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THE STAR COMPACTIFICATION

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ABSTRACT. The relationships between a convergence space and its star compactification is studied. Special attention is given to lifting properties of this compactification. In particular, it is shown that a natural extension of any continuous function to the respective compactification spaces is θ-continuous.

KEY WORDS AND PHRASES. *Convergence space, compactification, G-space, R-series, natural extension, θ-continuous function, proper map, open map.*

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1. INTRODUCTION.

We study a convergence space compactification which was introduced by one of the authors in 1970 (see [11]). The star compactification $\kappa^* = (X^*, i^*)$ of a convergence space X is constructed by adjoining to X the set X' of all non-convergent ultrafilters on X and constructing a compact convergence structure on $X^* = X \cup X'$ in a natural way. It is proved in [11] that a continuous function from a T_2 space X into a compact T_3 space Y has a continuous extension to X^*.

The authors published a survey paper, [7], concerning the existence of largest and smallest convergence space compactifications relative to various constraints. In all cases studied, the largest compactification, whenever it existed, turned out to be κ^*. In a more recent paper, [9], we showed that K^* can be used to characterize ω-regular and completely regular spaces. These results suggest that further investigation of the star compactification is appropriate.

In Section 2, we examine the relationship between the decomposition series of X and X^*, showing that the lengths of these series can differ by at most one. These results yield a method for constructing compact T_2 spaces with arbitrarily long decomposition series. In Section 3, the R-series of X and X^* are compared. By means of the R-series, the notion of θ-continuity and other θ-mapping properties (see [2], [3]) are extended to convergence spaces.

If f is a function from a space X to a space Y, then a "natural extension" $f_* : X^* \to Y^*$ is defined in Section 4. The natural extension is unique if Y is T_2 and coincides with the continuous extension constructed in [11] when Y is compact and T_3. The main result of Section 4 is that any natural extension f_* is θ-continuous whenever f is continuous. This result is used to obtain, among other things, an alternate construction of βX for a Tychonoff topological space X. Section 5 examines conditions on f, X, and Y under which f_* is continuous, and Section 6 gives conditions under which f_* preserves certain quotient-type mapping properties, such as "open", "proper", and "perfect".

2. DECOMPOSITION SERIES.

The reader is asked to refer to [7] for convergence space notation and terminology not given here, as well as additional information about the star compactification. As in [7], space will always mean convergence space, and the abbreviation "u.f." is often used for "ultrafilter". The separation axioms T_1 (singletons are closed), T_2 (convergent filters have unique limits), and T_3 (regular plus T_2) will be used, but no separation axioms are assumed unless such is explicitly stated.

Given a space X, let F(X) (resp. U(X)) be the set of all filters (resp., ultrafilters) on X. Let X' be the set of all non-convergent members of $\mathcal{U}(X)$, and $X^* = X \cup X'$. If $A \subseteq X$, define $A' = \{ \mathfrak{F} \in X' : A \in \mathfrak{F}\}$, and $A^* = A \cup A'$. If $\mathfrak{F} \in F(X)$, and $F' \neq \phi$ for all $F \in \mathfrak{F}$, then let $\mathfrak{F}'$ be the filter generated by $\{F' : F \in \mathfrak{F}\}$; let $\mathfrak{F}^*$ be the filter generated by $\{F^* : F \in \mathfrak{F}\}$. If $\mathfrak{F}'$ exists, then $\mathfrak{F}^* = \mathfrak{F} \cap \mathfrak{F}'$; otherwise, $\mathfrak{F}^* = \mathfrak{F}$. We omit the easy proofs of the first two propositions.

PROPOSITION 2.1. The following equalities hold for any subsets A, B of X:

$$A' \cup B' = (A \cup B)' \; ; \; A' \cap B' = (A \cap B)'; \; A^* \cup B^* = (A \cup B)^* \; ; \; A^* \cap B^* = (A \cap B)^*.$$

Let X be a space, $\mathcal{G} \in F(X^*)$, and $X' \in \mathcal{G}$. Define $\mathcal{G}^\wedge$ to be the filter on X generated by $\{A \subseteq X : A' \in \mathcal{G}\}$.

PROPOSITION 2.2. (a) If $\mathfrak{F} \in F(X)$, $\mathcal{S} \in F(X^*)$, and $\mathcal{S} \geq \mathfrak{F}^*$, then $\mathcal{S}^\wedge \geq \mathfrak{F}$.

(b) If $\mathfrak{F} \in \mathcal{U}(X)$ and $\mathfrak{F}'$ exists, then $(\mathfrak{F}')^\wedge = \mathfrak{F}$.

(c) If $\mathcal{G} \in \mathcal{U}(X^*)$ and $X' \in \mathcal{G}$, then $\mathcal{G}^\wedge \in \mathcal{U}(X)$.

A convergence structure is defined on X^* as follows:
For $x \in X$, $\mathcal{G} \to x$ in X^* iff there is $\mathfrak{F} \to x$ in X such that $\mathcal{G} \geq \mathfrak{F}^*$; for $\mathcal{S} \in X'$, $\mathcal{G} \to \mathcal{S}$ iff $\mathcal{G} \geq \mathcal{S}^*$. Let i^* denote the identity embedding of X into X^*; it is proved in [11] that $\kappa^* = (X^*, i^*)$ is a compactification of X which is T_2

whenever X is T_2. It is immediate from the construction that, for any non-compact space X, X^*-X is a T_2 pretopological space; thus X^* is pretopological whenever X is pretopological. The universal property of κ^* established in [11] will be obtained in Section 3 as a corollary of a much more general result.

A subset A of space X is <u>bounded</u> if each ultrafilter containing A is convergent. X is said to be <u>locally bounded</u> if each convergent filter contains a bounded set. X is <u>essentially bounded</u> if $\mathcal{F} \in X'$ implies that the filters $\mathcal{F}$ and $\cap \{ \mathcal{G} \in X' : \mathcal{G} \neq \mathcal{F} \}$ contain disjoint sets. The next proposition is proved in [7].

PROPOSITION 2.3. (a) X is locally bounded iff X is open in X^*.

(b) X is essentially bounded iff $X^* - X$ is discrete.

We shall next consider the relationship between the closure operators of X and X^*. Let cl_X be the closure operator on a space X. For an ordinal number α, we define:

$cl_X^\circ A = A$

$cl_X^1 A = cl_X A$

$\vdots$

$cl_X^\alpha A = cl_X(cl_X^{\alpha-1} A)$ if $\alpha-1$ exists

$cl_X^\alpha A = \bigcup_{\beta<\alpha} cl_X^\beta A$, if α is a limit ordinal.

