim}{r}_1, \underset{\sim}{r}_2, \underset{\sim}{r}_3, \dots, \underset{\sim}{r}_N) &= p_1(\underset{\sim}{r}_i)\, p_{N-1}(\underset{\sim}{r}_1, \underset{\sim}{r}_2, \dots, ', \dots, \underset{\sim}{r}_N) \\ &= p_2(\underset{\sim}{r}_i, \underset{\sim}{r}_j)\, p_{N-2}(\underset{\sim}{r}_1, \underset{\sim}{r}_2, \dots, ', \dots, ', \dots, \underset{\sim}{r}_N) \\ &= p_3(\underset{\sim}{r}_i, \underset{\sim}{r}_j, \underset{\sim}{r}_k)\, p_{N-3}(\underset{\sim}{r}_1, \underset{\sim}{r}_2, \dots, ', \dots, ', \dots, ', \dots, \underset{\sim}{r}_N) \text{ etc.} \end{aligned} \quad (3.1)$$

As in Lloyd [11] we shall work in terms of N-body correlation function $g_N(\underset{\sim}{r}_1, \underset{\sim}{r}_2, \dots, \underset{\sim}{r}_N)$ rather than the density functions themselves. This function is defined by the relation

$$p_N(\underset{\sim}{r}_1, \underset{\sim}{r}_2, \dots, \underset{\sim}{r}_N) = \frac{1}{S^N} g_N(\underset{\sim}{r}_1, \underset{\sim}{r}_2, \dots, \underset{\sim}{r}_N) \ . \quad (3.2)$$

The function vanishes when any one of the arguments lies outside the region S.

As in [1] we shall assume that the cylinders are uniformly distributed in S, so that

$$g_1(\underset{\sim}{r}_i) = 1 \ . \quad (3.3)$$

Under the same assumption, the two point correlation function is a function of the distance between them and we can write

$$g_2(\underset{\sim}{r}_i, \underset{\sim}{r}_j) = 1 - f(r_{ij}) = g(r_{ij}) \ . \quad (3.4)$$

Derivation of an expression for $f(r_{ij})$ based on probabilistic postulates is another difficult point in the scattering theory. However, we note that due to impossibility of interpenetration and independence at large separation

$$f(r_{ij}) = 1 \quad , \quad r_{ij} < 2a$$
$$\to 0 \quad , \quad r_{ij} \to \infty \; . \qquad (3.5ab)$$

Determination of higher order correlation functions is likewise difficult. We shall assume, as an approximation, that these functions can be split into two point correlation functions:

$$g_3(\underset{\sim}{r}_i, \underset{\sim}{r}_j, \underset{\sim}{r}_k) = g_2(\underset{\sim}{r}_i, \underset{\sim}{r}_j)\, g_2(\underset{\sim}{r}_j, \underset{\sim}{r}_k)\, g_2(\underset{\sim}{r}_k, \underset{\sim}{r}_i)$$
$$g_N(\underset{\sim}{r}_1, \underset{\sim}{r}_2, \ldots, \underset{\sim}{r}_N) = g_2(\underset{\sim}{r}_1, \underset{\sim}{r}_2)\, g_2(\underset{\sim}{r}_2, \underset{\sim}{r}_3) \cdots g_2(\underset{\sim}{r}_{N-1}, \underset{\sim}{r}_N) \; . \qquad (3.6ab)$$

We note that the first of these relations is the well known Kirkwood superposition approximation.

4. <u>AVERAGE FIELD IN THE COMPOSITE MEDIUM</u>.

To represent the total field in the composite medium we introduce as in [1] the symbol

$$\alpha(\underset{\sim}{r}, \underset{\sim}{r}_i) = 0 \quad , \quad \underset{\sim}{r} \text{ within the ith cylinder}$$
$$= 1 \quad , \quad \underset{\sim}{r} \text{ outside the ith cylinder.} \qquad (4.1ab)$$

We could then combine the fields in the different regions, in the form

$$W = [1 - \sum_{i=1}^{N} \{1-\alpha(\underset{\sim}{r},\underset{\sim}{r}_i)\}]w + \sum_{i=1}^{N} \{1-\alpha(\underset{\sim}{r},\underset{\sim}{r}_i)\}w_i \; . \qquad (4.2)$$

