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ON THE KORTEWEG-DE VRIES EQUATION: AN ASSOCIATED EQUATION

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ABSTRACT. The purpose of this paper is to describe a relationship between the Korteweg-de Vries (KdV) equation

$$u_t - 6uu_x + u_{xxx} = 0$$

and another nonlinear partial differential equation of the form

$$z_t + z_{xxx} - 3\frac{z_x z_{xx}}{z} = H(t)z.$$

The second equation will be called the Associated Equation (AE) and the connection between the two will be explained. By considering AE, explicit solutions to KdV will be obtained. These solutions include the solitary wave and the cnoidal wave solutions. In addition, similarity solutions in terms of Airy functions and Painlevé transcendents are found. The approach here is different from the Inverse Scattering Transform and the results are not in the form of solutions to specific initial value problems, but rather in terms of solutions containing arbitrary constants.

KEY WORDS AND PHRASES. Korteweg-de Vries equation. Painlevé transcendent. Solitary waves.
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1. INTRODUCTION

This paper will describe a relationship between the Korteweg-de Vries (KdV) Equation

$$u_t - 6uu_x + u_{xxx} = 0 \tag{1.1}$$

and another nonlinear partial differential equation. We shall indicate how this connection arises and then use the result to generate solutions to KdV. The connection between KdV and the Associated Equation (AE) permits us to obtain some explicit solutions of (1.1). We remark at the outset that this method of solution is different from the Inverse Scattering Transform developed by Gardner et al [1], and we shall indicate the specific initial values satisfied by the solutions we obtain.

Numerous articles, including surveys by Miura [2] and by Jeffrey and Kakutani [3] recount the rich history and the enthusiasm generated by the KdV Equation. These include bibliographies which document the discovery of soliton solutions, the existence of an infinite number of associated polynomial conservation laws, and the development of the Inverse Scattering Transform. The adaptation of this last method of solution to other nonlinear evolution equations ensures that vigorous growth will continue.

Our principal concern in this paper is the development of an associated partial differential equation which can be solved explicitly to yield further solutions to KdV. The method is an outgrowth of the Miura Transformation, which we use in a slightly different way. This allows us to solve KdV, not in the context of a specific initial value problem, but rather in a form which is more akin to a complete integral, as there are two arbitrary constants present. We first indicate how the associated equation arises and then proceed to use it to solve KdV in some cases. Some remarks on the asymptotic behavior of the solutions we have obtained will be included, especially those in terms of the Airy functions and the Painlevé transcendents. Some of these solutions remain bounded as x approaches infinity, some even approach zero, while some of the others are unbounded.

2. THE DERIVATION OF THE ASSOCIATED EQUATION (AE)

In 1968, Miura [3] published a result giving a connection between two nonlinear partial differential equations. He discovered this in his research on the conservation laws associated with KdV, and we state his result as a

THEOREM: (Miura) If $v(x,t)$ is a solution of the modified Korteweg-de Vries Equation (MKdV)

$$v_t - 6v^2v_x + v_{xxx} = 0 \tag{2.1}$$

then

$$u(x,t) = v^2(x,t) + v_x(x,t) \tag{2.2}$$

solves the KdV Equation

$$u_t - 6uu_x + u_{xxx} = 0$$

The full impact of this result was realized in the invention of the Inverse Scattering Transform by observing that (2.2) is, in fact, a Ricatti Equation for $v(x,t)$. The usual linearizing transformation

$$v(x,t) = \frac{z_x}{z} \tag{2.3}$$

gives

$$u(x,t) = \frac{z_{xx}}{z} \tag{2.4}$$

If one observes that KdV is Galilean invariant, then (2.4) can be considered as the time-independent Schrödinger equation and, by exploiting this to its fullest, Gardner et al [1] developed the Inverse Scattering Transform. This method exactly linearizes KdV, but the explicit solution of an initial value problem is still far from trivial requiring the solution of the Schrödinger equation and the Gel'fand-Levitan integral equation.