The smallest ordinal α such that $cl_X^\alpha A = cl_X^{\alpha+1} A$ for all $A \subseteq X$ is called the <u>length of the decomposition series</u> of X and denoted by $\ell_D(X)$. The relationship between $\ell_D(X)$ and $\ell_D(X^*)$ can be obtained with the help of several lemmas.

For the remainder of this section, we shall assume that X is an arbitrary space; (X^*,i^*) will always denote the star compactification of X. Let ω be the smallest infinite ordinal number.

LEMMA 2.4. If $A \subseteq X$, then $cl_{X^*}^n A = cl_X^n A \cup (cl_X^{n-1} A)'$.

PROOF. It suffices to prove this result for $n=2$. Note that $cl_X^2 A \cup (cl_X A)' \subseteq cl_{X^*}^2 A$ is obvious. If $x \in (cl_{X^*}^2 A) \cap X$, then there is $\mathcal{G} \in \mathcal{U}(X^*)$ such that $\mathcal{G} \geq \mathcal{J}^*$, and $X \in \mathcal{G}$ implies $\mathcal{G}|_X \geq \mathcal{J}$. Thus $x \in cl_X^2 A$. If $\mathfrak{F} \in cl_{X^*}^2 A \cap X'$, then it is easy to see that $cl_X A \in \mathfrak{F}$, and so $\mathfrak{F} \in (cl_X A)'$. ∎

We next describe $cl_{X^*}^n B$ for $B \subseteq X'$. For this purpose, it is necessary to introduce some additional terminology. If $B \subseteq X'$, let $\mathcal{H}_B = \cap \{ \mathfrak{F} : \mathfrak{F} \in B\}$; note that $\mathcal{H}_B \in F(X)$. Define $B^\sim = \{ \mathcal{J} \in X' : \mathcal{J} \geq \mathcal{H}_B\}$, and $B^\vee = \{x \in X : \exists\ \mathfrak{F} \in \mathcal{U}(X)$ such that $\mathfrak{F} \to x$ in X and $\mathfrak{F} \geq \mathcal{H}_B\}$. Note that $(B^\sim)^\sim = B^\sim$ and $(B^\sim)^\vee = B^\vee$.

LEMMA 2.5. If $B \subseteq X'$, then $cl_{X^*}^n B = B^\sim \cup (cl_X^{n-1} B^\vee) \cup (cl_X^{n-2} B^\vee)'$.

PROOF. For $n=1$, the statement becomes $cl_{X^*} B = B^\sim \cup B^\vee$. If $\mathcal{J} \in (cl_{X^*} B) \cap X'$, then there is $\mathcal{G} \in \mathcal{U}(X^*)$ such that $\mathcal{G} \to \mathcal{J}$ in X^* and $B \in \mathcal{G}$. Thus $\mathcal{G} \geq \mathcal{J}'$, and so $B \cap G' \neq \phi$ for all $G \in \mathcal{J}$. This implies $\mathcal{J} \geq \mathcal{H}_B$, and so $\mathcal{J} \in B^\sim$. If $x \in (cl_{X^*} B) \cap X$, then there is an u.f. $\mathcal{G}$ containing B and a filter $\mathfrak{F} \to x$ such that $\mathcal{G} \geq \mathfrak{F}'$. By Proposition 2.2, $\mathcal{G}^\wedge \geq \mathfrak{F}$, and so $\mathcal{G}^\wedge \to x$ in X. Also, $\mathcal{G}^\wedge \in \mathcal{U}(X)$ and $\mathcal{G} \geq (\mathcal{G}^\wedge)'$. Letting $\mathcal{J} = \mathcal{G}^\wedge$, we have $B \cap G' \neq \phi$ for all $G \in \mathcal{J}$. Thus $\mathcal{J} \geq \mathcal{H}_B$, and $x \in B^\vee$.

Conversely, if $x \in B^\vee$, then there is $\mathfrak{F} \in \mathcal{U}(X)$ such that $\mathfrak{F} \to x$ and $\mathfrak{F} \geq \mathcal{H}_B$. For each $F \in \mathfrak{F}$, choose $\mathcal{J}_F \in B$ such that $F \in \mathcal{J}_F$. Let $\mathcal{G}$ be the filter of sections of the net $(\mathcal{J}_F)_{F \in \mathfrak{F}}$. Then $\mathcal{G} \geq \mathfrak{F}^*$ implies $\mathcal{G} \to x$ in X^*, and so $x \in cl_{X^*} B$. A similar argument shows that $B^\sim \subseteq cl_{X^*} B$. This establishes the result for $n=1$.

If $n=2$, then $cl_{X^*}^2 B = cl_{X^*}(B^\sim \cup B^\vee) = (B^\sim)^\sim \cup (B^\sim)^\vee \cup (cl_X B^\vee)'$ follows with the help of Lemma 2.4. By the remarks preceding Lemma 2.5, $(B^\sim)^\sim = B^\sim$ and $(B^\sim)^\vee = B^\vee = B^\vee \subseteq cl_X B^\vee$. This establishes the result for $n=2$. The generalization to arbitrary n is now a trivial induction argument.

COROLLARY 2.6. If $A \subseteq X$, then $cl_{X^*} A = cl_{X^*} A^*$.

PROOF. $cl_{X^*} A^* = cl_{X^*} A \cup cl_{X^*} A'$. By Lemma 2.5, $cl_{X^*} A' = (A')^{\sim} \cup (A')^{\vee}$. It is easy to check that $(A')^{\sim} = A' \subseteq cl_{X^*} A$, and $(A')^{\vee} \subseteq cl_X A \subseteq cl_{X^*} A$. ∎

LEMMA 2.7 (a) If $A \subseteq X$ and α is a non-limit ordinal, then $cl^{\alpha}_{X^*} A = cl^{\alpha}_X A \cup (cl^{\alpha-1}_X A)'$.

(b) If α is a limit ordinal, then $cl^{\alpha}_{X^*} A = cl^{\alpha}_X A \cup (\cup\{(cl^{\beta}_X A)' : \beta < \alpha\})$.

PROOF. Using transfinite induction along with Lemma 2.4, the results follow easily from both limit and non-limit ordinals, except for the case $\alpha = \gamma + 1$, where γ is a limit ordinal. In this case we have

$$cl^{\gamma+1}_{X^*} A = cl_{X^*}(cl^{\gamma}_X A \cup B) = cl^{\gamma+1}_X A \cup (cl^{\gamma}_X A)' \cup B^{\sim} \cup B^{\vee},$$

where $B = \cup\{(cl^{\beta}_X A)' : \beta < \alpha\}$. It is not difficult to show that $B^{\vee} \subseteq cl^{\gamma+1}_X A$ and $B^{\sim} \subseteq (cl^{\gamma}_X A)'$; we omit the details. ∎

The symbol λX represents the topological modification of X, i.e., the finest topological space on X coarser than X.

