THE SECOND COEFFICIENT OF A FUNCTION WITH ALL DERIVATIVES UNIVALENT

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ABSTRACT. We consider the second coefficient of a class of functions, univalent

and normalized, and with all derivatives univalent in the unit disk D, and improve

on a known result. It is also shown that this bound is in a sense best possible.

KEY WORDS AND PHRASES. Univalent functions, Typically real functions, Entire
functions of exponential type.

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1. INTRODUCTION.

Let S be the family of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic and univalent in the unit disk D: $|z| < 1$. Let E be the sub-

family of S such that if $f \in E$, then each $f^{(n)}$ is univalent in D, n = 0,1,2,... .

Set

$$\alpha = \sup \{|a_2| : f \in E\}. \tag{1.2}$$

Then it is known that f must be an entire function of exponential type not exceed-

ing 2α [2,3] and [1,2]

$$1.5910 \leq \alpha < 1.7208. \tag{1.3}$$

We construct, in Section 2, a function f belonging to E for which

$$a_2 = \frac{f''(0)}{2} > \frac{3.18781}{2} = 1.593905. \tag{1.4}$$

Consider now a subfamily of E, E_+ defined as follows:

$$E_+ = \{f \mid f \in E, f^{(k)}(0) > 0, k = 1,2,\ldots\}.$$

Set

$$\alpha_+ = \sup \{a_2, f \in E_+\}.$$

It follows from an unpublished result of Ted Suffridge that

$$\alpha_+ < 1.62031. \tag{1.5}$$

Note that the function f defined below by (2.2) belongs to E_+, and so a lower bound

for α_+ is given by (1.4).

2. **MAIN RESULT.**

In the following we shall write

$$\zeta = \{(T + \pi)e^{-\pi} + (T - \pi)e^{\pi}\}/2$$

where T is a positive constant.

THEOREM. Let

$$\phi(z) = \{\exp(\pi(1 + z)) - \exp(-\pi(1 + z))\}/2\pi + Az$$

$$+ (AT + \zeta)\frac{z^2}{2} + Bz^3 \tag{2.1}$$

where A, B, T are nonnegative constants, and let

$$f(z;A,B,T) \equiv f(z) = \{\phi(z) - \phi(0)\}/\phi'(0). \tag{2.2}$$

(i)   If T = 3.18781018, A = 0.03666, B = 0 then f defined by (2.2) belongs to

E and

$$2a_2 = f''(0) = T > 3.18781. \tag{2.3}$$

(ii)   If T = 3.18782 then f defined by (2.2) does not belong to E for any

choice of nonnegative constants A,B.

PROOF. We require the following:

DEFINITION. Let f(z) be regular in D. Then F(z) is called typically-real if

and only if

$$\text{Im } f(z) = 0 \Longleftrightarrow \text{Im } z = 0.$$

The following theorem is due to S.Y. Trimble.

THEOREM A [4]. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be regular in D and suppose that all

a_n are real. Then a necessary and sufficient condition that $f \in E$ is that each

$f^{(k)}$ (k = 0,1,2,...) is typically real in D.

PROOF OF (i). It is easily seen that $\phi(0) = \dfrac{\sinh \pi}{\pi} > 0$, $\phi'(0) = \cosh \pi + A > 0$;

and

$$f(z) = z + \frac{T}{2} z^2 + \left\{ \left((e^\pi + e^{-\pi}) \frac{\pi^2}{12} + B \right) \Big/ \left(\frac{e^\pi + e^{-\pi}}{2} + A \right) \right\} z^3 + \cdots$$

is entire and of exponential type π and has all positive coefficients. Further,

writing z = x + iy, we have

$$\left\{ \frac{\operatorname{Im} \phi'(z)}{y} \right\} = \frac{\sin \pi y \, \sinh \pi(1 + x)}{y} + (AT + \zeta) + 6Bx.$$

Write

$$\psi_2(x) = \left(\frac{\operatorname{Im} \phi'(z)}{\operatorname{Im} z} \right) \bigg|_{|z| = 1}.$$

Then

$$\psi_2(x) = \frac{\sin \pi \sqrt{1 - x^2}}{\pi \sqrt{1 - x^2}} \, \pi \sinh \pi(1 + x) + (AT + \zeta) + 6Bx.$$

This function ψ_2 is positive for $-1 \le x \le 1$, since $0 = 6B < AT + \zeta$ by our choice of

T, A and B. Hence $\phi'(z)$ is typically real. Further,

$$\phi''(z) = \pi \sinh \pi(1 + z) + AT + \zeta + 6Bz,$$

and since B = 0

$$\operatorname{Im} \phi''(z) = \pi \sinh \pi y \, \cosh \pi(1 + x).$$

This shows that $\phi''(z)$ is typically real. Similarly each $\phi^{(k)}(z)$, k = 3,4,... is

typically real.