In seeking other relationships, the second author pursued a somewhat different tack in using (2.4) from the Miura transformation directly in the KdV equation. Thus, we choose

$$u(x,t) = \frac{z_{xx}}{z}$$

and substitute this into the KdV equation.

Computing the required derivatives of (2.4), followed by substitution into (1.1) yields

$$\frac{z_{xxt}}{z} - \frac{z_t z_{xx}}{z^2} - 10\,\frac{z_{xx} z_{xxx}}{z^2} + 12\,\frac{z_x z_{xx}^2}{z^3} + \frac{z^{(5)}}{z}$$

$$- 3\,\frac{z_x z^{(4)}}{z^2} + 6\,\frac{z_x^2 z_{xxx}}{z^3} - 6\,\frac{z_x^3 z_{xx}}{z^4} = 0. \tag{2.5}$$

where the superscript (i) indicates $\frac{\partial^i}{\partial x^i}$. After some algebraic rearrangement, we may write (2.5) as

$$\frac{1}{z}\left[z_{xxt} - 9\,\frac{z_{xx} z_{xxx}}{z} + 9\,\frac{z_x z_{xx}^2}{z^2} + z^{(5)}\right.$$

$$\left. - 3\,\frac{z_x z^{(4)}}{z} + 6\,\frac{z_x^2 z_{xxx}}{z^2} - 6\,\frac{z_x^3 z_{xx}}{z^3}\right]$$

$$\frac{-z_{xx}}{z^2}\left[z_t + z_{xxx} - 3\,\frac{z_x z_{xx}}{z}\right] = 0. \tag{2.6}$$

The first bracketed term in (2.6) is simply the second partial derivative with respect to x of the second term, so we may write

$$\frac{1}{z}\left[z_t + z_{xxx} - 3\,\frac{z_x z_{xx}}{z}\right]_{xx} - \frac{z_{xx}}{z^2}\left[z_t + z_{xxx} - 3\,\frac{z_x z_{xx}}{z}\right] = 0. \tag{2.7}$$

We now denote this bracketed term by R; that is,

$$z_t + z_{xxx} - 3\,\frac{z_x z_{xx}}{z} = R \tag{2.8}$$

This last equation first appeared in the analysis by Gardner et al [1] in describing the Galilean invariance of the KdV equation by setting

$$\frac{z_{xx}}{z} = u + \lambda \tag{2.9}$$

Returning to equation (2.7) in the form simplified with the use of (2.8), we have

$$\frac{1}{z}R_{xx} - \frac{z_{xx}}{z^2}R = 0 \tag{2.10}$$

We observe that (2.10) can be written as

$$\left[zR_x\right]_x - \left[z_x R\right]_x = 0 \tag{2.11}$$

which permits integration with respect to x. This yields an arbitrary function of t only, G(t).

$$zR_x - z_x R = G(t) \tag{2.12}$$

Dividing by z^2 and integrating with respect to x, results in the solution for R in terms of z

$$R = z\int_{x_0}^{x} \frac{G(t)}{z^2}\,dx \tag{2.13}$$

The resulting equation for z is then given by

$$z_t + z_{xxx} - \frac{3z_x z_{xx}}{z} = z\int_{x_0}^{x} \frac{G(t)}{z^2}\,dx \tag{2.14}$$

which we refer to as the Associated Equation (AE). From the above discussion, we have that if z(x,t) satisfies (2.14) then

$$u(x,t) = \frac{z_{xx}}{z}$$

satisfies KdV and

$$v(xt) = \frac{z_x}{z}$$

satisfies MKdV.

Finally, we observe that (2.14) was obtained by Gardner et al [1] where the right-hand side took the form below with C,D arbitrary functions of t

$$R = Cz + Dz \int_{x_0}^{x} \frac{dx}{z^2} \tag{2.15}$$

It is at this point where we diverge from the development of Gardner et al [1] by attempting to solve AE in its own right as opposed to pursuing the method of the Inverse Scattering Transform.

