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RESEARCH NOTES

A RELATIONSHIP BETWEEN THE MODIFIED EULER METHOD AND e

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ABSTRACT. Approximating solutions to the differential equation $dy/dx = f(x,y)$ where $f(x,y) = y$ by a generalization of the modified Euler method yields a sequence of approximates that converge to e. Bounds on the rapidity of convergence are determined, with the fastest convergence occuring when the parameter value is $\frac{1}{2}$, so the generalized method reduces to the standard modified Euler method. The situation is similarly examined when f is altered.

KEY WORDS AND PHRASES. *Euler method, modified Euler method.*

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1. INTRODUCTION.

The Euler method is known as a simple, but crude,method for approximating solutions to differential equations. The modified Euler method offers greater refinement, as shown in Ross [1]. Let us recall in this setting we wish to solve the equation $dy/dx = f(x,y)$ subject to the condition $y(x_0) = y_0$. We let h denote a positive increment in x and define $x_k = x_0 + kh$. To approximate the exact solution y at x_k, $y(x_k) = y_k$, we construct a sequence of approximates $y_k^{(1)}$, $y_k^{(2)}$, ... which converge to y_k. Proceeding inductively we get y_{k+1} by considering the sequence:

$$y_{k+1}^{(1)} = y_k + hf(x_k,y_k) \tag{1.1}$$

$$y_{k+1}^{(2)} = y_k + (h/2)[f(x_k,y_k) + f(x_{k+1},y_{k+1}^{(1)})] \tag{1.2}$$

while in general

$$y_{k+1}^{(n)} = y_k + (h/2)[f(x_k,y_k) + f(x_{k+1},y_{k+1}^{(n-1)})]. \tag{1.3}$$

When successive terms in this sequence are close enough, we set their common value equal to y_{k+1}. With this in mind, we can consider the equation

$$y_{k+1} = y_k + h[\tfrac{1}{2}f(x_k,y_k) + \tfrac{1}{2}f(x_{k+1},y_{k+1})] \tag{1.4}$$

as defining the solution points by the modified Euler method (MEM).

If we consider the specific differential equation with $f(x,y) = y$, and the side condition $y(0) = 1$, with an increment of $h = 1/n$ we get the values

$$y_n = [\frac{2n+1}{2n-1}]^n = (1 + \frac{1}{n - .5})^{(n - .5) + .5}. \tag{1.5}$$

This produces a sequence that converges to e (as was to be expected since $y' = y$).

The modified Euler method fits into a more general scheme given by

$$y_{k+1} = y_k + h[pf(x_k,y_k) + (1-p)f(x_{k+1},y_{k+1})] \tag{1.6}$$

where $0 \le p \le 1$. If we now apply this generalized method (call it M_pEM) to the same differential equation as above, we get a general term of

$$y_n = [\frac{n+p}{n-(1-p)}]^n = [1 + \frac{1}{n-(1-p)}]^n \tag{1.7}$$

and clearly y_n approaches e. We note here that $p = 1$ produces the same sequence as the Euler method, and $p = \frac{1}{2}$ produces the same sequence as the modified Euler method.

2. <u>MAIN RESULTS</u>.

One is now led to ask which value of p yields the sequence that best approximates e. The expression for y_n suggests one could examine the family of functions

$$f_p(x) = (1 + 1/x)^{x + (1-p)}. \tag{2.1}$$

These functions fall into one of three types, depending on the size of p. The function f_p is decreasing for $p \leq .5$, is increasing for $p > (-1 + \sqrt{5})/2$, and is decreasing at first then eventually increasing for $.5 < p < (-1 + \sqrt{5})/2$. The reader is referred to the articles by Darst, Dence and Polya [2-4] for further details on this. It follows that $p = .5$ yields the best approximation to e because any value p' greater than .5 can be improved upon by, say, $(p' + .5)/2$. Perhaps Euler knew something that we haven't given him credit for when he chose $p = \frac{1}{2}$ instead of an alternate weighting system!