Inserting the expressions (2.1) and (2.3) in the above and taking the mean value with density and correlation functions (3.1) and (3.2), we obtain

$$\langle W \rangle = (1-c)e^{ikx} + n_0 i \sum_{m=-\infty}^{\infty} C_m \Big[\int_{|\underset{\sim}{r}_1-\underset{\sim}{r}|>a} \langle F_{1m} \rangle_1 H_m(kR_1) e^{im\phi_1} d\underset{\sim}{r}_1$$
$$- n_0 \int_{|\underset{\sim}{r}_1-\underset{\sim}{r}|\le a} d\underset{\sim}{r}_1 \int_{|\underset{\sim}{r}_2-\underset{\sim}{r}_1|>2a} g(r_{12}) \langle F_{2m} \rangle_{12} H_m(kR_2) e^{im\phi_2} d\underset{\sim}{r}_2 \Big]$$
$$+ n_0 \sum_{m=-\infty}^{\infty} D_m \int_{|\underset{\sim}{r}_1-\underset{\sim}{r}|\le a} \langle F_{1m} \rangle_1 J_m(k'R_1) e^{im\phi_1} d\underset{\sim}{r}_1 \; , \qquad (4.3)$$

where $n_0 = N/S$ is the number of cylinders per unit area, $c = \pi a^2 n_0$ is the concentration of the cylinders and $\langle F_{1m} \rangle_1$, $\langle F_{2m} \rangle_{12}$ are the conditional mean values when cylinders 0_1 or 0_1 and 0_2 both are held fixed. The determination of the average field thus depends on their evaluation.

We compute these conditional means by using the forward scattering series (2.13). Taking the conditional mean with 0_1 held fixed, we get

$$i^{-n} \langle F_{1n} \rangle_1 = \int e^{ikx_2} \Big[\delta(\underset{\sim}{r}_1 - \underset{\sim}{r}_2) + n_0 \sum_m v_{21mn}\, g_2(\underset{\sim}{r}_1, \underset{\sim}{r}_2)$$

$$+ n_0^2 \sum_m \sum_p \int v_{23pm}\, v_{31mn}\, g_2(\underset{\sim}{r}_2, \underset{\sim}{r}_3)\, g_2(\underset{\sim}{r}_3, \underset{\sim}{r}_1)\, d\underset{\sim}{r}_3$$

$$+ n_0^3 \sum_m \sum_p \sum_q \iint v_{23qp} v_{34pm} v_{41mn}\, g_2(\underset{\sim}{r}_2, \underset{\sim}{r}_3) g_2(\underset{\sim}{r}_3, \underset{\sim}{r}_4) g_2(\underset{\sim}{r}_4, \underset{\sim}{r}_3)\, d\underset{\sim}{r}_3 d\underset{\sim}{r}_4$$

$$+ \, \ldots\ldots \Big] d\underset{\sim}{r}_2 \qquad (4.4)$$

We note that in the process of taking the mean of the summations over i,j,k,... degenerate into identical (N-1), (N-2), (N-3),... terms, which contribute the factors n_0, n_0^2 , n_0^3,... to the different terms, when N is large. A similar phenomenon takes place in equation (4.3). The series can be resummed into a compact form, if we introduce the Fourier integral (Lloyd [11])

$$u_{mn}(\underset{\sim}{K}) = \int v_{21mn}\, g_2(\underset{\sim}{r}_1, \underset{\sim}{r}_2)\, e^{i\underset{\sim}{K}\cdot(\underset{\sim}{r}_2 - \underset{\sim}{r}_1)}\, d(\underset{\sim}{r}_2 - \underset{\sim}{r}_1) \qquad (4.5)$$

$$v_{21mn}\, g_2(\underset{\sim}{r}_1, \underset{\sim}{r}_2) = \frac{1}{(2\pi)^2} \int u_{mn}(\underset{\sim}{K})\, e^{-i\underset{\sim}{K}\cdot(\underset{\sim}{r}_2 - \underset{\sim}{r}_1)}\, d\underset{\sim}{K} \, . \qquad (4.6)$$