3. EXPLICIT SOLUTIONS OF THE ASSOCIATED EQUATION (AE)

While AE appears at first, to be even more formidable than the original KdV equation, we remark that the homogeneity of the x and t derivatives permits a reduction of order in some cases. Specifically, we consider those cases for which $G(t) = 0$ and discuss either traveling wave solutions or similarity solutions. In these situations, direct integration yields explicit solutions for both KdV and MKdV. We distinguish two cases for explicit solution. These depend upon the form of the right-hand side in (2.14).

First Case: If we consider (2.8) with $R = 0$, then AE becomes

$$z_t + z_{xxx} - 3\frac{z_x z_{xx}}{z} = 0 \tag{3.1}$$

Suppose now that we inquire about traveling wave solutions to (3.1); that is, consider

$$z(x,t) = f(x - a^2 t) \tag{3.2}$$

Substitution of (3.2) into (3.1) results in the ordinary differential equation

$$- a^2 f' + f''' - 3\frac{f' f''}{f} = 0 \tag{3.3}$$

where ' indicates differentiation with respect to the argument, $x - a^2 t$.

If we divide by (3.3) by f^3, we may integrate once. Multiplication by $f'f^3$ permits an additional integration, which results in the first-order equation

$$(f')^2 = A^2 f^4 - \frac{a^2 f^2}{2} + B^2 \tag{3.4}$$

where A and B are arbitrary constants of integration. Specific choices for A and B give rise to explicit solutions for AE, KdV, and MKdV. We enumerate some of the possibilities below.

Case (1-a): If we choose $A^2 = B^2 = 0$ in (3.4), we obtain the solution to AE

$$z(x,t) = f(x - a^2 t) = x_0 \exp\left[\frac{ia}{\sqrt{2}} (x - a^2 t)\right] \tag{3.5}$$

where x_0 is a constant of integration.

Using (2.3) and (2.4), we obtain the corresponding constant solutions for KdV and MKdV.

Case (1-b): If we choose $A^2 = 0$ and $B^2 \neq 0$, we obtain the AE solution as

$$z(x,t) = f(x - a^2t) = \frac{\sqrt{2}\,B}{a} \sin\left[\frac{\pm a}{\sqrt{2}} (x - x_0 - a^2t)\right] . \tag{3.6}$$

The appropriate MKdV solution is

$$v(x,t) = \frac{a}{\sqrt{2}} \cot\left[\frac{a}{\sqrt{2}} (x - x_0 - a^2f)\right] \tag{3.7}$$

while the KdV solution again reduces to a constant.

Case (1-c): Suppose we now choose the integration constants in (3.4) as $A^2 \neq 0$ and $B^2 = 0$. The solution to AE has the form

$$z(x,t) = f(x - a^2t) = \frac{a}{\sqrt{2}A} \sec\left[\frac{aA}{\sqrt{2}} (x - x_0 - a^2t)\right] \tag{3.8}$$

Accordingly, the respective MKdV and KdV solutions become

$$v(x,t) = \frac{aA}{\sqrt{2}} \tan\left[\frac{aA}{\sqrt{2}} (x - x_0 - a^2t)\right] \tag{3.9}$$

and

$$u(x,t) = a^2A^2 \sec^2\left[\frac{aA}{\sqrt{2}} (x - x_0 - a^2t)\right] - \frac{a^2A^2}{2} . \tag{3.10}$$

Case (1-d): The choice of A and B such that $AB \neq 0$ yields the solution to AE in terms of the Jacobi elliptic function. For simplification, we denote

$$\alpha^2 = \frac{a^2}{4A^2} + \frac{\sqrt{a^4 - 4A^2B^2}}{2A^2} \tag{3.11}$$

and

$$\beta^2 = \frac{a^2}{4A^2} - \frac{\sqrt{a^4 - 4A^2B^2}}{2A^2} \tag{3.12}$$

The solution to AE is now

$$z(x,t) = f(x - a^2t) = \frac{1}{\alpha} \operatorname{sn}\left[\frac{x - x_0 - a^2t}{\beta} \,\middle|\, \frac{\beta^2}{\alpha^2}\right] \tag{3.13}$$

