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CLOSED SPECTRAL MEASURES IN FRÉCHET SPACES

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ABSTRACT. Closed spectral measures, which are often used in the theory of operators, have the desirable property that their L^1-space is complete. In this note criteria are given which assure the closedness of spectral measures acting in Fréchet spaces.

KEY WORDS AND PHRASES. *Fréchet space, spectral measure, closed measure, projective limit of measures.*
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1. INTRODUCTION.

Closed vector measures were introduced by I. Kluvánek in [1]. Their basic properties can be found in the monograph [2]; see also [3]. Applications, such as in [3], [4], [2] and [5] for example, are based mainly on the fact that the L^1 -space of a closed vector measure is complete for the topology of uniform convergence of indefinite integrals. In [5], the theory of integration with respect to closed spectral measures was used to obtain results in operator theory.

Applications of this kind make it desirable to have available criteria which guarantee the closedness of a given spectral measure. However, the criteria available for arbitrary vector measures are often difficult to apply to operator-valued measures. For example, any vector measure assuming its values in a complete metrizable locally convex space is necessarily closed [2; IV Theorem 7.1]. However, the space of continuous linear operators on such a space is metrizable for the pointwise convergence topology only if the underlying space is finite dimensional (this follows from [7; § 39 Proposition 4.6], for example).

In this note we prove two results which are concerned with the problem of determining the closedness of spectral measures. It is shown that if the underlying space is a *separable* Fréchet space, then any spectral measure is necessarily closed (cf. Theorem 1 below).

The second result is based on a relatively successful technique which is often used in the study of continuous linear operators on locally convex spaces, namely to realize the given locally convex space as a projective limit of Banach spaces. It is then possible to use the well developed theory for operators in Banach spaces. Using this approach it is shown that if a spectral measure in an (arbitrary) Fréchet space is interpreted as the projective limit of a sequence of spectral measures, each of which acts in a suitable Banach space, then the given measure is closed if and only if each of the induced measures is closed (cf. Theorem 2 below).

2. PRELIMINARIES AND STATEMENT OF RESULTS

Throughout this note X will denote a Fréchet space, that is, a complete metrizable locally convex Hausdorff space. Let $L(X)$ denote the space of continuous linear operators of X into itself, equipped with the topology of pointwise convergence.

The space of continuous linear functionals on X is denoted by X'. The correspondence $\sum_i x_i \otimes x_i' \to f \in (L(X))'$, defined by

$$f(T) = \sum_i \langle T(x_i), x_i' \rangle, \quad T \in L(X), \tag{2.1}$$

is an (algebraic) isomorphism of the tensor product, $X \otimes X'$, onto the dual of $L(X)$, [6; §39, Proposition 7.2].

An $L(X)$-valued operator measure is a σ-additive map $P : M \to L(X)$, whose domain M is a σ-algebra of subsets of a non-empty set Ω. It follows from the identification (2.1.) and the Orlicz-Pettis lemma that P is σ-additive if and only if the complex-valued set function

$$\langle P(\cdot)(x), x' \rangle : E \to \langle P(E)(x), x' \rangle, \quad E \in M,$$

is σ-additive for each $x \in X$ and $x' \in X'$. Since X is barrelled it follows that the range of P, that is, $P(M) = \{P(E); E \in M\}$, is an equicontinuous part of $L(X)$.

An operator-valued measure $P : M \to L(X)$ is said to be a spectral measure if it is multiplicative and $P(\Omega)$ is the identity operator on X. Of course, the multiplicativity of P means that $P(E \cap F) = P(E)P(F)$, for every $E \in M$ and $F \in M$.

A net $\{E_\alpha\}$ of sets in M is said to be P-convergent to an element E of M (respectively, to be P-Cauchy) if, for every neighbourhood U of zero in $L(X)$, there is an index α_U such that $P(F) \in U$, for every set $F \subset E_\alpha \Delta E = (E \cup E_\alpha) \setminus (E \cap E_\alpha)$ (respectively, $F \subset E_\alpha \Delta E_\beta$), $F \in M$, whenever $\alpha_U \leq \alpha$ (respectively $\alpha_U \leq \alpha$ and $\alpha_U \leq \beta$).

