QUALITATIVE PROPERTIES OF SOLUTIONS OF CERTAIN FOURTH ORDER LINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT.   This work considers differential equations of the form

$$(py'')'' + \underline{qy}'' + ry = 0$$

where p,q and r are positive continuous functions defined on $[0,\infty)$. The main

concentration is on the oscillatory and asymptotic behavior of the solutions.

Such an investigation is important because the above equation often arises in the

study of mechanical vibrations.

KEY WORDS AND PHRASES. *Oscillatory solutions, nonoscillatory solutions, strong
oscillation, oscillation numbers.*

1980 MATHEMATICS SUBJECT CLASSIFICATION CODE.   *34C10.*

1.   INTRODUCTION.

In this paper we are concerned with the asymptotic and oscillation proper-
ties of solutions of the differential equation

$$(py'')'' + \underline{qy}'' + ry = 0. \tag{1.1}$$

The following assumptions will be made:

(H_0) p, q, r are continuous real-valued functions defined on $[0,\infty)$.

(H_1) $p(x) > 0$, $\int_0^\infty \frac{dx}{p(x)} = \infty$, $r(x) > 0$, $q(x) \geq 0$.

(H_2) For each solution y of (1), $Q(y'',y) = py''^2 + qy''y + ry^2 \geq 0$.

The adjoint equation of (1.1) is

$$(pz'' + qz)'' + rz = 0. \tag{1.2}$$

Note that if q is a constant, then (1.1) is selfadjoint. A nontrivial solution

of (1.1){(1.2)}is termed <u>oscillatory</u> if it changes signs for arbitrarily large

values of x. We also say equation (1.1) {(1.2)} is oscillatory whenever it has

an oscillatory solution. <u>Nonoscillatory</u> solutions of (1.1) {(1.2)} are those

which fail to oscillate.

If it happens that all solutions of (1.1) {(1.2)} are oscillatory we say that

(1.1) {(1.2)} is <u>strongly</u> oscillatory. If no solution oscillates then (1.1) {(1.2)}

is termed <u>nonoscillatory</u>. In either of these cases the solutions of (1.1) {(1.2)}

are said to have the <u>same oscillatory character</u>. Finally, we say that (1.1) {(1.2)}

is (2,2)-disconjugate if no nontrivial solution satisfies $y(a) = y'(a) = 0 = y(b) = y'(b)$ for some a and b on $[0,\infty)$.

Some special cases of (1.1) have been studied in detail. In particular, the

selfadjoint equation

$$y^{iv} + r(x)y = 0, \tag{1.3}$$

has been studied extensively by Svec [1,2]. Svec examined the asymptotic pro-

perties of certain solutions and proved that all the solutions of (1.3) have the

same oscillatory character. Leighton and Nehari [3] in their fundamental work

concerning the more general selfadjoint equation

$$(p(x)y'')'' + r(x)y = 0, \tag{1.4}$$

obtained oscillation and nonoscillation criteria for solutions of (1.4).

Schneider [4] proved that solutions of a (2,2)-disconjugate selfadjoint fourth

order equation must all have the same oscillatory character. That the condition

of selfadjointness is necessary can be seen by studying the following example:

EXAMPLE. The (2,2)-disconjugate nonselfadjoint Euler differential equation

$$y^{iv} + \frac{1}{x^2}y'' + \frac{1}{x^4}y = 0 \tag{1.5}$$

has both oscillatory and nonoscillatory solutions, since the characteristic equation

for (1.5), $r^4 - 6r^3 + 12r^2 - 7r + 1 = 0$ has exactly two real roots.

We also refer to some recent related work of Kreith [5].

2. <u>PRELIMINARY RESULTS</u>.

Consider the following functional defined on the solution space of (1.1):

$$F[y(x)] = y(x)(py'')'(x) - y'(x)(py'')(x).$$

Computing the derivative of $F[y(x)]$ and making appropriate substitutions we find that

$$F'[y(x)] = -Q(y''(x),y(x))$$

from which it follows that $F[y(x)]$ is nonincreasing. This fact will be of consider-able importance for the remainder of this work. The monotone property of $F[y(x)]$ implies that (1.1) is (2,2)-disconjugate.