The MKdV solution is, therefore,

$$v(x,t) = \frac{1}{\beta} \frac{\mathrm{cn}\left[\dfrac{x - x_0 - a^2t}{\beta} \,\middle|\, \dfrac{\beta^2}{\alpha^2}\right] \mathrm{dn}\left[\dfrac{x - x_0 - a^2t}{\beta} \,\middle|\, \dfrac{\beta^2}{\alpha^2}\right]}{\mathrm{sn}\left[\dfrac{x - x_0 - a^2t}{\beta} \,\middle|\, \dfrac{\beta^2}{\alpha^2}\right]} \tag{3.14}$$

The solution to KdV is now the "cnoidal wave" as first discovered by Korteweg and de Vries [5]

$$u(x,t) = -\frac{1}{\beta^2} + \frac{\beta^2}{\alpha^4}\left[1 - 2\,\mathrm{cn}^2\left[\frac{x - x_0 - a^2t}{\beta} \,\middle|\, \frac{\beta^2}{\alpha^2}\right]\right] \tag{3.15}$$

Case (1-e): This is the last case we consider for traveling wave solutions to AE with R = 0. If we assume that the right-hand side of (3.4) is a perfect square, then we have

$$(f')^2 = K\,(f^2 - \sigma^2)^2 \tag{3.16}$$

We may further simplify the algebra by choosing K = 1 (or, equivalently, $\sigma = \frac{a}{2}$, A = 1, $B = \frac{a^2}{4}$). The solution to AE now has the form

$$z(x,t) = f(x - a^2t) = \frac{2}{a}\tanh\left[\frac{a}{2}(x - x_0 - a^2t)\right] . \tag{3.17}$$

The MKdV solution is, therefore,

$$v(x,t) = \frac{a}{2}\,\mathrm{sech}^2\left[\frac{a}{2}(x - x_0 - a^2t)\right]\ \mathrm{ctnh}\left[\frac{a}{2}(x - x_0 - a^2t)\right] . \tag{3.18}$$

Finally, we obtain the KdV solution here

$$u(x,t) = -\frac{a^2}{2}\,\mathrm{sech}^2\left[\frac{a}{2}(x - x_0 - a^2t)\right] . \tag{3.19}$$

We remark that equation (3.19) is the famous "one-soliton solution" obtained by Korteweg and de Vries [5]. We further observe that we have but one arbitrary constant x_0 in (3.19) due to the specific choices of A and B following (3.16). However, the solution in (3.15), the "cnoidal wave", reflects the integration process with three arbitrary constants, x_0, α, and β. Of course, A and B are used to define α and β. In the previous solutions, the choice of A or B as zero leads to two or fewer integration constants. The presence of these constants in the solution makes our results analogous to complete integral solutions.

Second Case: We now consider the solution of AE with $R = \frac{K(t)}{t} z$ where $K(t)$ is (as yet) undetermined. This arises from the choice of $G(t)$ in (2.13) still zero, however we permit the arbitrary constant (function of t) $K(t)$ to enter upon integration of the right-hand side of (2.14). The choice of $K(t)$ will be made shortly.

Suppose, therefore, we consider similarity solutions for AE of the form

$$z(x,t) = f(xt^{-1/3}) \ . \tag{3.20}$$

The motivation comes from the balance of x and t derivatives present in AE. Using (3.20), then we obtain the following

$$-\ 1/3\ f'\ xt^{-1/3}t^{-1} + f'''t^{-1} - \frac{3f'f''t^{-1}}{f} = \frac{K(t)}{t} f \tag{3.21}$$

where $' = \frac{d}{d\xi}$ with

$$\xi = xt^{-1/3} \tag{3.22}$$

Equation (3.21) can now be written in the form

$$\left[\frac{f''}{f^3}\right]' = \frac{K(t)}{f^2} + 1/3\ \frac{f'\xi}{f^3} \tag{3.23}$$

Now, by choosing $K(t) = \frac{-1}{6}$, we have

$$\left[\frac{f''}{f^3}\right]' = -\frac{1}{6}\left[\frac{\xi}{f^2}\right]' \tag{3.24}$$

which can be integrated to obtain

$$f'' = -\frac{1}{6}\xi f + Af^3 \tag{3.25}$$

where A is an arbitrary constant. By rescaling ξ as

$$\eta = (-6)^{-1/3}\xi = -x(6t)^{-1/3} \tag{3.26}$$

we obtain

$$\ddot{f} = \eta f + Af^3 \tag{3.27}$$

where $\cdot = \frac{d}{d\eta}$. Two cases now arise depending on the choice of the constant A.