The integrals over r_3, (r_3, r_4), etc. may be considered as convolution integrals, and then (4.4) assumes the form

$$i^{-n} \langle F_{1n} \rangle_1 = \int e^{ikx_2} \Big[\delta(\underset{\sim}{r}_1 - \underset{\sim}{r}_2) + \frac{1}{(2\pi)^2} \int [n_0 \sum_m u_{mn}(\underset{\sim}{K}) +$$

$$+ n_0^2 \sum_m \sum_p u_{pm}(\underset{\sim}{K})\, u_{mn}(\underset{\sim}{K}) + n_0^3 \sum_m \sum_p \sum_q u_{qp}(\underset{\sim}{K})\, u_{pm}(\underset{\sim}{K})\, u_{mn}(\underset{\sim}{K})$$

$$+ \ldots\ldots] e^{-i\underset{\sim}{K}\cdot(\underset{\sim}{r}-\underset{\sim}{r}_1)} d\underset{\sim}{K}\, d\underset{\sim}{r}_2$$

$$= \int e^{ikx_2}\left[\delta(\underset{\sim}{r}_1-\underset{\sim}{r}_2) + \frac{1}{(2\pi)^2}\sum_m \int n_0 U(E-n_0U)^{-1} e^{-i\underset{\sim}{K}\cdot(\underset{\sim}{r}_2-\underset{\sim}{r}_1)} d\underset{\sim}{K}\right] d\underset{\sim}{r}_2 ,$$

where E is the unit matrix, and similarly the matrix $U = [u_{mn}]$. The above result can be simplified into

$$i^{-n}\ \langle F_{1n}\rangle_1 = \frac{1}{(2\pi)^2}\int e^{ikx_2}\, d\underset{\sim}{r}_2 \sum_m \int (E-n_0U)^{-1}\ e^{-i\underset{\sim}{K}\cdot(\underset{\sim}{r}_2-\underset{\sim}{r}_1)} d\underset{\sim}{K}\ . \tag{4.7}$$

We evaluate the integral over $\underset{\sim}{r}_2$ first, noting that its range is finite over S. The integral over $\underset{\sim}{K}$ can be evaluated next by contour integration. The integral has a pole at

$$\det(E-n_0U) = 0\ . \tag{4.8}$$

Denoting the pole also by $\underset{\sim}{K}$, equation (4.7) has the form

$$i^{-n}\ \langle F_{1n}\rangle_1 = \int_0^{2\pi} F_n(K)\, e^{i\underset{\sim}{K}\cdot\underset{\sim}{r}_1}\, d\Omega \tag{4.9}$$

where $\underset{\sim}{K} = Ke^{i\Omega}$. Equation (4.9) has the form of superposition of plane waves with number $\underset{\sim}{K}$ incident at $\underset{\sim}{r}_1$. The form (4.9) is sufficient for the purpose of investigating the average wave.

If we similarly proceed to calculate $\langle F_{1n}\rangle_{12}$ from the series (2.13), we shall get the formal expression

$$i^{-n}\ \langle F_{1n}\rangle_{12} = \int_0^{2\pi} G_n(\underset{\sim}{K})e^{i\underset{\sim}{K}\cdot\underset{\sim}{r}_1}\, d\Omega + \sum_m v_{21mn}\int_0^{2\pi} F_n(\underset{\sim}{K})e^{i\underset{\sim}{K}\cdot\underset{\sim}{r}_2}\, d\Omega$$

$$+ e^{ikx_2}\int_0^{2\pi} H_n(\underset{\sim}{K})e^{-i\underset{\sim}{K}\cdot(\underset{\sim}{r}_2-\underset{\sim}{r}_1)}\, d\Omega \tag{4.10}$$

When we insert the forms in (4.9) and (4.10) in equation (4.3) and evaluate the integrals as [1], $\langle W\rangle$ has the form

$$\langle W\rangle = \int_0^{2\pi} W_0(\underset{\sim}{K})e^{i\underset{\sim}{K}\cdot\underset{\sim}{r}}d\Omega\ . \tag{4.11}$$

The incident wave term cancels away with similar terms emerging from the integrals in (4.3) (extinction theorem). Thus $\underset{\sim}{K}$ emerges as the wave number of the average waves in the composite medium.