Consider now

$$\psi_1(x) = \left(\frac{\operatorname{Im} \phi}{\operatorname{Im} z} \right) \bigg|_{|z| = 1}$$

$$= \{\cosh \pi(1 + x)\} \frac{\sin \pi \sqrt{1 - x^2}}{\pi \sqrt{1 - x^2}} + A + (AT + \zeta)x + B(4x^2 - 1). \qquad (2.4)$$

Note that $\psi_1(x) > 0$ for $0 \le x \le 1$. Write now $F(x) = \psi_1(-x)$. Then

$$F(x) = \{\cosh \pi(1 - x)\} \frac{\sinh \pi \sqrt{1 - x^2}}{\pi \sqrt{1 - x^2}} + A - (AT + \zeta)x + B(4x^2 - 1). \qquad (2.5)$$

Write

$$C(x) = \cosh \pi(1 - x), \quad H(x) = (\sinh \pi \sqrt{1 - x^2}) / \pi \sqrt{1 - x^2}$$

$$G(x) = A - (AT + \zeta)x, \quad S(x) = \sinh \pi(1 - x).$$

Then since B = 0,

$$F(x) = C(x)H(x) + G(x) ,$$

$$F'(x) = \{-\pi \sinh \pi(1 - x)\}H(x) + C(x)H'(x) + G'(x) ,$$

and

$$H(x) = x^2 \prod_{n=2}^{\infty} \left(\frac{n^2 - 1 + x^2}{n^2} \right) .$$

Now B = 0 and for $0 \le x \le 0.046 < \dfrac{A}{(AT + \zeta)}$, $G(x) = A - (AT + \zeta)x > 0$.

Further for $x \in [0.92,1]$ we have

$$\min C(x) = C(1) = 1 ,$$

$$\min H(x) = H(0.92) > 0.765$$

and

$$\min G(x) = G(1) > -0.752.$$

Thus $F(x) > 0$ for $x \in [0.92,1]$ and also for $x \in [0,0.04]$.

Differentiating log H(x), we have

$$\frac{H'(x)}{H(x)} = \frac{2}{x} + 2x \sum_{n=2}^{\infty} \frac{1}{n^2 - 1} - \varepsilon(x)$$

where

$$\varepsilon(x) = \sum_{n=2}^{\infty} (2x^3)/\{(n^2 - 1)(n^2 - 1 + x^2)\} .$$

Write $u(x) = (2/x) + (3x/2)$. Then

$$F'(x) = -\pi S(x)H(x) + C(x)u(x)H(x) - \varepsilon(x)C(x)H(x) - (AT + \zeta) .$$

Now $0 \le \varepsilon(x) < (0.27)x^3$ in the interval (0.040,1). Further in this interval,

$u\downarrow$, $C\downarrow$, $S\downarrow$, $H' > 0$ and so $H\uparrow$. Now $u(1) > 0$ and so $u(x) > 0$, $C(1) = 1$ and so

$C(x) > 0$, $S(1) = 0$ and so $S(x) > 0$, $H(x) \to 1$ as $x \to 1-0$, $H(0) = 0$, $H(x) > 0$ and

$\varepsilon(x) < 0.27x^3$ and $x^3\uparrow$ and positive.

Let $x_1 \le x \le x_2$ where x_1, x_2 belong to this interval. Denote by subscript 2

the value of these functions at x_2 and by subscript 1 at x_1. Let

$$T_1(x_1,x_2) \equiv T_1 = u_2 C_2 H_1 - H_2(\pi S_1 + C_1(0.27x_2^3)) - (AT + \zeta),$$

$$T_2(x_1,x_2) \equiv T_2 = u_1 C_1 H_2 - H_1(\pi S_2) - (AT + \zeta) .$$

Then for $x_1 \le x \le x_2$, $T_1 < F'(x) < T_2$. Hence we have

PROPOSITION 2. If $T_1 > 0$ or $T_2 < 0$ then $F(x)$ is monotone in this interval and

so if $F(x_1) > 0$, $F(x_2) > 0$ then $F(x)$ is positive.