Case (2-a): The easier case to consider in (3.27) is that for which $A = 0$. We obtain Airy's equation

$$\ddot{f} = \eta f \tag{3.28}$$

Hence, we have the solution to AE as

$$z(x,t) = C_1\ Ai(\eta) + C_2 Bi(\eta) \tag{3.29}$$

For simplicity, we choose $C_2 = 0$ and observe that the possible choice $C_1 = 0$ yields analogous results, while $C_1C_2 \neq 0$ provides a third result. Thus, we obtain the MKdV solution

$$v(x,t) = \frac{-(6t)^{-1/3}Ai'(\eta)}{Ai(\eta)} . \tag{3.30}$$

The KdV solution is

$$u(x,t) = \frac{-x}{6t} \tag{3.31}$$

where the Airy equation was used to simplify the result. Similar formulas arise for MKdV with $Bi(\eta)$ but the KdV solution is still the same.

Case (2-b): If we consider (3.27) with $A = 2$, we obtain the canonical form for the second Painlevé transcendent (with $\mu = 0$).

$$\ddot{f} = 2f^3 + \eta f \tag{3.32}$$

We denote the solution to (3.32) as $P(\eta)$ so that the solution to AE, in this case, is

$$z(x,t) = P(\eta) \tag{3.33}$$

The appropriate MKdV solution has the form

$$v(x,t) = \frac{-(6t)^{-1/3}P'(\eta)}{P(\eta)} \tag{3.34}$$

while the KdV solution is

$$u(x,t) = (6t)^{-2/3}P^2(\eta) - \frac{x}{6t} \tag{3.35}$$

The solutions that have been obtained in (3.34) and (3.35) in terms of the Painlevé transcendents have recently received considerable attention. Articles by Rosales [6], Miles [7] and [8], and by Ablowitz et al [9 - 11] have indicated additional relationships between KdV and these transcendents. We remark that our solutions do not arise in asymptotic formulas and are fundamentally different as they involve quotients and derivatives of the Painlevé functions.

This concludes our first set of solutions to KdV and MKdV. We now turn to AE itself with the intention of improving the utility of the equation.

4. A PROPERTY OF THE ASSOCIATED EQUATION (AE)

The treatment of KdV as a problem solvable via the methods available in inverse scattering problems was a significant step in the analysis of some nonlinear PDE's. The related problem from the inverse scattering setting is given in Ablowitz and Segur [12] by the pair of equations

$$z_{xx} + (\lambda^2 + u)\, z = 0$$

$$z_t - (\alpha(\lambda) + u_x)\, z - (4\lambda^2 - 2u)z_x = 0 \tag{4.1}$$

These equations in (4.1) also determine a Bäcklund Transformation from KdV to AE. See Wahlquist and Estabrook [13] for further details. The property which we exploit here is a symmetry condition that

$$w = \frac{1}{z} \tag{4.2}$$

leaves AE invariant. Thus, we can potentially increase the utility of AE twofold by using reciprocals of the AE solutions we have found in Section 3. Not all of these, however, generate new solutions for KdV and MKdV.