To determine how quickly $f_{.5}$ converges to e, we wish to find N such that $x > N$ implies $|f_{.5}(x) - e|$ is bounded above by $\varepsilon > 0$. To this end we have

$$f_{.5}(x) - f_{.51}(x) = (1 + \frac{1}{x})^x [e^{.50\ln(1 + 1/x)} - e^{.49\ln(1 + 1/x)}] \tag{2.2}$$

$$= (1 + \frac{1}{x})^x \sum_{i=0}^{\infty} (.50^i - .49^i)\ln^i(1 + 1/x)(1/i!) \tag{2.3}$$

$$< e \sum_{i=0}^{\infty} .01(i)(.50^{i-1})\ln^i(1 + 1/x)(1/i!) \tag{2.4}$$

$$< .01e \sum_{i=1}^{\infty} i(.50^{i-1})(1/x)^i(1/i!) \tag{2.5}$$

$$< .01e \sum_{i=1}^{\infty} 2^{1-i}x^{-i} \tag{2.6}$$

$$= e/[50(2x - 1)], \qquad \text{for } |2x| > 1. \tag{2.7}$$

Since $f_{.5}(x) - e < .5[f_{.5}(x) - f_{.51}(x)] < e/[100(2x - 1)]$ for all sufficiently large x, the difference between $f_{.5}$ and e can be made small enough by choosing x greater than $.5[1 + e/(100\varepsilon)]$. For example, with $\varepsilon = .0001$ we then choose $x > 136.4$ and get $f_{.5}(137) = 2.7182938$ and the difference $f_{.5}(137) - e = .000012$.

If we now consider the slightly more general initial value problem $dy/dx = f(x,y) = Ay$ with side condition $y(0) = 1$ then, using M_pEM, we get

$$y_1 = y_0 + (1/n)[pAy_0 + (1-p)Ay_1] = 1 + (1/n)[pA + (1-p)Ay_1] \tag{2.8}$$

so

$$y_1 = \frac{n + Ap}{n + (p-1)A} \tag{2.9}$$

and then

$$y_2 = y_1 + (1/n)[pAy_1 + (1-p)Ay_2] \tag{2.10}$$

so $y_2 = y_1^{\,2}$. The n-th term is given by $y_n = y_1^{\,n}$, or

$$y_n = \left[1 + \frac{A}{n - (1-p)A}\right]^n \tag{2.11}$$

so y_n converges to e^A. Furthermore, since y_n is of the form $(1 + A/x)^{x + (1-p)A}$, insight into the behavior of y_n can be gained by examining the related family of sequences

$$(1 + A/n)^{Bn + C} \tag{2.12}$$

with A,B,C real. We shall consider A as positive in what follows.

Case 1. Set $a_n = (1 + A/n)^{n + \alpha}$ and $b_n = (1 + A/n)^{-n + \alpha}$ and define the number $\gamma(A)$ by

$$\gamma(A) = \frac{2\ln(1 + A/2) - \ln(1 + A)}{\ln(1 + A) - \ln(1 + A/2)} > 0. \tag{2.13}$$

The motivation for this is that $\gamma(A)$ is the limiting value of α as n tends to ∞ for which $a_n = a_{n+1}$. By methods analagous to those used by the author in [3] we know that $\{a_n\}$ is increasing if $\alpha < \gamma(A)$, decreasing if $\alpha \geq A/2$, and initially decreasing then eventually increasing if $\gamma(A) < \alpha < A/2$. Because b_n is basically a reciprocal of a_n it follows that the monotonicity of $\{b_n\}$ is increasing if $\alpha \leq -A/2$, decreasing if $\alpha > -\gamma(A)$, and initially increasing then eventually decreasing if $-A/2 < \alpha < -\gamma(A)$.

Case 2. Set $c_n = (1 - A/n)^{n + \alpha}$ and $d_n = (1 - A/n)^{-n + \alpha}$, with $n > A$, and define the number $\gamma(A)$ by

$$\gamma(A) = \frac{([A] + 2)\ln(1 - \frac{A}{[A] + 2}) - ([A] + 1)\ln(1 - \frac{A}{[A] + 1})}{\ln(1 - \frac{A}{[A] + 1}) - \ln(1 - \frac{A}{[A] + 2})} < 0 \tag{2.14}$$

where the brackets denote the greatest integer function. Similar to above we have that $\{c_n\}$ is increasing if $\alpha \geq -A/2$, decreasing if $\alpha < \gamma(A)$, and initially increasing then eventually decreasing if $\gamma(A) < \alpha < -A/2$, and that $\{d_n\}$ is increasing if $\alpha > -\gamma(A)$, decreasing if $\alpha \leq A/2$, and initially decreasing then eventually increasing if