In [1], Bose and Mal obtained the average wave number by assuming Lax's "quasicrystalline approximation" and the existence of an average plane wave propagating in the direction of x-axis. It can be shown that this number is identical with that given by (4.8). For, the elements of U as given by equation (4.5) with (2.11) and (3.4) are

$$u_{mn}(\underset{\sim}{K}) = iC_m i^{m-n} \int_{|\underset{\sim}{r}_2-\underset{\sim}{r}_1|>2a} \{1-f(\underset{\sim}{r}_{12})\} e^{i\underset{\sim}{K}\cdot(\underset{\sim}{r}_2-\underset{\sim}{r}_1)} H_{m-n}(kr_{12}) e^{i(m-n)\theta_{21}} d(\underset{\sim}{r}_2-\underset{\sim}{r}_1). \quad (4.12)$$

For the uncorrelated term, we write

$$e^{i\underset{\sim}{K}\cdot(\underset{\sim}{r}_2-\underset{\sim}{r}_1)} H_{m-n}(kr_{12}) e^{i(m-n)\theta_{21}} = \frac{1}{k^2-K^2}\left[\nabla^2\left\{e^{i\underset{\sim}{K}\cdot(\underset{\sim}{r}_2-\underset{\sim}{r}_1)}\right\} H_{m-n}(kr_{12}) e^{i(m-n)\theta_{21}} - e^{i\underset{\sim}{K}(\underset{\sim}{r}_2-\underset{\sim}{r}_1)} \nabla^2\left\{H_{m-n}(kr_{12}) e^{i(m-n)\theta_{21}}\right\}\right],$$

and use Green's theorem to convert the surface integral to a contour integral around the circle $|\underset{\sim}{r}_2-\underset{\sim}{r}_1| = 2a$. The latter can be easily evaluated by using the plane wave expansion

$$e^{i\underset{\sim}{K}\cdot(\underset{\sim}{r}_2-\underset{\sim}{r}_1)} = \sum_{s=-\infty}^{\infty} i^s(-1)^s e^{-is\Omega} J_s(Kr_{12}) e^{is\theta_{21}} .$$

The above expansion can also be used to simplify the integral containing the correlation term $f(r_{12})$ and we finally obtain

$$u_{mn}(\underset{\sim}{K}) = 2\pi i C_m\left[\frac{a}{k^2-K^2}\left\{J_{m-n}(2Ka)\frac{\partial}{\partial a}H_{m-n}(2ka) - H_{m-n}(2ka)\frac{\partial}{\partial a}J_{m-n}(2Ka)\right\} - \int_{2a}^{\infty} f(r_{12}) J_{m-n}(Kr_{12}) H_{m-n}(kr_{12}) r_{12} dr_{12}\right] e^{i(m-n)\Omega} . \quad (4.13)$$

The solution of (4.8) then leads to the equation for the wave number obtained in [1]. The equation for K is a complicated transcendental equation which was solved for thin cylinders $ka \ll 1$, $|Ka| \ll 1$.

5. THE AVERAGE WAVE FOR UNCORRELATED THIN CYLINDERS.

The value of K obtained in [1] depends on $f(r_{12})$ as is apparent from (4.13). In order to obtain explicit results, we also assumed that $f(r_{12})$ has an exponential form

$$f(r_{12}) \approx e^{-r_{12}/L} \quad , \quad r_{12} > 2a . \tag{5.1}$$

It was found that due to correlation the average wave showed both dispersion and attenuation, which to the lowest order of small quantitites were approximately proportional to $(kL)^2$. In as much as the correlation length L can be a few multiples of a, it is imperative to look into the average wave number correct to the order $(ka)^2$. In view of this we drop the correlation term in (4.13), and expand the Bessel and Hankel functions in (2.7) to obtain

$$C_0 = \frac{\pi}{4}(ka)^2\, d[1 + \frac{\alpha_0}{8}(ka)^2] , \tag{5.2}$$

$$C_1 = - \frac{\pi}{4}(ka)^2\, m[1 + \frac{\alpha_1}{8}(ka)^2] , \tag{5.3}$$

$$C_2 = - \frac{\pi}{32}(ka)^4\, m , \tag{5.4}$$

where we have written $d = \rho'/\rho - 1$, $m = (\mu' - \mu)/(\mu' + \mu)$ and

$$\alpha_0 = 3 - 4\frac{\rho'}{\rho} - 4(\frac{\rho'}{\rho} - 1)^2\left[\ell n\, \frac{ka}{2} + \gamma - \frac{i\pi}{2}\right] + \frac{\mu}{\mu'}(\frac{\rho'}{\rho})^2 , \tag{5.5a}$$

$$\alpha_1 = 5 - 3\,\frac{\mu'}{\mu} - 4\,\frac{(1 - \mu'/\mu)^2}{1 + \mu'/\mu}\left[\ell n\, \frac{ka}{2} + \gamma - \frac{i\pi}{2}\right] + \frac{4\rho'/\rho}{1 + \mu'/\mu} . \tag{5.5b}$$