In the consideration of traveling wave solutions to AE in the first case above, we had found the solution in (1-e) as

$$z(x,t) = \frac{2}{a} \tanh\left[\frac{a}{2}(x - x_0 - a^2 t)\right]$$

We now have the reciprocal

$$w(x,t) = \frac{1}{z(x,t)} = \frac{a}{2} \operatorname{ctnh}\left[\frac{a}{2}(x - x_0 - a^2 t)\right] \tag{4.3}$$

as a solution to AE, which yields

$$v(x,t) = -\frac{a}{2} \operatorname{csch}^2\left[\frac{a}{2}(x - x_0 - a^2 t)\right] \tanh\left[\frac{a}{2}(x - x_0 - a^2 t)\right] \tag{4.4}$$

as the MKdV solution.

$$u(x,t) = \frac{a^2}{2} \operatorname{csch}^2\left[\frac{a}{2}(x - x_0 - a^2 t)\right] \tag{4.5}$$

is the resulting KdV solution.

It is unfortunate that the potential for new MKdV and KdV solutions is not fully realized when we treat traveling wave solutions. While we sometimes obtain new solutions to AE, the only additional KdV solution arises in case (1-e). The others repeat previous solutions.

In the case of similarity solutions, however, we do obtain different solutions by use of the symmetry property of AE. Recall from equation (3.29) that one solution of AE was

$$z(x,t) = C_1 Ai(\eta) \tag{4.6}$$

We have another solution to AE as

$$w(x,t) = \left[C_1 Ai(\eta)\right]^{-1} \tag{4.7}$$

The corresponding MKdV solution is

$$v(x,t) = (6t)^{-1/3} \frac{Ai'(\eta)}{Ai(\eta)} \tag{4.8}$$

while we obtain the KdV solution in this case

$$u(x,t) = \frac{x}{6t} + 2\ (6t)^{-2/3}\left[\frac{Ai'(\eta)}{Ai(\eta)}\right]^2 \tag{4.9}$$

The analogous solution involving Bi (η) and the linear combination of Ai (η) and Bi (η) also hold.

We may also utilize $w(x,t)$ to include the Painlevé solutions, too, by way of the solution to AE in (3.33). This yields

$$w(x,t) = \left[\ P(\eta)\ \right]^{-1} \tag{4.10}$$

as the AE solution. This implies that the solution to MKdV is

$$v(x,t) = (6t)^{-1/3}\ \frac{P'(\eta)}{P(\eta)} \tag{4.11}$$

The final KdV solution is

$$u(x,t) = \frac{x}{6t} - 2\ (6t)^{-2/3}\ P^2(\eta)$$

$$+ 2\ (6t)^{-2/3}\left[\frac{P'(\eta)}{P(\eta)}\right]^2 \tag{4.12}$$

We now have completed our set of explicit solutions to KdV and MKdV via the associated equation.

5. DISCUSSION OF RESULTS

It was noted from the start that our method here was fundamentally different from the Inverse Scattering Transform. It is not necessary to determine the scattering data nor the solution of the Gel'fand - Levitan integral equation. The implementation of an initial condition

$$u(x,0) = f(x) \tag{5.1}$$

may be introduced by means of the transformation in (2.4) with $t = 0$; that is,

$$u(x,0) = f(x) = \frac{z_{xx}(x,0)}{z(x,0)} \tag{5.2}$$

This direct scattering problem needs to be solved for $z(x,0)$ which is then used as the initial condition for AE.

In the explicit solutions we obtained above, we did not approach the problem as a specific initial value problem, but rather we derived the traveling wave solutions and similarity solutions as a result of direct integration of the ordinary differential equation which arose from AE. The appropriate initial values for the traveling wave solutions may be read *a posteriori* from the solutions in the first case by setting $t = 0$. The appropriate initial values which correspond to the similarity solutions may be interpreted as those arising when t is replaced by $t + k$. For example, the KdV similarity solution (3.31) becomes

$$u(x,t) = -\frac{x}{6(t+k)} \tag{5.3}$$

with the corresponding initial value

$$u(x,0) = -\frac{x}{6k} \tag{5.4}$$

This same approach allows the assignment of initial values to the other similarity solutions in the second case.

