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ON THE STRONG MATRIX SUMMABILITY OF DERIVED FOURIER SERIES

K. N. MISHRA and R. S. L. SRIVASTAVA

Department of Mathematics
Indian Institute of Technology
Kanpur 208016, India

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ABSTRACT. Strong summability with respect to a triangular matrix has been defined and applied to derived Fourier series yielding a result which extends some known results under a general criterion.

KEY WORDS AND PHRASES. Strong Summability, Toeplitz matrix, Fourier Series.
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1. INTRODUCTION.

The triangular matrix $A = [a_{n,k}]$, $n, k = 0,1,\ldots$ and $a_{n,k} = 0$ for $k > n$ is regular if

$$\lim_{n\to\infty} a_{n,k} = 0 ,$$

$$\sum_{k=0}^{n} |a_{n,k}| \le M, \ M \text{ is independent of } n$$

and

$$\lim_{n\to\infty} \sum_{k=0}^{n} a_{n,k} = 1$$

Denoting the sum $\sum_{r=1}^{k} u_r$ by s_k, Fekete [1], defined that the series Σu_r is strongly summable to the sum s, provided

$$\sum_{k=1}^{n} |s_k - s| = o(n) .$$

This type is now known as strong Cesàro summability of order unity with index 1 or [C,1] summability.

The series Σu_r is said to be strongly summable by Cesàro means, with index q, or summable [C,q], or summable H_q to the sum s if

$$\sum_{k=1}^{n} |s_k - s|^q = o(n) .$$

A special point of interest in the method of summability H_q lies in the fact that it is given neither by Toeplitz matrix nor by a sequence to function transformation. The relationship between summability H_q and some regular methods of summation given by A- matrices has been investigated by Kuttner, [2], who proved that if A is any regular Toeplitz method of summability then for any q ($0 < q < 1$) there is a series which is not summable A but summable H_q.

In the present paper we shall define strong summability of series Σu_k with the help of a matrix.

DEFINITION. The series Σu_k is said to be strongly summable by the regular method A determined by the matrix $[a_{n,k}]$ with index $q (q > o)$ to the sum s if

$$\sum_{k=o}^{n} a_{n,k}|s_k - s|^q = o(1), \quad \text{as} \quad n \to \infty .$$

For $a_{n,k} = \frac{1}{n+1}$, $k \le n$, we get (C,1) matrix.

2. MAIN RESULTS.

Let f(x) be a periodic function with period 2π and integrable (L) over $(-\pi,\pi)$. Let

$$f(x) \sim \frac{1}{2} a_o + \sum_{1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{2.1}$$

be the Fourier series of f(x) and

$$\sum_{1}^{\infty} n(b_n \cos n x - a_n \sin nx) \tag{2.2}$$

be the first derived series of (2.1) obtained by term by term differentiation. Write

$$g(u) = f(x+u) - f(x-u) - 2uf'(x) \quad , \tag{2.3}$$

where $f'(x)$ is the derivative of f(x),

$$G(t) = \int_{0}^{t} |dg(u)| \; . \tag{2.4}$$

Here we shall take $q = 1,2$. Since the case $q = 1$ is included in the strong summability for $q = 2$, we omit the same. Precisely we prove the following:

THEOREM. Let g(u), G(t) be defined as in (2.3) and (2.4). If g(u) is a continuous function of bounded variation over $[0,\pi]$ and for some $\beta \ge 1$

$$G(t) = o\,[t\,\lambda^{\beta}(t)] \; , \quad \text{as } t \to o \; , \tag{2.5}$$

where $\lambda^{\beta}(t)$ is a positive function of t such that

$$\lambda^{\beta}(t) \to o \quad \text{as} \quad t \to o \; , \tag{2.6}$$

it is monotonic in (n^{-1},δ) (δ being small but fixed) and

$$\int_{n^{-1}}^{\delta} \frac{\lambda^{2\beta}(t)}{t}\, dt = O(1) \tag{2.7}$$

then the derived series (2.2) is strongly summable to $f'(x)$ by the matrix (C,1) with index 2.