An operator-valued measure $P : M \to L(X)$ is said to be closed if M is P-complete, that is, if every P-Cauchy net of sets in M is P-convergent to a member of M; see [1].

Let $P : M \to L(X)$ be a spectral measure. A complex-valued, M-measurable function f on Ω is said to be P-integrable if it is integrable with respect to every measure $\langle P(\cdot)(x), x' \rangle$, $x \in X$ and $x' \in X'$, and if, for every $E \in M$, there exists an element $(fP)(E)$ of $L(X)$ such that

$$\langle (fP)(E)(x),x'\rangle = \int_E f d\langle P(\cdot)(x),x'\rangle,$$

for each $x \in X$ and $x' \in X'$. The element $(fP)(\Omega)$ is denoted simply by $P(f)$. The multiplicativity of P implies that $(fP)(E) = P(E)P(f) = P(f)P(E)$, for every $E \in M$.

Since the space $L(X)$ is quasi-complete it follows that every bounded, measurable function on Ω is P-integrable [2; II Lemma 3.1].

The topology of X can be specified by a sequence of seminorms q_n, $n = 1,2,\ldots$, satisfying the following compatibility conditions with respect to P, [8; Proposition 2.3].

(i) If f is measurable and $|f(\omega)| = 1$, $\omega \in \Omega$, then

$$q_n(P(f)(x)) = q_n(x), \quad x \in X,$$

for each $n = 1,2,\ldots$.

(ii) If f and g are bounded, measurable functions on Ω such that $0 \le f \le g$, then

$$q_n(P(f)(x)) \le q_n(P(g)(x)), \quad x \in X,\ n = 1,2,\ldots.$$

Moreover, the seminorms can be chosen so that $q_n \le q_{n+1}$, $n = 1,2,\ldots$.

If f is a bounded, measurable function on Ω, *then*

$$q_n(P(f)(x)) \le \|f\|_\infty q_n(x), \quad x \in X, \tag{2.2}$$

for each $n = 1,2,\ldots$; see [8; Proposition 2.4]. In particular, each of the closed subspaces $q_n^{-1}(\{0\})$, $n = 1,2,\ldots$, is invariant for $P(f)$.

Denote by X_n, $n = 1,2,\ldots$, the quotient space of X modulo the closed subspace $q_n^{-1}(\{0\})$. The image of an element $x \in X$, under the natural map of X onto X_n, is denoted by $[x]_n$, $n = 1,2,\ldots$. The space X_n is a normed space with respect to the norm

$$\|[x]_n\| = q_n(x), \quad [x]_n \in X_n,$$

for each $n = 1,2,\ldots$. The completion of X_n with respect to this norm is denoted by $\tilde{X}_n$, $n = 1.2,\ldots$.

For each $m \ge n$, $n = 1,2,\ldots$, the map $\psi_{m,n}: X_m \to X_n$ given by

$$\psi_{m,n} : [x]_m \to [x]_n, \quad [x]_m \in X_m,$$

is a well defined continuous linear map of X_m onto X_n with norm not exceeding one and hence, has a unique continuous extension $\tilde{\psi}_{m,n}$, onto the whole of $\tilde{X}_m$.

If $E \in M$, then each subspace $q_n^{-1}(\{0\})$, $n = 1,2, \ldots$, is invariant for $P(E)$ and hence, there is induced a sequence of linear operators $P_n(E): X_n \to X_n$, $n = 1,2,\ldots$, given by

$$P_n(E)([x]_n) = [P(E)(x)]_n, \quad [x]_n \in X_n. \tag{2.3}$$

It is clear from (2.2) that each operator $P_n(E)$, $n = 1,2,\ldots$, is continuous with norm not exceeding one. Hence, each of the induced operators has a unique continuous extension to $\tilde{X}_n$ denoted by $\tilde{P}_n(E)$, $n = 1,2,\ldots$. It is easily verified that the map $\tilde{P}_n : M \to L(\tilde{X}_n)$ given by