$A/2 < \alpha < -\gamma(A)$. Because of cases 1 and 2 we can determine the monotonicity of (2.12) from the identity

$$(1 + \frac{A}{n})^{Bn + C} = \left[(1 + \frac{A}{n})^{(\mathrm{sgn}\ B)n + C/|B|}\right]^{|B|} . \tag{2.15}$$

Furthermore, since (2.11) is of the form

$$(1 + A/x)^{x + (1-p)A} \tag{2.16}$$

it follows that the fastest convergence to e^A is when $(1-p)A = A/2$, or $p = \frac{1}{2}$. This is because $(1 + A/n)^{n + \alpha}$ is decreasing to e^A for $\alpha \geq A/2$, with the fastest convergence at $\alpha = A/2$. We remark here that some of the above monotonicity properties could be alternately derived by examining the logarithm of $(1 + A/x)^{Bx + C}$.

The rapidity of this convergence can be discussed by considering the functions $f_p(x)$, given by (2.16), and noting (same technique as before) that,

$$f_{.5}(x) - f_{.51}(x) < e^A[e^{.50A\ln(1 + A/x)} - e^{.49A\ln(1 + A/x)}] \tag{2.17}$$

$$< .01e^A \sum A^i(A/x)^i 2^{1-i} \tag{2.18}$$

$$= A^2e^A/[50(2x - A^2)]. \tag{2.19}$$

Table 1 lists some data for this situation.

x	$f_{.50}(x)$	$f_{.51}(x)$	$f_{.50}(x)-f_{.51}(x)$	$\frac{A^2e^A}{50(2x - A^2)}$
10	20.43377	20.27357	.16020	.32867
50	20.10259	20.06748	.03511	.03972
100	20.08992	20.07211	.01781	.01892
400	20.08581	20.08131	.00450	.00457

Table 1 (A = 3)

For large enough x we have

$$f_{.50}(x) - e^A < \tfrac{1}{2}[f_{.50}(x) - f_{.51}(x)] < \frac{A^2e^A}{100(2x-A^2)} \tag{2.20}$$

and for this difference to be less than $\varepsilon > 0$ just choose x greater than $.5[A^2 + A^2e^A/(100\ )]$. For example, with $\varepsilon = .001$, we choose x = 999 and get $f_{.50}(x) - e^3 = .00004$.

3. CONCLUDING REMARKS.

Noticing how critical the value $p = \frac{1}{2}$ is on the efficiency of convergence prompts one to characterize those functions f(x,y) which fall under this classification. Knowing this to be true for f(x,y) = Ay, we can now show it to be true for the elementary functions $f(x,y) = x^m$, with the side condition (0,0), and for m = 0,1, 2,3,... : (we know $y = x^{m+1}/(m+1)$ and $y(1) = 1/(m+1)$). Using M_pEM of (1.6) and $h = 1/n$ we get

$$y_n = \frac{(1-p)\sum_{i=1}^{n} i^m + p\sum_{i=1}^{n-1} i^m}{n^{m+1}} = \frac{\sum_{i=1}^{n} i^m - pn^m}{n^{m+1}} . \qquad (3.1)$$

But $\sum_{i=1}^{n} i^m$ is expressable as a polynomial $p(n) = \sum_{i=1}^{m+1} a_i n^i$ with $a_{m+1} = 1/(m+1)$ and $a_m = \frac{1}{2}$. Thus y_n can be written as

$$y_n = \frac{\left(\frac{1}{m+1} n^{m+1} + a_{m-1}n^{m-1} + \ldots + a_1 n\right) + \left(\frac{1}{2} n^m - pn^m\right)}{n^{m+1}} \qquad (3.2)$$

and this expression converges to $1/(m+1)$ fastest when $p = \frac{1}{2}$. Likewise it follows that $p = \frac{1}{2}$ whenever $f(x,y)$ is a polynomial in x. Further classifications of f appear to be more difficult to obtain.

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