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ON THE EXISTENCE OF EQUATIONS OF EVOLUTION

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ABSTRACT. A time-independent, non-autonomous non-linear system governed by a principle of determinism (the state at a given time is determined by the initial state and by the control history during the intervening closed interval) is shown to obey a generalized evolution equation (1.2), where n is such that the state is continuously differentiable with respect to time whenever the control is of class C^n.

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1. INTRODUCTION.

The equations of evolution of time-independent, non-autonomous non-linear systems are almost universally taken as

$$u' = f(u,x) \tag{1.1}$$

where u is the output (or state) and x is the input (or control), both functions of time t with values in finite-dimensional vector spaces (say U and X, respectively), and the prime denotes differentiation with respect to time.

However, equation (1.1) is not the most general evolution equation. It may be regarded as a special case (corresponding to $n = 0$) of

$$u' = f(u,x,x',\ldots,x^{(n)}). \tag{1.2}$$

An example of the need for an equation of the form (2) occurs in the mechanics of inelastic continua. Here u is the vector whose components are the internal variables (or the inelastic strain), while x is stress or strain. Viscoelastic and viscoplastic materials are described by equation (1.1). Plastic materials, on the other hand, require equation (1.2) with $n = 1$. In particular, for a rate-independent material the function f must be first-degree homogeneous in x'.

The difference between systems described by equations (1.1) and (1.2) (or, more generally, by equations (1.2) with different values of n) lies in the character of the solutions u(t). If f is continuous, then the solution of (1.1) is continuously differentiable whenever x is a continuous function of time. On the other hand, solution of (1.2) will not in general be continuously differentiable unless x is of class C^n. In other words, the choice of n in equation (1.2) depends on the way in which the system smooths the input: the greater the smoothing, the lower the value of n.

However, the existence of an equation of evolution cannot be assumed *a priori* for an arbitrary system. In this note we shall try to find sufficient conditions for the existence of an equation of evolution, and to relate the value of n to the smoothing property of the system.

2. MAIN RESULTS

We shall assume the system to be governed by a principle of determinism as follows: the value of u at time $t+\tau$ is determined by its value at time t and by the history of x during the interval $[t,t+\tau]$. We shall express this mathematically as follows. Let x^t be defined by $x^t(s) = x(t+s)$, and let x^t_τ denote the restriction of x^t to $[0,\tau]$. The pair $(u(t), x^t_\tau)$ may be regarded as determining a *process* of duration τ in the system; let P_τ denote the set of all such pairs determining possible processes in the system. Then there exists a mapping $\phi_\tau : P_\tau \to U$ such that

$$u(t+\tau) = \phi_\tau(u(t), x^t_\tau) \tag{2.1}$$

Furthermore, let C^n_τ denote the Banach space $C^n([0,\tau];X)$, let $P^n_\tau = P_\tau \cap (U \times C^n_\tau)$. Then the smoothing property of the system may be expressed by saying that the left-hand side of (2.1) is differentiable with respect to τ whenever $(u(t), x^t_\tau) \in P^n_\tau$, that is, that the limit

$$\lim_{\tau\to 0+} \frac{1}{\tau}[\phi_\tau(u(t), x^t_\tau) - u(t)] \tag{2.2}$$

exists and is continuous in t under that condition. It is clear that this limit is determined by u(t) and by the behavior of x in a neighborhood of t. It is not immediately clear that it should take the form of the right-hand side of (1.2). In fact, in order to derive this result we need to assume some properties of the restriction to P^n_τ of the mapping ϕ_τ, as given in the following theorem.

THEOREM. Let $\phi_\tau : P^n_\tau \to U$ be such that $\phi_\tau(a,\cdot)$ is locally Lipschitz (with respect to the C^n_τ norm), the local Lipschitz norm F_τ being $O(\tau)$ as $\tau\to 0+$. Then the limit

$$\lim_{\tau\to 0+} \frac{1}{\tau}[\phi_\tau(a,y) - a],$$

when it exists, depends only on $a, y, y', \ldots, y^{(n)}$.