A solution $y(x)$ of (1.1) is called a __type I__ solution if $F[y(x)] \geq 0$ for all $x \geq 0$. A __type II__ solution is one which fails to be type I. It is clear that (1.1) always has type II solutions. However, it is not obvious that nontrivial type I solutions exist. Before proceeding further, we establish the existence of such a solution.

THEOREM 2.1. There exists a nontrivial Type I solution.

PROOF. Suppose $a > 0$. Let $\{x_n\}_{n=1}^{\infty}$ be an increasing sequence of numbers such that $a < x_1 < x_2 \ldots < x_n \ldots$ and $\lim_{n \to \infty} x_n = \infty$.

Let $y_1(x)$, $y_2(x)$, $y_3(x)$ be three linearly independent solutions of (1.1) which vanish at $x = a$, then let

$$u_n(x) = c_{1n}y_1(x) + c_{2n}y_2(x) + c_{3n}y_3(x)$$

be a solution of (1.1) satisfying

$$u_n(a) = 0$$

$$u_n(x_n) = u_n'(x_n) = 0$$

$$c_{1n}^2 + c_{2n}^2 + c_{3n}^2 = 1.$$

Note that $F[u_n(x)] > 0$ on $[0,x_n)$ for each n.

The sequences $\{c_{in}\}_{n=1}^{\infty}$ for $i = 1,2,3$ are bounded and hence there exists an increasing sequence of positive integers $\{n_j\}_{j=1}^{\infty}$ such that $\{c_{inj}\}_{j=1}^{\infty}$ is con-vergent for $i = 1,2,3$.

Let $c_i = \lim_{j \to \infty} c_{inj}$, for $i = 1,2,3$ and let $u(x) = c_1y_1(x) + c_2y_2(x) + c_3y_3(x)$.

Since $c_1^2 + c_2^2 + c_3^2 = 1$, $u(x)$ is a nontrivial solution of (1.1).

We now show that $u(x)$ is a type I solution. Suppose the contrary, then there is a number $c > a$ such that $F[u(c)] < 0$. Since $u_{nj}(c) \to u(c)$, we can infer that

$F[u_{nj}(c)]$ → $F[u(c)] < 0$. Choose a positive integer N such that for all $j > N$,

$F[u_{nj}(c)] < 0$ and $x_{nj} > c$. Since $F[u_{nj}(x)]$ is decreasing and $F[u_{nj}(x_{nj})] = 0$, for

$j > N$, we get $0 = F[u_{nj}(x_{nj})] < F[u_{nj}(c)] < 0$, a contradiction. Thus $u(x)$ is a

type I solution of (1.1).

If $x = b$ is a number such that $u(b) \neq 0$, then using the same procedure we can

construct a type I solution which is independent of $u(x)$. Consequently, (1.1) has

at least two linearly independent type I solutions.

Before we proceed with some results on type II solutions we state the so-called

"fundamental" lemma used so extensively in [5].

LEMMA. Suppose $u, v \in C'[a,b]$ with $u(x_1) = u(x_2) = 0$, $a < x_1 < x_1 < b$, and

$u(x) \neq 0$ for $x_1 < x < x_2$. If $v(x) \neq 0$ on $[x_1, x_2]$ then some linear combination of

$u(x)$ and $v(x)$ has a double zero in (x_1, x_2).

THEOREM 2.2. All type II solutions have the same oscillatory character.

PROOF. Let $y_a(x)$ be the solution of (1.1) satisfying

$$y_a(a) = y_a'(a) = y_a''(a) = 0, \quad (py_a'')'(a) = 1$$

where $a > 0$. We assert that if $y_a(x)$ is oscillatory and $b > a$, then $y_b(x)$ is

oscillatory. Suppose the contrary, i.e. assume $y_a(x)$ is oscillatory and that

there is a $c > b$ such that $y_b(x) > 0$ for $x \geq c$. We can assume without loss of

generality that $y_a(b) \geq 0$.