Similarly for (4.13) we obtain

$$J_n(2Ka)\, \frac{\partial}{\partial a}\, H_n(2ka) - H_n(2ka)\, \frac{\partial}{\partial a}\, J_n(2Ka) = \frac{2i}{a\pi}(\frac{K}{k})^{|n|}\, [1 - \beta_n (ka)^2 (\frac{K^2}{k^2} - 1)] \tag{5.6}$$

where

$$\left.\begin{aligned} \beta_n &= 1 - 2\left[\ell n\,(ka) + \gamma - i\pi/2\right], \quad n=0 \\ &= \frac{1}{|n|} \quad , \; n = \pm 1, \pm 2, \pm 3, \ldots . \end{aligned}\right\} \tag{5.7ab}$$

γ is the Euler's constant. The final result from (4.8), corrected to the order $(ka)^2$, is found in the form

$$\frac{K^2}{k^2} = \frac{(1-cm)(1+cd)}{1-cm} + \frac{\alpha_0}{8}(ka)^2\, cd\, \frac{(1-cm)}{1+cm} - \frac{\alpha_1}{4}(ka)^2\, \frac{cm(1+cd)}{(1+cm)^2} +$$

$$+ \frac{\beta_0 (ka)^2}{1-c^2m^2}\left[cd(1-4cm+3c^2m^2) - 2cm(1-cm) + \frac{cm(2+c^2md)(1-cm)(1+cd)}{1+cm}\right]$$

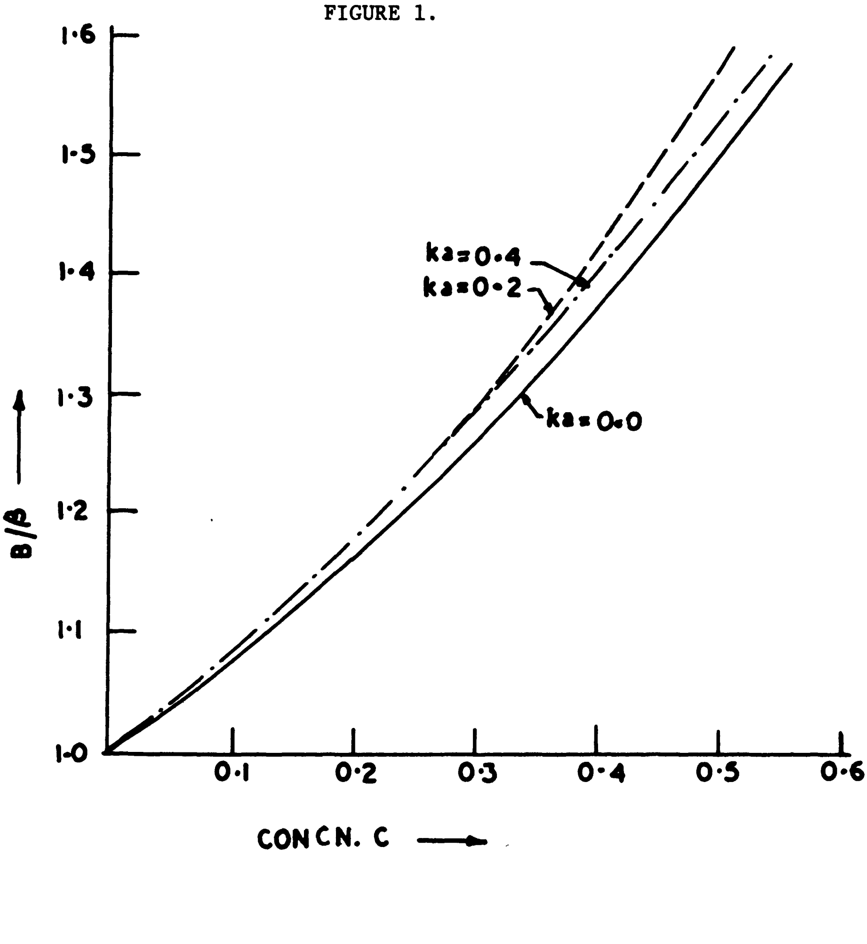
$$- \frac{cm}{1-c^2m^2}\frac{(ka)^2}{8}\left[1 + (1+cd)(1-cm)^2 + cd(1-cm)^2 - cm(2-cm)\right.$$