$$\tilde{P}_n : E \to \tilde{P}_n(E), \quad E \in M,$$

is a spectral measure for each $n = 1,2,\ldots$, with norms uniformly bounded by one. For each $E \in M$ and $m \geq n$, $n = 1,2,\ldots$, the formula

$$\tilde{\psi}_{m,n} \circ \tilde{P}_m(E) = \tilde{P}_n(E) \circ \tilde{\psi}_{m,n}, \tag{2.4}$$

is valid. Since X is the projective limit of the spaces $\tilde{X}_n$, $n = 1,2,\ldots$, the measure P can be interpreted as the projective limit of the measures P_n, $n = 1,2,\ldots$; see [8; Theorem 2.7].

THEOREM 1. *Let X be a separable Fréchet space. Then any $L(X)$-valued measure is a closed measure.*

THEOREM 2. *Let X be a Fréchet space. A spectral measure $P:M \to L(X)$, where M is a σ-algebra of sets, is closed if and only if each of the induced spectral measures $\tilde{P}_n : M \to L(\tilde{X}_n)$, $n = 1,2,\ldots$, is closed.*

In view of Theorem 1 and the fact that X is separable if and only if each of the spaces $\tilde{X}_n$, $n = 1,2,\ldots$, is separable, it is clear that the criterion given in Theorem 2 is of use mainly for spectral measures acting in a non-separable space. It reduces the question of closedness for such a spectral measure to the same question in Banach spaces.

3. PROOFS OF THE THEOREMS.

The following result [5; Proposition 3] is needed.

PROPOSITION. *A spectral measure $P : M \to L(X)$ is closed if and only if the range $P(M) = \{P(E);\ E \in M\}$ is a closed set in $L(X)$.*

It is interesting to note that the above Proposition is peculiar to spectral measures. It is not valid for general operator-valued measures (hence, also not valid for arbitrary vector measures).

For example, let X denote the Banach space of continuous functions on [0,1] vanishing at zero, equipped with the uniform norm. For each Borel set E of [0,1] define an element $P(E)$ of $L(X)$ by

$$P(E)(f) : s \to \int_0^s \chi_E(t)f(t)dt, \quad s \in [0,1],\ f \in X.$$

Then P is σ-additive but not multiplicative. It follows from [2; IV Theorem 7.3] that P is a closed measure because it is absolutely continuous with respect to Lebesgue measure. However, the range of P is not closed in $L(X)$. To see this, define a sequence of Borel sets in [0,1] by

$$E_n = \bigcup_k \{[k/2^n, (k+1)/2^n];\ 1 \leq k \leq 2^n,\ k \text{ an odd integer}\},$$

for each $n = 1,2,\ldots.$ We show that $\{P(E_n)\}_{n=1}^{\infty}$ has a limit in $L(X)$ equal to the operator $\frac{1}{2}P([0,1])$, which does not belong to the range of P.

Let Y denote the Banach space of bounded measurable functions on $[0,1]$ with the uniform norm. Then X is a closed subspace of Y. Each of the operators $P(E_n)$, $n = 1,2,\ldots$, and $\frac{1}{2}P([0,1])$, has a natural extension to a continuous linear operator $\tilde{P}(E_n)$, $n = 1,2,\ldots$, and $\frac{1}{2}\tilde{P}([0,1])$, on Y, respectively (without an increase in norm).

Let D denote the linear span of the family of characteristic functions based on the dyadic intervals $I(k,r) = (k2^{-r},(k+1)2^{-r})$, $0 \le k < 2^r$, and $r = 1,2,\ldots.$ Since elements of X can be approximated in Y by elements of D, to show that $\tilde{P}(E_n) \to \frac{1}{2}\tilde{P}([0,1])$ in $L(X)$, it suffices to show that $\tilde{P}(E_n)(f) \to \frac{1}{2}\tilde{P}([0,1])(f)$ for each $f \in D$. This can be easily verified for each f of the form $\chi_{I(k,r)}$, $0 \le k < 2^r$, $r = 1,2,\ldots$, and the result follows.