THEOREM 2.8. λX is a subspace of λX^*.

PROOF. Let $X_1 = \lambda X^*|_X$. Then $\lambda X \geq X_1$ is clear. Let $A \subseteq X$ be λX-closed. Then $cl_{X^*} A = cl_X A \cup A' = A \cup A' = A^* = cl_X A^*$, by Corollary 2.6. Thus A^* is closed in λX^* and $A = A^* \cap X$, which implies A is X_1 -closed. ∎

THEOREM 2.9. (a) If $1 \leq \ell_D(X) < \omega$, then $\ell_D(X) \leq \ell_D(X^*) \leq \ell_D(X) + 1$.

(b) If $\ell_D(X) \geq \omega$, then $\ell_D(X) = \ell_D(X^*)$.

PROOF. Let $A \subseteq X^*$ and $A = B \cup C$, where $B = A \cap X$, $C = A \cap X'$. Then, by Lemmas 2.4 and 2.5, $cl^n_{X^*} A = cl^n_X B \cup (cl^{n-1}_X B)' \cup (cl^{n-1}_X C^{\vee}) \cup (cl^{n-2}_X C^{\vee}) \cup C^{\sim}$. If cl^k_X is idempotent, it follows that $cl^{k+1}_{X^*}$ must be idempotent. Thus $\ell_D(X^*) \leq \ell_D(X) + 1$. It follows easily from Lemma 2.4 that cl^k_X is idempotent if $cl^k_{X^*}$ is idempotent. Thus (a) is established.

Statement (b) follows easily from Lemma 2.6. ∎

COROLLARY 2.10. If X is a topological space, then $\ell_D(X^*) \leq 2$.

We define a G-space to be a regular space with the property that $cl_X \mathfrak{F} = \mathfrak{F}$ for all $\mathfrak{F} \in X'$; this concept (but not the terminology) was introduced by Gazik [4]. Discrete spaces and compact regular spaces are the most obvious examples of G-spaces. Another class of G-spaces are the ultraspaces, which are the topological spaces having exactly one convergent ultrafilter. The next theorem is proved in [4] in the case where X is T_2; removal of the T_2 assumption causes no problems in the proof.

THEOREM 2.11. X^* is regular iff X is a G-space.

A regular space is symmetric if $\mathfrak{F} \to x$ whenever $\mathfrak{F} \to y$ and $\dot{y} \to x$. Examples of symmetric spaces include T_3 spaces, regular topological spaces, and regular convergence groups. It is shown in [14] that a compact symmetric space has the same ultrafilter convergence as a compact regular topological space.

PROPOSITION 2.12. If X is a symmetric G-space, then X^* is symmetric.

PROOF. Let $\mathcal{G} \to \alpha$ and $\dot{\alpha} \to \beta$ in X^*. By the construction of X^*, we can conclude that both α and β are in X. Since $\mathcal{G} \to \alpha$, there is $\mathfrak{F} \to \alpha$ in X such that $\mathcal{G} \geq \mathfrak{F}^*$. Since X is symmetric, $\mathfrak{F} \to \beta$ in X, and so $\mathcal{G} \to \beta$ in X^*. ∎

THEOREM 2.13. (a) If X is a G-space, then $\ell_D(X^*) \leq 2$ and $\ell_D(X) \leq 2$.

(b) If X is a symmetric G-space, then $\ell_D(X^*) \leq 1$.

(c) If X is a T_2 topological space, then $\ell_D(X^*) \leq 1$ iff X is a G-space.

PROOF. (a) If X is a G-space, then X^* is a compact regular space by Theorem 2.11, and it follows by Theorem 2.4(a), [14], that $\ell_D(X^*) \leq 2$. The second inequality follows from the first and Theorem 2.9.

Statement (b) follows immediately from Proposition 2.12 and Theorem 2.4(b), [14].

(c) If X is a topological G-space, then X is symmetric, and so $\ell_D(X^*) \leq 1$ by (b). If $\ell_D(X^*) \leq 1$, then X^* is a compact T_2 topological space, and hence X^* is regular. By Theorem 2.11, X is a G-space. ∎

It should be noted that $\ell_D(Y) = 0$ iff Y is discrete, and consequently $\ell_D(X^*) = 0$ iff X is a finite discrete space. If X is not a finite discrete space, we can replace "$\ell_D(X^*) \leq 1$" by "$\ell_D(X^*) = 1$" in (b) and (c) of Theorem 2.13.

3. R-SERIES.

In this section, we summarize some results on the R-series of a space (originally studied in [13] and [14]), obtain a few results on the R-series of X^*, and lay the groundwork for many of the results of Section 4.

Starting with a space X, an ordinal family of spaces $\{r_\alpha X\}$ is defined on the same underlying set as follows: $r_0(X) = X$

$\mathcal{F} \to x$ in $r_1 X$ iff there exist $n \in N$ and $\mathcal{G} \to x$ in X such that $\mathcal{F} \geq cl_X^n \mathcal{G}$.

$\mathcal{F} \to x$ in $r_\alpha X$ iff there exist $n \in N$, $\mathcal{G} \to x$ in X and $\beta < \alpha$ such that $\mathcal{F} \geq cl_{r_\beta X}^n \mathcal{G}$.

The family $\{r_\alpha X\}$ is called the <u>R-series</u> of X. If γ is the least ordinal such that $r_\gamma X = r_{\gamma+1} X$, then $r_\gamma X$ is denoted X_r, and γ is called the <u>length of the R-series</u> of X and denoted by $\ell_R(X)$. Note that X is regular iff $\ell_R(X) = 0$. It is shown in [13] that X_r is the finest regular space coarser than X, and X_r is called the <u>regular modification</u> of X.

Of course, X_r will not in general be T_2, even when X is T_2. A T_3 space associated with X (we use T_3 to mean regular plus T_1) is constructed as follows. First, define an equivalence relation among the elements of X by : $x \sim y$ iff $\dot{x} \to y$ in X_r. Let X_s be the set of equivalence classes with the quotient convergence structure derived from X_r.

If $f : X \to Y$, let $\bar{f} : X_s \to Y_s$ be the (unique) function which makes the following diagram commute:

$$\begin{array}{ccccc} X & \longrightarrow & X_r & \xrightarrow{\varphi_X} & X_s \\ \downarrow f & & \downarrow f & & \downarrow \bar{f} \\ Y & \longrightarrow & Y_r & \xrightarrow{\varphi_Y} & Y_s \end{array}$$

where the maps from X to X_r and Y to Y_r are the respective identity maps, and φ_X, φ_Y are the respective quotient maps.