A few remarks concerning the assignment of specific initial values are in order. First, for the solution of KdV and MKdV via AE, we are still faced with the formidable task of solving (5.2) first and then using this in the solution of AE. Second, the solution of (5.2) will include two arbitrary constants arising from boundary conditions on x. Perhaps this may be utilized in solution of some initial-boundary value problems for KdV. Some investigations of KdV on finite intervals have been considered by Bubnov [14]. The advantage here, of course, is that we need not rely on asymptotic values of x, but rather we might prescribe values for finite x.

Finally, we consider the behavior of the solutions which we have obtained. The traveling wave solutions have already received extensive treatments in Gardner et al [1], Miura [2], and Korteweg and de Vries [5]. It was, of course, the discovery of the solitary waves which initially stirred the interest in KdV. The similarity solutions, most notably the Painlevé functions, have been discussed in many recent articles, Rosales [6], Miles [7-8], Ablowitz and Segur [9], Ablowitz et al [10-11]. We observe that the asymptotic behavior of our solutions (as $t \to \infty$) can be considered most reasonably in terms of the size of $|x|$ versus t, with the distinction of bounded solutions pertaining to that part of the domain for which $|x| \leq (t^{1/3})$.

In the case for which $|x| < (t^{1/3})$, we remark that the quotients in the similarity solutions approach constants, so that the multiplicative factor t^{-r} forces the solutions to zero. For example, in (3.30) as $t \to \infty$, we obtain

$$\lim_{\substack{t \to \infty \\ |x| < (t^{1/3})}} \left[-(6t)^{-1/3} \frac{Ai'(\eta)}{Ai(\eta)} \right] = 0 \tag{5.5}$$

since $Ai(0) = 3^{-2/3}/\Gamma(2/3) \approx .35502$ and $Ai'(0) = -(3)^{-1/3}/\Gamma(1/3) \approx -.25881$ as given by Abramowitz and Stegun [15]. Notice that the quotient of the Airy functions approaches a constant as $t \to \infty$ and the $t^{-1/3}$ drives the solution to zero. Similar results hold for the solutions in (3.34), (4.8), and (4.11). With an additive term of $x(6t)^{-1}$ or a multiplicative factor of $t^{-2/3}$, we still obtain solutions which approach zero eventually. In the case where $|x| = (t^{1/3})$, the only difference is that the quotients of the Airy functions, or the quotients of the Painlevé functions, remain essentially constant with the t^{-r} forcing the solutions to zero.

We obtain unbounded solutions in those situations for which $|x| > (t^{1/3})$. For large values of the arguments $\eta = x(6t)^{-1/3}$, the quotients of the Airy functions grow at the rate of $\eta^{1/2}$. When we consider $|x| > (t^{1/3})$, this yields an algebraic pattern of growth on the order of $x^{1/2}(t)^{-1/6}$ for the Airy functions which is them tempered by $t^{-1/3}$ multiplier in the MKdV solution. This results in a solution which just escapes when $|x|$ is only slightly larger then $t^{1/3}$. Of course, as $|x|$ increases with respect to the relative size of $t^{1/3}$, we expect the greater rate of algebraic growth. This result is valid for the KdV and MKdV solutions involving the Airy Function. The results for the Painlevé solutions require further analysis. Preliminary numerical results, however, indicate that these solutions are also unbounded.

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APPENDIX I

TABLE I

$z(x,t)$ is the solution to AE with $R = 0$. $v(x,t)$ is the solution to MKdV. $u(x,t)$ is the solution of KdV. $\xi = x - x_0 - a^2 t$.