To prove Theorem 1, let the space X be *separable* and $P : M \to L(X)$ be an operator-valued measure. Let $\{x_n\}_{n=1}^{\infty}$ be a countable dense set in X. It follows from [2, II Corollary 2] applied to the X-valued vector measure $P(\cdot)(x_n) : E \to P(E)(x_n)$, $E \in M$, that for each $n = 1,2,\ldots$, there exists a finite, positive measure λ_n on M, such that if $E \in M$ is a λ_n-null set, then $P(E)(x_n) = 0$. Let $\lambda = \sum_{n=1}^{\infty}\alpha_n\lambda_n$ where $\alpha_n > 0$, $n = 1,2,\ldots$, are chosen such that $\lambda(\Omega) < \infty$. It follows from the density of $\{x_n\}_{n=1}^{\infty}$ in X that $P(E) = 0$ for each λ-null set E of M. Hence, P is a closed measure by [2; IV Theorem 7.3]. This completes the proof of Theorem 1.

To prove Theorem 2, let X be an arbitrary Fréchet space. Suppose that $P : M \to L(X)$ is a closed spectral measure. It follows from [9; Corollary 13], that there exists a localizable measure λ on M such that $\langle P(\cdot)(x),x'\rangle$ is absolutely continuous with respect to λ for each $x \in X$ and $x' \in X'$. In particular, if $E \in M$ and $\lambda(E) = 0$, then $P(E) = 0$. Since X_n is dense in $\tilde{X}_n$, it follows from (2.3) that $\tilde{P}_n(E) = 0$, for each $n = 1,2,\ldots.$ Hence, $\tilde{P}_n$ is closed for each $n = 1,2,\ldots$; see [2; IV Theorem 7.3].

Conversely, suppose that $\tilde{P}_n$ is closed for each $n = 1,2,\ldots.$ Let $\{P(E_\alpha\}_{\alpha\in A}$ be a Cauchy net in $P(M)$. By the Proposition, it suffices to show that there exists a set $E \in M$ such that $P(E_\alpha) \to P(E)$, $\alpha \in A$, in L(X).

If $x \in X$ and n is a positive integer, then the identity

$$\|\tilde{P}_n(E_\alpha)([x]_n) - \tilde{P}_n(E_\beta)([x]_n\ \| = q_n(P(E_\alpha)(x) - P(E_\beta)(x)),$$

for each $\alpha,\beta, \in A$, shows that $\{\tilde{P}_n(E_\alpha)(\xi)\}_{\alpha\in A}$ is a bounded Cauchy net in $\tilde{X}_n$ for each $\xi \in X_n$. Since $\{\tilde{P}_n\ E_\alpha); \alpha \in A\}$ is uniformly bounded in $L(\tilde{X}_n)$, it follows that $\{\tilde{P}_n(E_\alpha)\}_{\alpha\in A}$ is a bounded Cauchy net in $L(\tilde{X}_n)$. As $\tilde{P}_n$ is closed, there is a set $E_n \in M$ such that

$$\tilde{P}_n(E_\alpha) \to \tilde{P}_n(E_n), \quad \alpha \in A, \tag{3.1}$$

in $L(\tilde{X}_n)$.

If $m \geq n$, then it follows from (2.4) that

$$\tilde{\psi}_{m,n} \circ \tilde{P}_m(E_\alpha) = \tilde{P}_n(E_\alpha) \circ \tilde{\psi}_{m,n}, \quad \alpha \in A.$$

Taking the limit with respect to $\alpha \in A$ gives

$$\tilde{\psi}_{m,n} \circ \tilde{P}_m(E_m) = \tilde{P}_n(E_n) \circ \tilde{\psi}_{m,n}, \tag{3.2}$$

for all $m \geq n$ and $n = 1,2,\ldots.$ It follows from (3.2) that

$$\tilde{P}_n(E_n) = \tilde{P}_n(E_k), \quad k \geq n, \; n = 1,2,\ldots. \tag{3.3}$$