If $\alpha < \beta$ are consecutive zeros of y_a in (c, ∞) for which $y_a(x) > 0$ for $\alpha < x < \beta$,

then there is a constant $k > 0$ such that the solution $z(x) = y_a(x) - ky_b(x)$ has a

double zero in (α, β). Thus $F[z(b)] > 0$, since $b < \alpha$. But $F[z(b)] = F[y_a(b)] -$

$ky_a(b) < 0$. This contradiction proves our assertion.

Now let $u(x)$ be an arbitrary type II solution of (1.1). If $u(x)$ is nonoscilla-

tory and positive on some interval $[e, \infty)$ where e is chosen large enough so that

$F[u(e)] < 0$. The $y_e(x)$ is oscillatory, assuming $y_a(x)$ oscillatory. Consequently,

there is a linear combination $W(x) = y_e(x) - ku(x)$, $k > 0$, have a double zero at

$e_1 \geq e$. Thus $F[W(e_1)] = 0$. But $F[W(e)] = -k\ U(e) + k^2 F[U(e)] < 0$, a contradiction

since $e_1 \geq e$ and $F[W(x)]$ is decreasing.

Thus, an arbitrary type II solution is oscillatory whenever $y_a(x)$ is oscillatory

for some a $\geq$ 0. In a similar manner it can be shown that the existence of an
oscillatory type II solution implies that $y_a(x)$ is oscillatory and our theorem
follows.

3. PROPERTIES OF NONOSCILLATORY SOLUTIONS.

We now focus on the asymptotic behavior of nonoscillatory solutions. We begin
by relating the zero behavior of a solution $y(x)$ with the zero behavior of $y''(x)$.
This theorem is essential to the study of the nonoscillatory solutions of (1.1).
Note the role played by the functional $F[y(x)]$.

THEOREM 3.1. Let $y(x)$ be a nontrivial solution of (1.1). There exists a
point $c > 0$ such that either $F[y(x)] \neq 0$ on $[c,\infty)$ or $F[y(x)] \equiv 0$ on $[c,\infty)$. If
$F[y(x)] \neq 0$ on $[c,\infty)$, then the zeros of $y(x)$ and $y''(x)$ separate on $[c,\infty)$. If
$F[y(x)] \equiv 0$ on $[c,\infty)$, then the zeros of $y(x)$ and $y''(x)$ coincide.

PROOF. Suppose $F[y(x)] \neq 0$ for all $x > c$ and let $x_0 < x_1$ be consecutive zeros
of $y(x)$ on $[c,\infty)$. Assume $y''(x) > 0$ on $[x_0,x_1]$. Then

$$(\frac{y}{py''})' = - \frac{F[y]}{(py'')^2}. \tag{3.1}$$

Integrating (3.1) from x_0 to x_1 yields

$$0 = \frac{y(x_1)}{(py'')(x_1)} - \frac{y(x_0)}{(py'')(x_0)} = - \int_{x_0}^{x_1} \frac{F[y(t)]dt}{(py'')^2(t)} \neq 0$$

which is a contradiction. Thus between consecutive zeros of $y(x)$, $y''(x)$ has a
zero. Exchanging the roles of $y(x)$ and $y''(x)$ we can prove that the zeros of
$y(x)$ separate the zeros of $y''(x)$ for sufficiently large x.

Now consider the case $F[y(x)] \equiv 0$ on $[c,\infty)$. Then

$$F'[y(x)] = -p(x)y''^2(x) - q(x)y''(x)y(x) - r(x)y^2(x) \equiv 0$$

on (c,∞). From which we see that if $d > c$ and $y(d) = 0$, then $y''(d) = 0$.
Similarly if $y''(d) = 0$, then $y(d) = 0$. So the zeros of $y(x)$ and $y''(x)$ coincide
on (c,∞). This completes the proof of our theorem.

For nonoscillatory solutions of (1.1) it follows immediately that there is a
number c such that

$$y(x)y'(x)y''(x) \neq 0 \text{ for all } x \geq c.$$

More explicit information concerning the nonoscillatory solutions of (1.1) can now be given.