Let σX denote the symmetric modification of X, i.e., the finest symmetric space coarser than X. The next proposition follows immediately from results of [13] and [14].

PROPOSITION 3.1. If $f : X \to Y$ is continuous, then each map in the following commutative diagram (in which all non-labeled vertical maps are f and all non-labeled horizontal maps are identities) is continuous.

$$\begin{array}{ccccccccccccc}
X & \to & R_1 X & \to & \cdots & \to & r_\alpha X & \to & \cdots & \to & X_r & \to & \sigma X & \xrightarrow{\varphi_X} & X_s \\
\downarrow & & \downarrow & & & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow \bar{f} \\
y & \to & r_1 Y & \to & \cdots & \to & r_\alpha Y & \to & \cdots & \to & Y_r & \to & \sigma Y & \xrightarrow{\varphi_Y} & Y_s
\end{array}$$

PROPOSITION 3.2. For any space X, $r_1 X$ is a subspace of $r_1 X^*$.

PROOF. Let X_1 be the restriction of $r_1 X^*$ to X. Since $i^* : X \to X^*$ is continuous, it follows from Proposition 3.1 that $i^* : r_1 X \to r_1 X^*$ is continuous, and thus $r_1 X \geq X_1$. On the other hand, let $\mathfrak{F} \to x$ in X_1. Then there is $\mathcal{G} \to x$ in X^* and $n \in N$ such that $\mathfrak{F} \geq \mathrm{cl}^n_{X^*} \mathcal{G}$. Since $\mathcal{G} \to x$ in X^*, there is $\mathcal{H} \to x$ in X such that $\mathcal{G} \geq \mathcal{H}^*$, and so we have $\mathfrak{F} \geq \mathrm{cl}^n_{X^*} \mathcal{H}^* = \mathrm{cl}^n_{X^*} \mathcal{H} = \mathrm{cl}^n_X \mathcal{H} \cap (\mathrm{cl}^{n-1}_X \mathcal{H}')$, by Lemma 2.4 and Corollary 2.6. Since $\mathfrak{F} \in F(X)$, $\mathfrak{F} \geq \mathrm{cl}^n_X \mathcal{H}$, and so $\mathfrak{F} \to x$ in $r_1 X$. ∎

PROPOSITION 3.3. If X is a locally compact T_3 space, then $r_2 X$ is a subspace of $r_2 X^*$.

PROOF. It is sufficient to show that for any $A \subseteq X$, $\mathrm{cl}^n_{r_1 X} A = (\mathrm{cl}^n_{r_1 X^*} A) \cap X$ for all $n \in N$, and this will be proved by induction. For $n = 1$, the equality follows by Proposition 3.2. If the equality holds for n and $x \in (\mathrm{cl}^{n+1}_{r_1 X^*} A) \cap X$, then there is $\mathfrak{F} \to x$ and $k \in N$ such that $(\mathrm{cl}^k_{X^*} F) \cap (\mathrm{cl}^n_{r_1 X^*} A) \neq \phi$ for all $F \in \mathfrak{F}$. Since X is locally compact and T_3, we may assume without loss of generality that $\mathfrak{F}$ has a filter base of compact sets, and so $cl^k_{X^*} \mathfrak{F} = cl^k_X \mathfrak{F}$. From this observation, along with the induction hypothesis, we may conclude that $x \in cl^{n+1}_{r_1 X} A$. ∎

It is not true in general that $r_\alpha X$ is a subspace of $r_\alpha X^*$. To establish this fact, we need to make use of some theorems from [9].

A space X is defined to be <u>completely regular</u> if it is a subspace of a symmetric compact space. It is shown in [9] that X is completely regular iff it is a symmetric space with the same ultrafilter convergence as a completely regular topological space. Let ωX denote the finest completely regular space coarser than X . A space X is defined to be <u>ω-regular</u> if $\mathfrak{F} \to x$ implies $cl_{\omega X} \mathfrak{F} \to x$. It is proved in [9] that X is ω-regular iff X is a subspace of a compact regular space. The ω-regular spaces include the completely regular spaces and also the c-embedded spaces of Binz [1].

PROPOSITION 3.4. (a) If X is a regular space which is not ω-regular, then X_r is not a subspace of X^*_r , and $\ell_R(X^*) \geq 2$.

(b) If X is a locally compact T_3 space which is not ω-regular, then $\ell_R(X^*) \geq 3$.

PROOF. The first part of (a) follows from the aforementioned characterization of ω-regular spaces as subspaces of compact regular spaces. The two statements concerning $\ell_R(X^*)$ follow by Proposition 3.2 and 3.3, respectively. ∎

For any space X, let C(X) be the set of all continuous real-valued functions on X. A T_3 topological space X for which C(X) consists only of constant functions is an example of a regular space which is not ω-regular; for this space, $\ell_R(X) = 0$ and $\ell_R(X^*) \geq 2$. An example of a regular space X for which $\ell_R(X^*) \geq 3$ is obtained with the help of the following lemma.

LEMMA 3.5. A locally compact T_3 space X is ω-regular iff C(X) separates points in X.

PROOF. If $X = \omega X$, then ωX is T_2 and so C(X) separates points in X. Conversely, if $\mathfrak{F} \to x$ in X, then $\mathfrak{F}$ contains a compact set A. Since C(X) separates points in X, ωX is T_2 and so the subspaces $X|_A$ and $\omega X|_A$ have the same ultrafilter convergence. It follows that $cl_X \mathfrak{F} = cl_{\omega X} \mathfrak{F} \to x$ in X, and therefore X is ω-regular. ∎

EXAMPLE 3.6. Let X be the set [0, 1]. If s is a sequence on the set X, let $\mathfrak{F}_s$ denote the filter generated by s in the usual way. Let $\mathfrak{F}$ be the filter generated by the sequence $(\frac{1}{n})$. Define a convergence on X as follows:
(1) If $x \notin \{0, 1\}$, then $\mathfrak{F} \to x$ iff there is a sequence s converging to x in the usual topology such that $\mathfrak{F} \geq \mathfrak{F}_s$; (2) $\mathfrak{F} \to 0$ iff there is a sequence s converging to 0 in the usual topology, but not a subsequence of $(\frac{1}{n})$, such that $\mathfrak{F} \geq \mathfrak{F}_s$; (3) $\mathfrak{F} \to 1$ iff $\mathfrak{F} \geq \mathfrak{F}_s$, where s is a sequence converging to 1 in the usual topology, or else $\mathfrak{F} \geq \mathfrak{F} \cap \dot{1}$. One may easily verify that the space X is locally compact and T_3, but C(X) will not separate the points 0 and 1. Thus, by Lemma 3.5, X is not ω-regular, and it follows from Proposition 3.4 that $\ell_R(X^*) \geq 3$, whereas $\ell_R(X) = 0$; this result contrasts with the conclusion of Theorem 2.9. One can also show (we omit the details) that $\dot{0} \to 1$ in $r_3 X^*$. Since $r_3 X = X$, it follows that $r_3 X$ is not a subspace of $r_3 X^*$. This shows that the conclusion of Proposition 3.3 cannot be improved without imposing additional conditions. ∎

Gazik showed in [4] that a T_3 pretopological G-space is a completely regular topological space. Another result along these lines is

PROPOSITION 3.7. (a) A symmetric G-space is completely regular.
(b) Every G-space is ω-regular.