Define measurable sets $F_n = \bigcap_{k=n}^{\infty} E_k$ for each $n = 1,2,\ldots.$ It follows from (3.3), the multiplicativity and σ-additivity of $\tilde{P}_n$ and the fact that the sequence of sets $\bigcap_{k=n}^{r} E_k$, $r \geq n$, decreases monotonically to F_n, that

$$\tilde{P}_n(E_n) = \tilde{P}_n(F_n) = \tilde{P}_n(F_k), \quad k \geq n, \; n = 1,2,\ldots. \tag{3.4}$$

Let $E = \bigcup_{n=1}^{\infty} F_n$. The identity (3.4) and the fact that $F_n \uparrow E$, implies that $\tilde{P}_n(E_n) = \tilde{P}_n(E)$ *for each* $n = 1,2,\ldots.$ It follows easily from (3.1) that $P(E_\alpha) \to P(E)$, $\alpha \in A$, in $L(X)$. This completes the proof.

Remarks. It is clear that Theorem 2 is valid for any sequence of seminorms q_n, $n = 1,2,\ldots$, determining the topology of X, provided it is compatible with respect to P.

If X is an arbitrary complete, locally convex Hausdorff space, not necessarily metrizable, then any spectral measure $P : M \to L(X)$ can be interpreted as the projective limit of a directed set of spactral measures $\tilde{P}_\alpha$, $\alpha \in A$, acting on Banach spaces $\tilde{X}_\alpha$ respectively, [8; Theorem 2.7]. The above proof shows that if $L(X)$ is quasi-complete and P is a closed measure, then so is each of the induced measures $\tilde{P}_\alpha$, $\alpha \in A$. This part of the proof does not rely on the fact that the topology of X is determined by a *countable* family of seminorms. However, the following example shows that this condition is essential for the validity of the converse statement.

Let $X = C^{[0,1]}$ denote the space of all complex-valued functions on [0,1], equipped with the topology of pointwise convergence. Let M be the σ-algebra of all Borel sets of [0,1]. Define a spectral measure $P : M \to L(X)$ by the formula

$$P(E) : f \to \chi_E f, \quad f \in X,$$

for each $E \in M$.

For each finite subset F of [0,1] define a continuous seminorm q_F on X, by the formula

$$q_F(f) = max\{|f(\omega)|; \; \omega \in F\}, \quad f \in X.$$

The family of seminorms $\{q_F; \; F \subset [0,1], \; F \textit{ finite}\}$, determines the topology of X and is compatible with respect to P. For each finite subset F of [0,1], the space $\tilde{X}_F$ can be identified with the finite dimensional space C^F and hence, the spectral measure $\tilde{P}_F$ is closed. However, the measure P is not closed.

REFERENCES

1. KLUVÁNEK, I. The range of vector-valued measure, *Math. Systems Theory* 7 (1973), 44-54.

2. KLUVÁNEK, I. and KNOWLES, G., Vector Measures and Control Systems, North Holland, Amsterdam, 1976.

3. BROOK, C.H. and GRAVES, W.H., Closed measures, Proc. Conf. on integration, Topology and Geometry in Linear Spaces, Amer. Math. Soc., Contemporary Math. 2 (1980), 145-160.

4. BROOK, C.H. and GRAVES, W.H., The range of a vector measure, *J. MATH. ANAL. AND Appl.* 73 (1980), 219-237.

5. RICKER, W., On Boolean algebras of projections and scalar-type spectral operators, *Proc. Amer. Math. Soc.* 87 (1983), 73-77.

6. KLUVÁNEK, I. Conical measures and vector measures, *Ann. Inst. Fourier (Grenoble)* 27 (1977), 83-105.

7. KÖTHE, G. Topological Vector Spaces II, Grundlehren der matematischen Wissenschaften 237, Springer-Verlag, New York-Heidelberg-Berlin, 1979.

8. WALSH, B. Structure of spectral measures on locally convex spaces, *Trans. Amer. Math. Soc.* 120 (1965), 295-326.