COROLLARY 3. Suppose that the quantities $\{T_i\}$ and $\{F(x_i)\}$ are machine calculated with at least 6-digit accuracy and we have:

(i) $T_1 \geq 10^{-6}$ or $T_2 \leq -10^{-6}$,

(ii) $F(x_1) > 10^{-6}$, $F(x_2) > 10^{-6}$

then $F(x)$ is positive in $[x_1,x_2]$.

COROLLARY 4. By machine calculations we have:

(i)   For

$$[x_1,x_2] = [0.040000,0.054844] \qquad (2.6)$$

$$T_2 = -0.0410121, \quad F(x_1) = 0.0133178$$

$$F(x_2) = 0.0081411 \ .$$

(ii)   For

$$[x_1,x_2] = [0.847885,0.918943] \qquad (2.7)$$

$$T_1 = 0.0373463, \quad F(x_1) = 0.0354489$$

$$F(x_2) = 0.1001402 \ .$$

(iii)   For

$$[x_1,x_2] = [0.918943,0.990000] \qquad (2.8)$$

$$T_1 = 0.8434650, \quad F(x_1) = 0.1001402$$

$$F(x_2) = 0.2242320 \ .$$

Hence $F(x)$ is positive in the intervals (2.6), (2.7) and (2.8). Thus it is now enough to show that $F(x) > 0$ in $[0.054844,0.847885] = [X_1,X_2]$ say. We now consider when $T_1 < 0$, $T_2 > 0$. Write $M(x_1,x_2) \equiv M = -T_1 \cdot (x_2 - x_1)$. By the Mean-Value Theorem, $F(x_0) = F(x_1) + F'(\xi)(x_0 - x_1)$, where $x_1 \leq x_0 \leq x_2$ and $\xi \in (x_1,x_0)$. Hence

$$F(x_0) \geq F(x_1) + T_1 \cdot (x_0 - x_1)$$

$$> F(x_1) + T_1 \cdot (x_2 - x_1) = F(x_1) - M$$

for every x_0 in $[x_1,x_2]$. Hence we have

PROPOSITION 5. Let $T_1 < 0$ and $F(x_1) > 0$, $F(x_2) > 0$. If $F(x_1) > M(x_1,x_2)$ then $F(x) > 0$ in $[x_1,x_2]$.

COROLLARY 6. Suppose that $\{F(x_i)\}$ and $M(x_1,x_2)$ are machine calculated with at leat 6-digit accuracy and we have

(i) $F(x_1) > 10^{-6}$, $F(x_2) > 10^{-6}$

(ii) $F(x_1) > M(x_1,x_2) + 10^{-6}$

then $F(x)$ is positive in $[x_1,x_2]$.

PROOF THAT $F(x) > 0$ IN $[X_1,X_2]$. This was carried out by running the following simple program and repeatedly applying Corollary 6, as indicated.

(1)   Set $x_1 = X_1$, $x_2 = X_2$.

(2)   Apply test of Corollary 6. Is the hypothesis satisfied? If not, then go to (5).

(3) $F(x)$ is now positive in $[x_1,x_2]$. Is $X_2 - x_2 < 10^{-6}$? If it is, then stop. We have proved $F(x) > 0$ in $[X_1,x_2]$ and x_2 is as close to X_2 as is reasonable. If $X_2 - x_2 \geq 10^{-6}$, then continue.

(4)   Start with the next interval by replacing x_1 by x_2 and x_2 by X_2. Go to (2).

(5) $[x_1,x_2]$ is too large. Replace x_2 by $(x_1 + x_2)/2$. Is $(x_2 - x_1) < 10^{-6}$? If it is, then stop. This method cannot be reliably continued beyond x_1. If $x_2 - x_1 \geq 10^{-6}$, then go to (2).

This program terminated by x_2 reaching X_2. In fact the program was run from 0.04 to 0.99 by incorporating the test of Corollary 4, as well as another mean value test comparing $F(x_2)$ and $T_2 \cdot (x_2 - x_1)$ when $T_2 > 0$. The numbers presented here were generated by the program and are not intrinsically significant. The table for the relevant calculations is available upon request.

This completes the proof of the statement that $F(x) > 0$ in $-1 \leq x \leq 1$ and hence that of (i).