PROOF. These statements follow immediately from Theorem 2.11, Proposition 2.12, and the characterization of ω-regular spaces obtained in [9]. ∎

THEOREM 3.8. Let X be a space.

(a) X is regular iff X is a subspace of $r_1 X^*$.

(b) X is ω-regular iff X is a subspace of X^*_r.

(c) X is completely regular iff X is a subspace of σX^*.

Proof. Statement (a) follows immediately from Proposition 3.2. Statements (b) and (c) are proved in [9]. ∎

For topological spaces X and Y, function $f : X \to Y$ is defined to be <u>θ-continuous</u> if, for every $x \in X$ and every neighborhood V of f(x), there is a neighborhood $\mathcal{U}$ of f(x) such that $f(cl_X \mathcal{U}) \subseteq cl_Y V$.

PROPOSITION 3.9. Let $f : X \to Y$, where X and Y are topological spaces. Then $f : X \to Y$ is θ-continuous iff $f : r_1 X \to r_1 Y$ is continuous.

PROOF. Let $f : X \to Y$ be θ-continuous, and let $\mathcal{F} \to x$ in $r_1 X$. Then there is $\mathcal{G} \to x$ in X such that $\mathcal{F} \geq cl_X \mathcal{G} \geq cl_X \mathcal{V}(x)$, where $\mathcal{V}(x)$ is the neighborhood filter at x. By θ-continuity, $f(cl_X \mathcal{V}(x) \geq cl_Y \mathcal{V}(f(x))$, and so $f(\mathcal{F}) \geq cl_Y \mathcal{V}(f(x))$. The latter filter $r_1 Y$-converges to f(x), and so $f : r_1 X \to r_1 Y$ is continuous. For the converse argument, one easily see that $f : r_1 X \to r_1 Y$ implies $f(cl_X \mathcal{V}(x)) \geq cl_Y \mathcal{V}(f(x))$ for all $x \in X$, which is equivalent to θ-continuity of $f : X \to Y$. ∎

The characterization of θ-continuity given in Proposition 3.9 is not suitable for a purely topological investigation, since $r_1 X$ may fail to be topological even when X is topological. Perhaps this suggests that convergence spaces are the natural realm for the study of θ-continuity. But in any event, we shall define a function $f : X \to Y$ between arbitrary convergence spaces to be <u>θ-continuous</u> if $f : r_1 X \to r_1 Y$ is continuous.

More generally, if P is any function property, then $f : X \to Y$ is defined to have property <u>θ -P</u> if $f : r_1 X \to r_1 Y$ has property P. Thus, one can speak of θ-open maps, θ-quotient maps, etc. Some of these "θ-properties" will be studied briefly in Section 6.

4. NATURAL EXTENSIONS.

Let X and Y be spaces and consider a function $f : X \to Y$. A function $f_* : X^* \to Y^*$ is called a natural extension of f if the following conditions are satisfied:

(1) $f_* |_X = f$.

(2) If $\mathfrak{F} \in X'$ and $f(\mathfrak{F})$ is convergent in Y, then $f_*(\mathfrak{F})$ is an element of Y to which $f(\mathfrak{F})$ converges.

(3) If $\mathfrak{F} \in X'$ and $f(\mathfrak{F}) \in Y'$, then $f_*(\mathfrak{F}) = f(\mathfrak{F}) \in Y'$.

If $\mathfrak{F} \in X'$ implies $f(\mathfrak{F}) \in Y'$, then f is said to be weakly proper. If $f : X \to Y$ is a weakly proper function, or if Y is T_2, then the natural extension f_* is unique; in general, f may have many natural extensions. In the proposition and theorem that follow, when $f : X \to Y$, f_* will be assumed to be an arbitrary natural extension of f.

The proof of the next lemma is straightforward and will be omitted.

LEMMA 4.1. If $f : X \to Y$ and $A \subseteq X$, then :

(a) If f is continuous, then $(f(A))^* \subseteq f_*(A^*) \subseteq f_*(cl_{X^*} A) \subseteq cl_{Y^*} f(A)$;

(b) If f is continuous and Y is T_2, then $f_*(cl_{X^*} A) = cl_{Y^*} f(A)$;

(c) If f is weakly proper, then $f_*(A^*) \subseteq (f(A))^*$.

THEOREM 4.2. If $f : X \to Y$ is continuous, then $f_* : X^* \to Y^*$ is θ-continuous.

PROOF. It is sufficient to show that, for each $\mathfrak{F} \in F(X)$, $f_*(cl^n_{X^*} \mathfrak{F}^*) \geq cl^n_{Y^*} f(\mathfrak{F})$. If $F \in \mathfrak{F}$, then $cl^n_{X^*} F = cl^n_{X^*} F^* = cl^n_X F \cup (cl^{n-1}_X F)'$ by Lemma 2.4 and Corollary 2.6. By continuity of f, $f_*(cl^n_X F) = f(cl^n_X F) \subseteq cl^n_Y f(F)$, and $f_*(cl^{n-1}_X F)^* \subseteq cl_{Y^*} f(cl^{n-1}_X F) \subseteq cl_{Y^*} (cl^{n-1}_Y f(F)) \subseteq cl^n_{Y^*} f(F)$ follows with the help of Lemma 4.1. Thus $f_*(cl^n_{X^*} F^*) \subseteq cl^n_{Y^*} f(F)$, and the theorem is proved. ∎

COROLLARY 4.3. If $f : X \to Y$ is continuous, then each map in the following commutative diagram (in which all non-labeled vertical maps are f_* and all non-labeled horizontal maps are identities) is continuous.

$$\begin{array}{ccccccccccc} r_1 X^* & \to \cdots \to & r_\alpha X^* & \to \cdots \to & X^*r & \to & \sigma X^* & \xrightarrow{\varphi_{X^*}} & X^* s \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow (f_*)^- \\ r_1 Y^* & \to \cdots \to & r_\alpha Y^* & \to \cdots \to & Y^*r & \to & \sigma Y^* & \xrightarrow[\varphi_{Y^*}]{} & Y^* s \end{array}$$

The next result closely resembles, but is more general than, the extension property of the star compactification obtained in [11].