PROOF. Define $\bar{y} \in C^n_\tau$ by

$$\bar{y}(s) = y(0) + y'(0)s + \ldots + y^{(n)}(0)\frac{s^n}{n!}$$

Let $\|\cdot\|_\tau$ denote the C^n_τ norm and $|\cdot|$ any finite-dimensional norm. Then

$$\|y-\bar{y}\|_\tau = \max\{ \sup_{s\,[0,\tau]} |y(s) - y(0) - \ldots - y^{(n)}(0)s^n/n!|,$$

$$\sup_{s\,[0,\tau]} |y'(s) - y'(0) - \ldots - y^{(n)}(0)s^{n-1}/(n-1)!|, \ldots, \sup_{s\,[0,\tau]} |y^{(n)}(s) - y^{(n)}(0)| \}.$$

Since the argument of each $|\cdot|$ is a continuous function of s that vanishes at 0, each supremum goes to zero as $\tau \to 0$, so that $\|y-\bar{y}\|_\tau \to 0$ as $\tau \to 0$. Now

$$\frac{1}{\tau}[\phi_\tau(a,y) - a] = \frac{1}{\tau}[\phi_\tau(a,y) - \phi_\tau(a,\bar{y})] + \frac{1}{\tau}[\phi_\tau(a,\bar{y}) - a];$$

but for the first term on the right-hand side we have

$$\frac{1}{\tau}\left|\phi_\tau(a,y) - \phi_\tau(a,\bar{y})\right| \le \frac{F_\tau}{\tau}\|y - \bar{y}\|_\tau,$$

so that this term goes to zero as $\tau \to 0$. Consequently,

$$\lim_{\tau\to 0+} \frac{1}{\tau}[\phi_\tau(a,y) - a] = \lim_{\tau\to 0+} \frac{1}{\tau}[\phi_\tau(a,\bar{y}) - a],$$

and this last limit, whenever it exists, depends only on a and on the parameters defining $\bar{y}$, that is, $y(0)$, $y'(0),\ldots,y^{(n)}(0)$. Q.E.D.

By the assumed smoothing property of the system the limit exists and is a continuous function of its arguments, $f(a,y(0),y'(0),\ldots,y^{(n)}(0))$; it is this f that furnishes the right-hand side of equation (2).

As a very simple example, consider ϕ_τ given by

$$\phi_\tau(a,y) = a + y(\tau) - y(0).$$

This $\phi_\tau(a,\cdot)$ is Lipschitz (since it is a continuous linear mapping) on C^0_τ, but the Lipschitz norm over this space is 2. On the other hand, we may rewrite it as

$$\phi_\tau(a,y) = a + \int_0^\tau y'(s)ds,$$

so that it is clearly a continuous linear mapping of C^1_τ, and its (Lipschitz) norm is τ. Consequently, $n = 1$, and indeed we have $f(a,y,y') = y'$.

A more sophisticated example, relevant to plasticity theory, is

$$\phi_\tau(a,y) = a + \int_0^\tau |y'(s)|ds.$$

This $\phi_\tau(a,\cdot)$ is not Lipschitz on C^0_τ but it is Lipschitz on C^1_τ:

$$|\phi_\tau(a,y) - \phi_\tau(a,\bar{y})| = \left|\int_0^\tau (|y'(s)| - |\bar{y}'(s)|)ds\right| \le \int_0^\tau \big||y'(s)| - |y'(s)|\big| ds$$

$$\le \tau \sup_{s\,[0,\tau]} \big||y'(s)| - |\bar{y}'(s)|\big| \le \tau \sup_{s\,[0,\tau]} |y'(s) - \bar{y}'(s)| \le \tau\|y - \bar{y}\|_\tau,$$

so that once again $F_\tau = \tau$.