THEOREM 3.2. Let $y(x)$ be a nonoscillatory solution of (1.1). Then there exists a number $a > 0$ such that for all $x > a$, either

(i) $\text{sgn } y(x) = \text{sgn } y'(x) = \text{sgn } y''(x) = \text{sgn } (py'')'(x)$

or

(ii) $\text{sgn } y(x) = \text{sgn } y'(x) \neq \text{sgn } y''(x).$

Moreover, if $y(x)$ satisfies (i) then $\lim_{x \to \infty}|y(x)| = \lim_{x \to \infty}|y'(x)| = \infty$.

In case $y(x)$ satisfies (ii) then $\lim_{x \to \infty} y'(x)$ exists and $\lim_{x \to \infty} \inf|y''(x)| = 0$.

THEOREM 3.3. A nonoscillatory solution $y(x)$ of (1.1) is type I if and only if

$$\text{sgn } y(x) = \text{sgn } y'(x) \neq \text{sgn } y''(x),$$

for all x sufficiently large.

PROOF. Let $y(x)$ be a nonoscillatory solution satisfying

$$y(x) > 0, \ y'(x) > 0, \ y''(x) < 0.$$

Consider the functional $F[y(x)] = y(x)(py'')'(x) - y'(x)(py'')(x)$.

If there is a number $x > c$ such that $F[y(c)] < 0$, then $F[y(x)] < 0$ on $[c,\infty)$, but

$$y(x) > 0, \ y'(x) > 0 \text{ and } y''(x) < 0$$

for $x > c$, together with $F[y(x)] < 0$ for $x > c$, implies $(py'')'(x) < 0$ for $x > c$. This inequality, however, will force $y(x)$ to become negative eventually, contradicting $y(x) > 0$ on $[c,\infty)$. Thus $F[y(x)] \geq 0$ for all x. To prove the converse, let $y(x)$ be a nonoscillatory type I solution, then from a preceding theorem there is a number b such that either

(i) $\text{sgn } y(x) = \text{sgn } y'(x) = \text{sgn } y''(x) = \text{sgn } (py'')'(x)$, or

(ii) $\text{sgn } y(x) = \text{sgn } y'(x) \neq \text{sgn } y''(x)$ for all $x \geq b$.

We will show that being type I excludes the possibility of (1.1). Suppose the contrary.

Let $y(x)$ be a solution of (1.1) satisfying (i) and suppose $F[y(x)] > 0$ on $[c,\infty)$.

Then $F[y(x)] > 0$ and $y(x) > 0$ implies

$$(py")'(x) > y'(x) \cdot \frac{(py")(x)}{y(x)} \qquad (3.2)$$

The positivity of $F[y(x)]$ also implies $\frac{(py")(x)}{y(x)}$ is positive and increasing, so

(3.2) implies $(py")'(x) \to \infty$ as $x \to \infty$. But this is impossible since $(py")"(x) < 0$

on $[c,\infty)$. Consequently (i) is inconsistent with being type I.

COROLLARY. A nonoscillatory solution $y(x)$ is type II if and only if there

exists $c > 0$ such that

$$\operatorname{sgn} y(x) = \operatorname{sgn} y'(x) = \operatorname{sgn} y"(x) = \operatorname{sgn} (py")'(x)$$

for all $x > c$.

We now compare $y(x)$ and $(py")(x)$ for nonoscillatory solutions. The proofs

are easy and will be omitted.

THEOREM 3.4. Let $y(x)$ be a nonoscillatory solution of (1.1). Then $\lim\limits_{x\to\infty} \frac{(py")(x)}{y(x)}$

exists. Moreover, in case $y(x)$ is type I, then $\lim\limits_{x\to\infty} \frac{(py")(x)}{y(x)} = 0$.

The next result yields a necessary and sufficient condition for (1.1) to be

nonoscillatory.

THEOREM 3.5. These are equivalent:

(A) Equation (1.1) is nonoscillatory.

(B) Equation (1.1) has both a type I nonoscillatory solution and a

type II nonoscillatory solution.

PROOF. We prove only (B) implies (A) since (A) implies (B) is obvious.