COROLLARY 4.4. If $f : X \to Y$ is continuous and Y is compact and regular, then $f_* : X^* \to Y$ is continuous. If Y is also T_2, then the extension f_* is unique.

PROOF. Under the given assumptions, $Y = Y^* = r_1 Y^*$; since $X^* \geq r_1 X^*$, the first statement is established. The second follows from an earlier remark. ∎

In the next section we shall see that continuity of $f : X \to Y$ does not guarantee the continuity of $f_* : X^* \to Y^*$. If X is a regular space, let $X^\sim = r_1 X^*$; then by Proposition 3.2 $\kappa^\sim = (X^\sim, i^*)$ is a compactification of X. Our study of the compactification $\kappa^\sim$ will be limited to the following proposition.

PROPOSITION 4.5. Let X and Y be regular spaces.

(a) If $f : X \to Y$ is continuous, then $f_* : X^\sim \to Y^\sim$ is continuous.

(b) $X^\sim$ is T_2 iff X is a T_2 G-space.

PROOF. Statement (a) follows immediately from Theorem 4.2.

(b) If X is a T_2 G-space, then X^* is regular by Proposition 2.10, and so $X^\sim = X^*$ is T_2. If X is not a G-space, then there is $\mathcal{F} \in X'$ such that $\mathcal{G} > cl_X \mathcal{F}$, where $\mathcal{G} \in \mathcal{U}(X)$ and $\mathcal{G} \neq \mathcal{F}$. If $\mathcal{G} \xrightarrow{X} x$, then $\dot{x} \to \mathcal{F}$ in $X^\sim$. If $\mathcal{G} \in X'$, then the filter $\mathcal{G}_1$ on X^* generated by $\mathcal{G}$ converges in X^* to both $\mathcal{G}$ and $\mathcal{F}$. ∎

It is shown in [12] that every completely regular T_2 space has a Stone-Čech compactification. This compactification is regular and T_2, has the universal property relative to the class of completely regular spaces, and agrees with the

COROLLARY 4.7. If X is a completely regular T_2 space, then (X^*_s, i_*^-) is the Stone-Čech compactification of X.

If X is a Tychonoff topological space, then Corollary 4.7 gives a new method for constructing βX. Indeed, βX is in this case the pretopological modification of X^*_s.

5. CONTINUITY OF NATURAL EXTENSIONS.

We next consider conditions under which a natural extension f_* of a continuous function f is continuous. For this purpose, we use some additional notation and terminology.

Let $f: X \to Y$ be a continuous function, and let $f_*: X^* \to Y^*$ be a natural extension of f. For $A \subseteq X$, define $A'_f = \{\mathfrak{F} \in A' : f(\mathfrak{F})$ converges in $Y\}$. Let $\Gamma_{f_*}(A) = f(A) \cup f_*(A'_f)$; note that $\Gamma_{f_*}(A) = f_*(A^*) \cap Y$. If $\mathfrak{F} \in F(X)$ define $\Gamma_{f_*}(\mathfrak{F}) \in F(Y)$ to be the filter generated by $\{\Gamma_{f_*}(F) : F \in \mathfrak{F}\}$; $\mathfrak{F} \in F(X)$ is said to be f_*-closed if $\Gamma_{f_*}(\mathfrak{F}) = f(\mathfrak{F})$.

PROPOSITION 5.1. Let f be a continuous map.

(1) f_* is continuous at $x \in f^{-1}(Y)$ iff $\mathfrak{F} \to x$ in X implies that $\Gamma_{f_*} \mathfrak{F} \to f(x)$ in Y.

(2) f_* is continuous at $\mathfrak{F} \in f_*^{-1}(Y) \cap X$ iff $\Gamma_{f_*}(\mathfrak{F}) \to f_* \mathfrak{F}$ in Y.

(3) f_* is continuous at $\mathfrak{F} \in f_*^{-1}(Y')$ iff $\mathfrak{F}$ is f_*-closed.

PROOF. If $\mathfrak{F} \in F(X)$ then one can easily show that $\Gamma_{f_*} \mathfrak{F} \geq (\Gamma_{f_*}(\mathfrak{F}))^*$; (1) and (2) follow from these inequalities.

(3) If f_* is continuous at $\mathfrak{F} \in f_*^{-1}(Y')$, then $\Gamma_{f_*} \mathfrak{F} \geq f_*(\mathfrak{F}^*) \geq (f(\mathfrak{F}))^*$, and hence $\Gamma_{f_*} \mathfrak{F} = f(\mathfrak{F})$. Conversely, $f_*(\mathfrak{F}^*) \geq (\Gamma_f \mathfrak{F})^* = (f\mathfrak{F})^* \to f_* \mathfrak{F}$ in Y^*, and thus f_* is continuous at $\mathfrak{F} \in f_*^{-1}(Y')$. ∎

COROLLARY 5.2. Let $f: X \to Y$ be continuous, and Y a regular space. Then

(1) f_* is continuous at all points of $f_*^{-1}(Y)$.

(2) f_* is continuous iff each $\mathfrak{F} \in f_*^{-1}(Y')$ is f_*-closed.

topological Stone-Čech compactification relative to ultrafilter convergence when X is topological. We shall now give an alternate construction of this compactification using κ^*.

For any space X, X^*_s is a compact, regular, T_2 space. However it is not generally true that X_s is a subspace of X^*_s. Recall the notation σX for the symmetric modification of X.

THEOREM 4.6. If X is a space such that σX is a subspace of σX^*, then σX is completely regular, X_s is completely regular and T_2, and $(X^*_s, \bar{i}_*)$ is the Stone-Čech compactification of X_s.

PROOF. By assumption, σX is a subspace of a compact symmetric space, and hence completely regular. X_s is T_2 by construction. In the diagram that follows

$$\begin{array}{ccc} \sigma X & \xrightarrow{i_*} & \sigma X^* \\ \varphi_X \downarrow & & \downarrow \varphi_{X^*} \\ X_s & \xrightarrow{\bar{i}_*} & X^*_s \end{array}$$

the maps φ_X and φ_{X^*} are strongly open (see Proposition 2.2, [14]). This means that if $\mathcal{G} \to \alpha$ in X_s and $x \in \varphi_X^{-1}(\alpha)$, then there is a filter $\mathfrak{F}$ on σX such that $\mathfrak{F} \to x$ in σX and $\varphi_X(\mathfrak{F}) = \mathcal{G}$. Using this property and the fact that σX and σX^* are symmetric, one can easily show that X_s is densely embedded in X^*_s.