$z(x,t)$	$v(x,t)$	$u(x,t)$	Reference
$a_0 \exp\left[\frac{ia}{\sqrt{2}}\xi\right]$	$\frac{ia}{\sqrt{2}}$	$-\frac{a^2}{2}$	(3.5)
$\frac{\sqrt{2}B}{a}\sin\left[\frac{a}{\sqrt{2}}\xi\right]$	$\frac{a}{\sqrt{2}}\cot\left[\frac{a}{\sqrt{2}}\xi\right]$	$-\frac{a^2}{2}$	(3.6) (3.7)
$\frac{a}{\sqrt{2}A}\sec\left[\frac{aA}{\sqrt{2}}\xi\right]$	$\frac{aA}{\sqrt{2}}\tan\left[\frac{aA}{\sqrt{2}}\xi\right]$	$a^2A^2\sec^2\left[\frac{aA\xi}{\sqrt{2}}\right] - \frac{a^2A^2}{2}$	(3.8) (3.9) (3.10)
$\frac{1}{\alpha}\,\mathrm{sn}\left[\frac{\xi}{\beta} \mid \frac{\beta^2}{\alpha^2}\right]$	$\frac{1}{\beta}\,\frac{\mathrm{cn}\left[\frac{\xi}{\beta} \mid \frac{\beta^2}{\alpha^2}\right]\mathrm{dn}\left[\frac{\xi}{\beta} \mid \frac{\beta^2}{\alpha^2}\right]}{\mathrm{sn}\left[\frac{\xi}{\beta} \mid \frac{\beta^2}{\alpha^2}\right]}$	$\frac{\beta^2}{\alpha^4} - \frac{1}{\beta^2} - \frac{2\beta^2}{\alpha^4}\,\mathrm{cn}^2\left[\frac{\xi}{\beta} \mid \frac{\beta^2}{\alpha^2}\right]$	(3.13) (3.14) (3.15)
$\frac{2}{a}\tanh\left[\frac{a}{2}\xi\right]$	$\frac{a}{2}\,\mathrm{sech}^2\left[\frac{a}{2}\xi\right]\,\mathrm{ctnh}\left[\frac{a}{2}\xi\right]$	$-\frac{a^2}{2}\,\mathrm{sech}^2\left[\frac{a}{2}\xi\right]$	(3.17) (3.18) (3.19)
$\frac{a}{2}\,\mathrm{ctnh}\left[\frac{a}{2}\xi\right]$	$-\frac{a}{2}\,\mathrm{csch}^2\left[\frac{a}{2}\xi\right]\tanh\left[\frac{a}{2}\xi\right]$	$\frac{a^2}{2}\,\mathrm{csch}^2\left[\frac{a}{2}\xi\right]$	(4.3) (4.4) (4.5)

TABLE II

$z(x,t)$ is the solution to AE with $R = \frac{K(t)}{t} z$; $v(x,t)$ is the solution to MKdV; $u(x,t)$ is the solution to KdV; $\eta = -x(6t)^{-1/3}$.

$z(x,t)$	$v(x,t)$	$u(x,t)$	Reference
$Ai(\eta)$	$-(6t)^{-1/3} \frac{Ai'(\eta)}{Ai(\eta)}$	$-\frac{x}{6t}$	(3.29) (3.30) (3.31)
$Bi(\eta)$	$-(6t)^{-1/3} \frac{Bi'(\eta)}{Bi(\eta)}$	$-\frac{x}{6t}$	(3.29) (3.30) (3.31)
$\frac{1}{Ai(\eta)}$	$(6t)^{-1/3} \frac{Ai'(\eta)}{Ai(\eta)}$	$\frac{x}{6t} + 2(6t)^{-2/3} \left[\frac{Ai'(\eta)}{Ai(\eta)}\right]^2$	(4.7) (4.8) (4.9)
$\frac{1}{Bi(\eta)}$	$(6t)^{-1/3} \frac{Bi'(\eta)}{Bi(\eta)}$	$\frac{x}{6t} + 2(6t)^{-2/3} \left[\frac{Bi'(\eta)}{Bi(\eta)}\right]^2$	(4.7) (4.8) (4.9)
$P(\eta)$	$-(6t)^{-1/3} \frac{P'(\eta)}{P(\eta)}$	$-\frac{x}{6t} + 2(6t)^{-2/3} P^2(\eta)$	(3.33) (3.34) (3.35)
$\frac{1}{P(\eta)}$	$(6t)^{-1/3} \frac{P'(\eta)}{P(\eta)}$	$\frac{x}{6t} + 2(6t)^{2/3} \left[\frac{P'(\eta)}{P(\eta)}\right]^2 - 2(6t)^{-2/3} P^2(\eta)$	(4.10) (4.11) (4.12)