Suppose (1.1) has an oscillatory solution $u(x)$. Then it follows from Theorem 2.2

the $u(x)$ is type I. Let $z(x)$ be a type I nonoscillatory solution. Then there is

a combination $y(x) = kz(x) + u(x)$, $k \neq 0$, which has a double zero. Note that $y(x)$

is nonoscillatory, since all type II solutions are nonoscillatory. Solving for

$u(x)$ we find that $u(x) = y(x) - kz(x)$. However, $\lim\limits_{x\to\infty}|y'(x)| \to \infty$ and $z'(x)$ is bounded

on some ray $[c,\infty)$, so $u(x)$ is not oscillatory. This contradiction shows that

(1.1) is nonoscillatory.

EXAMPLE. The equation

$$y^{iv} + \frac{5}{48x^2} y" + \frac{35}{64x^4} y = 0 \qquad (3.3)$$

has both a type I nonoscillatory solution $y(x) = x^{3/4}$ and a type II solution non-oscillatory solution $y(x) = x^{5/2}$, hence by Theorem 3.5, Equation (3.3) is non-oscillatory.

We now examine some relationship between (1.1) and its adjoint (1.2) in the form

$$(pz'' + qz)'' + rz = 0.$$

As a notational convenience we define the differential operators

$$D_2z = pz'' + qz, \quad D_3z = (D_2z)', \quad D_4z = (D_3z)'.$$

For a solution of (1.2), assuming $Q(z'',z) = pz''^2 + qz''z + rz^2 \geq 0$ we see that the functional

$$G[z(x)] = z(x) D_3z(x) - z'(x)D_2z(x)$$

is nonincreasing. This follows immediately from the fact that $G'[z] = -Q(z'',z) \leq 0$.

Proceeding as before, we shall say that a solution $z(x)$ of (1.2) is a type I* or if and only if $G[z(x)] \geq 0$ for all $x \geq 0$. All other solutions of (1.2) will be called type II* solutions.

THEOREM 3.6. All type II* solutions have the same oscillatory character. The proof of this theorem is analogous to the proof of Theorem 2.2 and will be omitted.

The following lemma will be used to establish a connection between the oscillation of (1.1) and (1.2).

LEMMA. Suppose y is a solution of (1.1) and z is a solution of (1.2), then

$$z(p(x)y'')' - z'(p(x)y'') + y'D_2z - yD_3z \equiv k_1$$

k_1, constant.

THEOREM 3.7. If type II* solutions oscillate, then type I solutions oscillate.

PROOF. Suppose type II* solutions are oscillatory while $y(x)$ is a type I non-oscillatory solution of (1.1). Assume $y(x) > 0$ on $[a,\infty)$ for some $a \geq 0$. Let $z_a(x)$ denote the solution of (1.2) satisfying

$$z_a(a) = z_a'(a) = D_2z_a(a) = 0, \quad D_3z_a(a) = 1.$$

Then $z_a(x)$ is oscillatory. Let α and β be consecutive zeros of $z_a(x)$ and suppose $z_a(x) > 0$ on (α,β). Consider the following

$$\left(\frac{D_2z_a}{y}\right)' = \frac{yD_3z_a - y'D_2z_a}{y^2}.$$

(3.4)

From the previous lemma

$$yD_3 z_a - y'D_2 z_a + z_a'(py'') - z_a(py'')' \equiv k.$$

Thus we see that $yD_3 z_a - y'D_2 z_a = z_a(py'')' - z_a'(py'') + k$ so making a substitution

in (3.4) we obtain

$$(\frac{D_2 z_a}{y})' = \frac{z_a(py'')' - z_a'(py'') + k}{y^2} ,\qquad (3.4a)$$

Integrating (3.4a) by parts from $x = \alpha$ to $x = \beta$ we get

$$\frac{(pz_a'')(\beta)}{y(\beta)} - \frac{(pz_a'')(\alpha)}{y(\alpha)} = \int_\beta^\alpha \frac{2z_a(x)F[y(x)] + ky(x)}{y^3(x)} dx, \qquad (3.5)$$

where $k = y(a) \geq 0$.