$$- \{2cd(15 - 17cm + c^2m^2) + cm(1+2cm)\}\frac{(1-cm)(1+cd)}{1+cm}$$

$$\left. + 3cm\left\{\frac{(1-cm)(1+cd)}{1+cm}\right\}^2 - cd\left\{\frac{(1-cm)(1+cd)}{1+cm}\right\}^3\right]. \quad (5.8)$$

If $(ka)^2$ is ignored the expression agrees with that obtained in [1]. For sparse distribution correct to O(c), it also agrees with the result obtained in [4] and the result of Twersky [13]. The waves show both dispersion and attenuation which are roughly proportional to $(ka)^2$. This is so, to the order of calculation undertaken here.

FIGURE 1.



In Figures 1,2, we present the result of computation for aluminium reinforced by boron fibers. For this combination we have $\rho'/\rho = 2.53/2.72$ and $\mu'/\mu = 25/3.87$. If K_1 and K_2 are the real and imaginary parts of K, the average wave velocity is given by $B/\beta = k/K_1$, and the specific damping capacity is $\psi = 4\pi K_2/K_1$. These have been plotted in the figures against the concentration c for different values of ka.

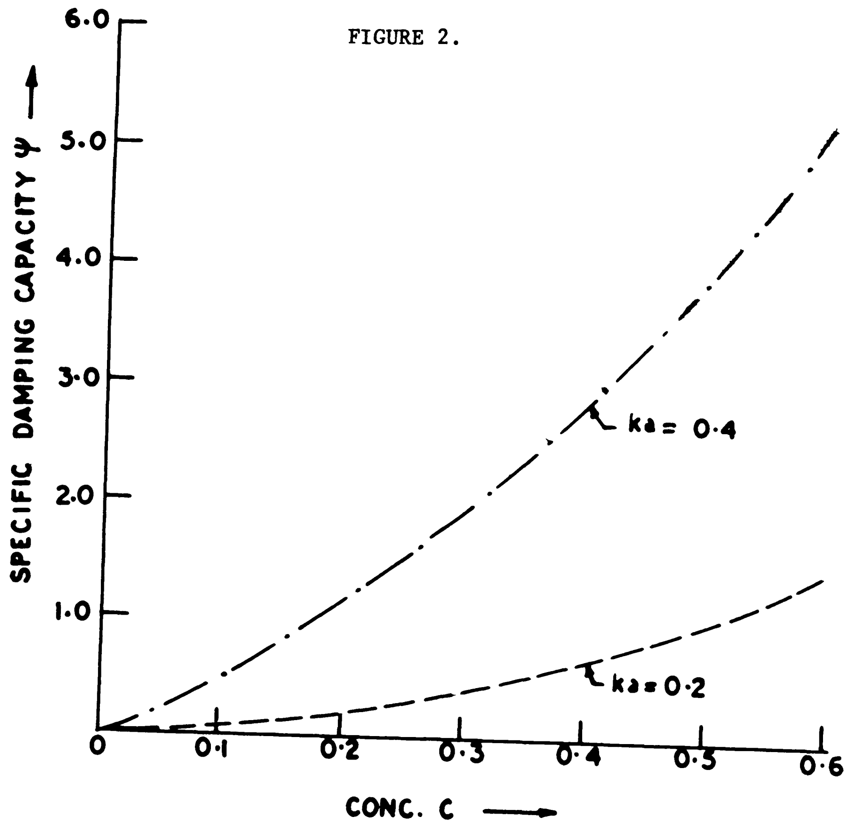


FIGURE 2.

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