If Y is a regular, compact, T_2 space and $f : X_s \to Y$ is continuous, then define $F : X \to Y$ by $F(x) = f([x])$, where $[x]$ is the equivalence class in X_s defined by x. It is easy to check that $F : X \to Y$ is continuous, and so by Corollary 4.3, $(F_*)^- : X^*_s \to Y^*_s = Y$ is continuous. $(F_*)^-$ is clearly an extension of f, and this extension is unique because Y is Hausdorff. Thus by the uniqueness of the Stone-Čech compactification established in [12], it follows that this compactification is equivalent to (X^*_s, i_*^-). ∎

(3) If X is essentially bounded, then f_* is continuous.

PROOF. Statements (1) and (2) follow immediately from Proposition 5.1 and the fact that Y is regular. The assumption that X is essentially bounded (see Section 2 for this definition) guarantees that each $\mathfrak{F} \in f_*^{-1}(Y')$ is f_*-closed. ∎

PROPOSITION 5.3. If $f : X \to Y$ is continuous and weakly proper, then f_* is continuous.

PROOF. If f is weakly proper and $A \subseteq X$, then $A'_f = \phi$. Thus each filter $\mathfrak{F} \in F(X)$ is f_*-closed. The conditions (1) and (3) of Proposition 5.1 for continuity of f_* are thus satisfied, while condition (2) is satisfied vacuously. ∎

COROLLARY 5.4. If X is a closed subspace of Y and $f : X \to Y$ is the identity embedding, then $f_* : X^* \to Y^*$ is also an embedding.

PROOF. Since X is closed in Y, f is weakly proper; thus f_* is continuous by Proposition 5.3. f_* is clearly one-to-one, and by Lemma 4.1, $f_*(\mathfrak{F}^*) = f(\mathfrak{F})^*$ for all $\mathfrak{F} \in F(X)$. From this equality, it follows easily that f_* is an embedding. ∎

PROPOSITION 5.5. The following statements about a regular space Y are equivalent.

(a) Y is a G-space.

(b) Every natural extension of every continuous function into Y is continuous.

(c) If Z and Y have the same set, Z is discrete, and $f : Z \to Y$ is the identity, then f_* is continuous.

PROOF. (a) ⇒ (b). If $f : X \to Y$ is continuous and Y is a G-space, then Y^* is regular by Theorem 2.11 and so $f_* : X^* \to Y^*$ is continuous by Corollary 4.4.

(b) ⇒ (c). Obvious.

(c) ⇒ (a). If Y is not a G-space, then there is $\mathfrak{F} \in Y'$ such that $cl_Y \mathfrak{F} \neq \mathfrak{F}$. Since Y is T_1 and Z is discrete, it follows that $\Gamma_{f_*} \mathfrak{F} \neq \mathfrak{F}$ for some natural extension f_*. Thus $\mathfrak{F}$ is not f_*-closed, and by Proposition 5.1 f_* is not continuous. ∎

The final result in this section is analogous to Proposition 5.1, but involves θ-continuity rather than continuity. Since this result is of marginal interest, we shall omit the proof.

PROPOSITION 5.6. Let $f : X \to Y$ be a θ-continuous map.

(a) f_* is θ-continuous at each point $x \in X$.

(b) f_* is θ-continuous at $\mathfrak{F} \in f_*^{-1}(Y) \cap X'$ iff $f(cl^n_X \mathfrak{F}) \to f_* \mathfrak{F}$ in $r_1 Y$ for each $n \geq 1$.

(c) f_* is θ-continuous at $\mathfrak{F} \in f_*^{-1}(Y')$ iff, for each $n \geq 1$, there is $m \geq 1$ such that $f(cl^n_X \mathfrak{F}) \geq cl^m_Y f(\mathfrak{F})$.

6. QUOTIENT EXTENSIONS.

In this concluding section, we shall consider the circumstances under which f_* will possess certain quotient-type properties. We begin with definitions of the properties to be considered.

The term map will be used to mean a continuous, onto function. Note that if $f : X \to Y$ is onto, then any natural extension $f_* : X^* \to Y^*$ is also onto.

1. f is proper if f is a map and, whenever $\mathcal{G} \to y$ in Y, and $\mathfrak{F}$ is an u.f. on X such that $f(\mathfrak{F}) = \mathcal{G}$, then there is $x \in f^{-1}(y)$ such that $\mathfrak{F} \to x$.

2. f is a convergence quotient map if f is a map and, whenever $\mathcal{G} \to y$ in Y, there is $x \in f^{-1}(y)$ and $\mathfrak{F} \to x$ in X such that $f(\mathfrak{F}) = \mathcal{G}$.

3. f is perfect if f is a proper convergence quotient map.

4. f is open if f is a map and whenever $\mathcal{G}$ is an u.f. on Y which converges to y, and $x \in f^{-1}(y)$, then there is an u.f. $\mathfrak{F} \to x$ such that $f(\mathfrak{F}) = \mathcal{G}$.

5. f is closure-preserving if $A \subseteq X$ implies $f(cl_X A) = cl_Y f(A)$.

Further information about these properties may be found in [6], [8], and [10]. Recall that if P represents any of the above properties, then $f : X \to Y$ is said to have property θ-P if $f : r_1 X \to r_1 Y$ has property P.

It is clear that a proper map is weakly proper, and that a weakly proper map onto a T_2 space is proper.

In all of the propositions that follow, f denotes a function from a space X into a space Y, and $f_* : X^* \rightarrow Y^*$ denotes a natural extension of f.

PROPOSITION 6.1. If f is a proper map, then f_* is also a proper map.

PROOF. Suppose that $\mathcal{G}$ is an u.f. on X^* such that $f_*(\mathcal{G}) \rightarrow \alpha$ in Y^*. Then there exists an u.f. $\mathcal{K}$ on Y such that $f_*(\mathcal{G}) \geq \mathcal{K}^*$, where $\mathcal{K}^* \rightarrow \alpha$ in Y^*. Suppose that $\mathcal{G} \rightarrow \beta$ in X^*; then there exists a filter $\mathcal{S}$ on X such that $\mathcal{S} \rightarrow \beta$ in X^* and $\mathcal{G} \geq \mathcal{S}^*$. Hence $f_*(\mathcal{G}) \geq f_*(\mathcal{S}^*) = f(\mathcal{S})^*$ since f is a proper map, and thus $f(\mathcal{S})^*$ and $\mathcal{K}^*$ are not disjoint filters on Y^*. This implies that $f(\mathcal{S})$ and $\mathcal{K}$ are not disjoint filters on Y; consequently, $f(\mathcal{S}) = \mathcal{K}$.