Since $z_a(x) > 0$ on (α,β), $G[z_a(\alpha)] = -p(\alpha)(z_a' z_a'')(\alpha) < 0$ and $G[z_a(\beta)] =$

$-p(\beta)z_a'(\beta)z_a''(\beta) < 0$, we conclude that $z_a'(\alpha) > 0$, $z_a''(\alpha) > 0$, $z_a'(\beta) < 0$ and

$z_a''(\beta) < 0$. This implies that the left side of (3.5) is negative, but note that

the right side of (3.5) is positive. This contradiction proves that y(x) must

oscillate.

COROLLARY. All type I solutions have the same oscillatory character. The

companion Theorem of Theorem 3.7 is also true. We state it without proof.

THEOREM 3.8. If type I* solutions oscillate, then type II solutions oscillate.

Immediately it follows that if all solutions of (1.1) have the same oscillatory

character then all solutions of (1.2) have the same oscillatory character.

OSCILLATION CRITERIA. In this section we establish some sufficient conditions

for all solutions of (1.1) to be oscillatory. One such condition makes use of the

nonoscillation numbers r_{ij} studied by Peterson [6]. We describe these briefly.

Let i and j be nonnegative integers such that $i + j = 4$. A solution y(x) is said

to have an (i,j)-distribution of zeros on an interval I if there are two numbers

x_i and x_j in I such that $x_i < x_j$ and y(x) has a zero of multiplicity at least i

at x_i and a zero of multiplicity at least j at x_j and $y(x) \neq 0$ in (x_i, x_j). If on

$[t,\infty)$ there are solutions with (i,j)-distributions of zero then $r_{ij}(t)$ for equation

(1.1) is the infimum of the numbers $b > t$ such that there exists a solution y(x)

of (1.1) having an (i,j) distribution of zeros on [t,b). If no such distribution

exists on I then we write $r_{ij} = \infty$ on I.

THEOREM 3.9. If $r_{31}(t) < \infty$ and $r_{13}(t) < \infty$ for all t on some ray $[\alpha,\infty)$, $\alpha < 0$,

then all solutions of (1.1) are oscillatory.

PROOF. We only show type I solutions oscillate, since the proof is similar to

show type II solutions oscillate.

Suppose $y(x) > 0$ on $[b,\infty)$ for some $b \geq \alpha$ and $F[y(x)] > 0$ on $[b,\infty)$. Let $u(x)$

be a nontrivial solution of (1.1) having a (1,3)-distribution at $x = c$ and $x = d$,

$b < c < d$; we assume $(pu'')'(d) = 1$. Since $y(x) > 0$ on $[b,\infty)$ and $u(x) < 0$ on (c,d),

there is a positive constant k such that $v(x) = y(x) + k\,u(x)$ has a double zero in

(c,d). Since $F[v(x)]$ is decreasing and $F[v(x)]$ vanishes in (c,d) we conclude that

$F[v(d)] < 0$. But $F[v(d)] = F[y(d)] + k\,y(d) > 0$. This contradiction proves that

$y(x)$ must have a zero between $x = c$ and $x = d$. Now, since $r_{13}(t)$ exists for every

$t > 0$ it is easy to show that type I solutions have infinitely many zeros on $[\alpha,\infty)$.

THEOREM 3.10. If $\int_{0}^{\infty} r(x)dx = \infty$ and $\frac{q(x)}{p(x)} \leq r(x)$ and $p(x)$ bounded, then all

solutions of (1.2), hence (1.1), are oscillatory.

PROOF. Suppose $z(x)$ is a nonoscillatory solution of (1.2) satisfying $z(x) > 0$

on $[a,\infty)$. Then $D_4 z(x) < 0$ on $[a,\infty)$ and $D_3 z(x)$ is decreasing. If $D_3 z(b) < 0$ for

some $b > a$ then $D_3 z(x) < 0$ for $x > b$ and $D_2 z(x) \to -\infty$. Choose $c \geq b$ such that

$p(x)z''(x) \leq -1$ for $x > c$. Then $z''(x) \leq -\frac{1}{p(x)}$ on $[c,\infty)$. Integrating this

inequality we obtain $z'(x) \leq z'(c) - \int_{c}^{x} \frac{dt}{p(t)}$, which implies $z'(x) \to -\infty$ as $x \to \infty$.