(ii)   We require the following:

LEMMA. Consider a family of functions of two parameters (a,b) defined by

$$\phi(x,a,b) = af(x) - bg(x) + h(x). \tag{2.9}$$

Suppose that there exist a_o, x_1, x_2 satisfying

$$\phi(x_i,a_o,0) < 0, i = 1,2 , \tag{2.10}$$

$$f(x_1), g(x_1) > 0 ; f(x_2), g(x_2) < 0 , \tag{2.11}$$

and

$$f(x_1)/g(x_1) \leq f(x_2)/g(x_2) \ . \tag{2.12}$$

Then for all a, b $\geq$ 0 we must have either

$$(x_1,a,b) < 0 \quad \text{or} \quad (x_2,a,b) < 0 \ . \tag{2.13}$$

REMARK. In this lemma and proof of (ii), f is different from f in (2.2).

PROOF. Assume the contrary. We will show that the resulting system of inequalities has no solution. Set $\alpha = a - a_o$, $\beta = b$. Then we get from (2.10) and our assumption,

$$0 < \phi(x_i,a,b) - \phi(x_i,a_o,0) = \alpha f(x_i) - \beta g(x_i), \quad i = 1,2. \tag{2.14}$$

Also by our assumption

$$\beta = b \geq 0 \ . \tag{2.15}$$

From (2.15), (2.14) and (2.11) we get $\alpha \geq 0$. Now we claim that $\beta \neq 0$; for otherwise (2.14) (with i = 2) and (2.11) give $\alpha f(x_2) > 0$, that is $\alpha < 0$ leading to a contradiction.

Thus $\beta > 0$ and from (2.14) (with i = 1) and (2.11) we get $\alpha > 0$. Now (2.14) (with i = 1,2) and (2.11) yield

$$\frac{f(x_1)}{g(x_1)} > \frac{\beta}{\alpha} > \frac{f(x_2)}{g(x_2)} \qquad \text{`}$$

in contradiction to (2.12). The lemma is proved.

PROOF OF (ii). Let, in the lemma,

$$\phi(x,a,b) = C(x)H(x) - \zeta x + a(1 - Tx) - b(1 - 4x^2) \ ,$$

$$a = A, \quad b = B, \quad f(x) = 1 - Tx, \quad g(x) = 1 - 4x^2, \quad T = 3.18782 \ ,$$

$$h(x) = C(x)H(x) - \zeta x \ .$$

Then from (2.5) and (2.9) we see that

$$\phi(x) \equiv \phi(x,a,b) = \phi(x,A,B) = F(x) \ .$$

Let $x_1 = 0.1160000$, $x_2 = 0.7033334$, $a_o = 0.0366100$. Then the following are correct up to 6 decimals at least.

$$\phi(x_1) = -0.0000062, \quad \phi(x_2) = -0.0000037 \ ,$$

$$f(x_1) = 1 - Tx_1 = 0.6302128, \quad f(x_2) = -1.2421002 \ ,$$

$$g(x_1) = 1 - 4x_1^2 = 0.9461760, \quad g(x_2) = -0.9787114 ,$$

$$\frac{f(x_1)}{g(x_1)} = 0.6660630, \quad \frac{f(x_2)}{g(x_2)} = 1.2691179 .$$

Hence the conditions of the lemma are satisfied. This means that $\phi(x_1) = F(x_1) < 0$ or $\phi(x_2) = F(x_2) < 0$ for all choices of A, B $\geq$ 0 and consequently the function f defined by (2.2) cannot be in the class E, if T = 3.18782, for any choices of A $\geq$ 0 and B $\geq$ 0.

REMARK. The choices of T and A, in part (i) of the theorem are not intrinsically significant. Another computer program was set up which, for any given T,A and given interval $[x_1, x_2]$, could quickly check the values of the function at several points in the interval and calculate relevant quantities in the lemma. The actual T,A,B were then found simply by trial and error by taking some value of T and eliminating possible choices of A,B by applying the lemma. If for a given T all A,B got eliminated, then a lower T was chosen. Finally, all calculations were rechecked as before for the chosen T,A,B.

The fact that the computer used for this process   (Radio Shack TRS 80) was interactive and totally dedicated was very vital in the success of this trial and error method.

REFERENCES

1. LACHANCE, M. Remark of functions with all derivatives univalent, Inter. J. of Math. and Math. Sciences 3 (1980), 193-196.

2. SHAH, S.M. and TRIMBLE, S.Y. Univalent functions with univalent derivatives, Bull. Amer. Math. Soc. 75 (1969), 153-157.

3. SHAH, S.M. and TRIMBLE, S.Y. Analytic functions with univalent derivatives, Indian J. of Math. 20 (1978), 1-35.

4. TRIMBLE, S.Y. An extension of close-to-convex functions with an application to entire functions with univalent derivatives, to appear.