If $\alpha \notin Y$, then it follows that $f_*(\beta) = \alpha$. If $\alpha \in Y$, then since f is a proper map, $\mathcal{S} \rightarrow x$ in X for some $x \in f^{-1}(\alpha)$, and thus $\mathcal{G} \rightarrow x$ in X^*. It follows that f_* is a proper map. ∎

PROPOSITION 6.2. If f is a convergence quotient map, then f_* is a convergence quotient map iff f_* is continuous.

PROOF. The relation $f(A)^* \subseteq f_*(A^*)$ is satisfied for each subset A of X, and hence $f_*(\mathcal{F}^*) \leq f(\mathcal{F})^*$ for each filter $\mathcal{F}$ on X. Suppose that $\mathcal{S}^* \rightarrow \alpha$ in Y^* and $\alpha \notin Y$; let $\mathcal{H}$ be any u.f. on X such that $f(\mathcal{H}) = \mathcal{S}$. Then $f_*(\mathcal{H}) = \alpha$ and $f_*(\mathcal{H}^*) \leq \mathcal{S}^*$. If $\alpha \in Y$, then there exists $\mathcal{F} \in F(X)$ and $x \in f^{-1}(\alpha)$ such that $\mathcal{F} \rightarrow x$ in X and $f(\mathcal{F}) = \mathcal{S}$, since f is a convergence quotient map. Since $f_*(\mathcal{F}^*) \leq \mathcal{S}^*$, it follows that f_* is a convergence quotient map precisely when f_* is a continuous map. ∎

COROLLARY 6.3. f_* is a perfect map whenever f is a perfect map.

PROPOSITION 6.4. If f is an open, proper map, then f_* is open, proper, θ-open, and θ-proper.

PROOF. It follows by Proposition 6.1 that f_* is proper. If f_* is also open, then it follows from Theorem 4.2, [14], that f_* is also σ-open and σ-proper. Thus it remains only to show that f_* is open.

Let $\mathcal{G} \in \mathcal{U}(Y^*)$ such that $\mathcal{G} \to \alpha$ in Y^*, and let $\beta \in f_*^{-1}(\alpha)$. Then there is $\mathcal{S} \in \mathcal{U}(Y)$ such that $\mathcal{G} \geq \mathcal{S}^*$ and $\mathcal{S}^* \to \alpha$ in Y^*. If $\alpha \notin Y$, then $\beta \notin X$, and hence there exists exactly one u.f. $\mathcal{F}$ on X such that $\mathcal{F} \to \beta$ in X^*. Thus $f(\mathcal{F}) = \mathcal{S}$, and since f is a proper map, $f_*(\mathcal{F}^*) = \mathcal{S}^* \leq \mathcal{G}$. Let $\mathcal{B} \in \mathcal{U}(X^*)$ contain both $\mathcal{F}^*$ and $f_*^{-1}(\mathcal{G})$; then $f_*(\mathcal{B}) = \mathcal{G}$, and $\mathcal{B} \to \beta$ in X^*.

If $\alpha \in Y$ then, since f is a proper map, $\beta \in X$, and since f is an open map, there is $\mathcal{F} \in \mathcal{U}(X)$ such that $\mathcal{F} \to \beta$ in X and $f(\mathcal{F}) = \mathcal{S}$. Then, as in the argument of the preceding paragraph, there exists an u.f. $\mathcal{B} \to \beta$ in X^* such that $f_*(\mathcal{B}) = \mathcal{G}$ this establishes that f_* is an open map. ∎

We omit the straightforward proof the next proposition.

PROPOSITION 6.5. If f is perfect θ-proper map, then f_* is a θ-perfect map.

Propositions 6.4 and 6.5 yield the following corollary.

COROLLARY 6.6. If f is an open perfect map, then f_* is a θ-perfect map.

PROPOSITION 6.7. If f is a convergent quotient map which is closure preserving, then f_* is a θ-convergence quotient map.

PROOF. Suppose that $\mathcal{G} \to \alpha$ in r_1Y^*. Then there is $\mathcal{S} \in F(Y)$ such that $\mathcal{S}^* \to \alpha$ and $\mathcal{G} \geq cl^n_{Y^*} \mathcal{S}$. If $\alpha \notin Y$, then it may be assumed that $\alpha = \mathcal{S} \in Y'$. Let $\mathcal{F} \in \mathcal{U}(X)$ such that $f(\mathcal{F}) = \mathcal{S}$; then $\beta = \mathcal{F} \in X'$ and $f_*(\beta) = \alpha$. By Lemma 2.4 and the assumption that f is closure-preserving, it follows that $f_*(cl^{n+1}_{X^*} \mathcal{F}^*) \leq f_*((cl^n_X \mathcal{F})^*) \leq cl_{Y^*} f(cl^n_X \mathcal{F}) = cl_{Y^*} cl^n_Y f(\mathcal{F}) \leq cl^n_{Y^*} \mathcal{S}^*$. If $\alpha \in Y$ then, since f is a convergence quotient map, there is $\mathcal{F} \to \beta \in f^{-1}(\alpha)$ such that $f(\mathcal{F}) = \mathcal{S}$. Again, $f_*(cl^{n+1}_{X^*} \mathcal{F}) \leq cl^n_{Y^*} \mathcal{S}^*$. In both cases $cl^{n+1}_{X^*} \mathcal{F}^* \to \beta$ in r_1X^*. If $\mathcal{K} = f_*^{-1}(\mathcal{G}) \vee cl^{n+1}_{X^*} \mathcal{F}$, then $\mathcal{K} \to \beta$ in r_1X^* and $f_*(\mathcal{K}) = \mathcal{G}$. Thus f_* is a θ-convergence quotient map. ∎

The final proposition follows immediately from Proposition 5.6.

PROPOSITION 6.8. If f is θ-continuous and closure preserving, then f_* is θ-continuous.

We conclude by citing, without detail, some examples which place limitations on the types of results obtained in this section. The function f constructed in Example 4.3 of [14] is perfect but not θ-proper; it is also not difficult to show that in this case f_* is not θ-proper. Thus, in Corollary 6.6, one cannot drop the assumption that f is open. There are other examples which show that f_* may fail to be continuous when f is an open, convergence quotient map, and that f_* may fail to be open when f is open and f_* is continuous.

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