But this contradicts $z(x) > 0$ on $[c,\infty)$. So $D_3 z(x) > 0$ for all $x > a$. Note how-

ever that $D_3 z(x) > 0$ and decreasing implies

$$\int_{a}^{\infty} r(x)z(x)dx < \infty.$$

Now let us examine $D_2 z(x)$. We know $D_2 z(x)$ is increasing. If $D_2 z(x) < 0$ on $[b,\infty)$,

$b \geq a$, then $z(x)$ is increasing on $[c,\infty)$ for some $c > b$.

So $z(x) > z(c)$ for $x > c$ and

$$D_3 z(x) - D_3 z(c) = -\int_{c}^{x} r(t)z(t)dt < -z(c)\int_{c}^{x} r(t)dt$$

As $x \to \infty$, $\int_c^x r(t)\ dt \to \infty$, but this forces $D_3 z(x)$ to be negative at some point

greater than a, clearly a contradiction. Therefore, $D_2 z(x) < 0$ on some ray $[b,\infty)$

is impossible. Now consider the case where $D_2 z(x) > 0$ on $[b,\infty)$. If M is an

upper bound for $p(x)$ then using our hypothesis we obtain

$$M[z'' + r(x)z] \geq M(z''_{(x)} + \frac{q(x)}{p(x)} z(x)) \geq p(x)[z''(x) + \frac{q(x)}{p(x)} z(x)] =$$

$$D_2 z(x) > D_2 z(b) \text{ for } x > b. \tag{3.6}$$

Integrating (3.6) we see that

$$0 < D_2 z(b)(x-b) < M \int_b^x r(t)z(t)dt + Mz'(x) - Mz'(b)$$

which implies $z'(x) \to \infty$ as $x \to \infty$. Thus $z(x)$ is increasing on some ray $[d,\infty)$,

$d \geq c$.

But this leads us to a contradiction, since $\int_a^\infty r(x)dx = \infty$. Consequently, all

solutions of (1.2) are oscillatory.

THEOREM 3.11. If $\int_0^\infty x\ r(x)\ dx = \infty$ and $xq(x) \leq M$, then all solutions (1.1)

are oscillatory.

PROOF. Suppose (1.1) has a nonoscillatory solution $y(x)$. Assume without loss

of generality that $y(x) > 0$, for all $x > b$. If $y(x)$ is a type I solution then

there is a number c such that for all $x > c$

$$y(x) > 0, \ y'(x) > 0, \ y''(x) < 0.$$

Multiplying (1.1) by x and integrating we get

$$x(py'')'(x) - c(py'')'(c) + \int_c^x t\ q(t)y''(t)dt + \int_c^x t\ r(t)y(t)dt = 0.$$

Let M be an upper bound for $xq(x)$. Then we have

$$x(py'')'(x) - c(pu'')'(c) + M[y'(x) - \dot{y}'(c)] + \int_c^x t\ r(t)y(t)dt \leq 0.$$

Since $y(x)$ is positive and increasing the integral tends to infinity. Moreover,

since $y''(x) < 0$, $y'(x)$ has a finite limit. Thus $(py'')(x)$ is negative for large x

and $(py'')(x)$ is decreasing. Let $B < 0$ be a number such that $y''(x) < \frac{B}{p(x)}$ for

$x > c$. Integrating this inequality from c to x we conclude that $\lim_{x\to\infty} y'(x) = -\infty$,

which is absurd since $y'(x)$ is bounded.

If $y(x)$ is a positive type II solution, then since $y'(x)$ and $y''(x)$ are

positive there exists a positive number in such that $y(x) \geq mx$ for large x.

Substituting in (1.1) and integrating we get

$$(py'')'(x) - (py'')'(c) \leq - \int_c^x q(t)y''(t)dt - M \int_c^x t\, r(t)dt.$$

We conclude from this inequality that $(py'')'(x) \rightarrow -\infty$ as $x \rightarrow \infty$, a contradiction.

Thus type II solutions must be oscillatory.

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