Note (2.7) is equivalent to $\frac{\lambda^2(t)}{t} \in L(o,\delta)$.

In order to prove the theorem we need the following lemma.

LEMMA. If $G(t) = o(t)$ as $t \to o$ then for small but fixed δ

$$\text{(i)} \quad \int_{n^{-1}}^{\delta} \frac{|dg(t)|}{t}\, dt \int_{n^{-1}}^{\delta} \frac{|dg(u)|}{u}\, du = o(n)$$

and

$$\text{(ii)} \quad \int_{n^{-1}}^{\delta} \frac{|dg(t)|}{t^2}\, dt \int_{n^{-1}}^{t} \frac{|dg(u)|}{u}\, du = o(n) .$$

PROOF. Since

$$\int_{n^{-1}}^{\delta} \frac{|dg(u)|}{u}\, du = \left[\frac{G(u)}{u}\right]_{n^{-1}}^{\delta} + \int_{n^{-1}}^{\delta} \frac{G(u)}{u^2}\, du$$

$$= o(1) + \int_{n^{-1}}^{\delta} o(\tfrac{1}{u})\, du, \text{ in view of (2.4) },$$

$$= o(\log n) ,$$

Therefore

$$\int_{n^{-1}}^{\delta} \frac{|dg(t)|}{t}\, dt \int_{n^{-1}}^{\delta} \frac{|dg(u)|}{u}\, du = o(\log n)^2 = o(n) .$$

Again

$$\int_{n^{-1}}^{\delta} \frac{|dg(t)|}{t^2}\, dt \int_{n^{-1}}^{\delta} \frac{|dg(u)|}{u}\, du$$

$$= \int_{n^{-1}}^{\delta} \frac{|dg(t)|}{t^2} \left\{ \left[\frac{G(u)}{u}\right]_{n^{-1}}^{t} + \int_{n^{-1}}^{t} \frac{G(u)}{u^2}\, du\right\} dt$$

$$= \int_{n^{-1}}^{\delta} \frac{|dg(t)|}{t^2} \left\{ \frac{G(t)}{t} + o(1) + o(\log n\, t)\right\} dt$$

$$= o(1) \left\{ \int_{n^{-1}}^{\delta} \frac{dg(t)}{t^2} \log nt\right\}$$

$$= o \left\{\left[\frac{G(t)}{t^2} \log nt\right]_{n^{-1}}^{\delta} - \int_{n^{-1}}^{\delta} \frac{G(t)}{t^3}\, dt + 2 \int_{n^{-1}}^{\delta} \frac{G(t)}{t^3} \log nt\, dt \right\}$$

$$= o(n) + o \left(\int_{1}^{n\delta} (1/u^2)\, du\right) + o \left[\int_{1}^{n\delta} (\log u/u^2)\, du\right]$$

$$= o(n).$$

3. PROOF OF THE THEOREM.

The kth partial sum $\sigma_k(x)$ of the series (2.2) is given by [3],

$$\sigma_k(x) - f'(x) = \frac{1}{2\pi}\int_0^{\pi} \frac{\sin (k+1/2)t}{\sin \frac{1}{2} t}\, dg(t) .$$

Further, simplifying certain steps as given by [3] and [4] we have

$$\sigma_k(x) - f'(x) = \frac{1}{\pi}\int_{n^{-1}}^{\pi} \frac{\sin kt}{t}\, dg(t) + o(1)$$

$$= \frac{1}{\pi}\left\{ \int_{n^{-1}}^{\delta} + \int_{\delta}^{\pi} \right\} \frac{\sin kt}{t}\, dg(t) + o(1) .$$

Therefore

$$\sum_{k=1}^{n} \{\sigma_k(x) - f'(x)\}^2 = \frac{1}{\pi^2}\int_{n^{-1}}^{\delta} \frac{dg(t)}{t}\int_{n^{-1}}^{\delta} \left\{ \sum_1^n \sin kt \sin ku \right\} \frac{dg(u)}{u} + o(n)$$

$$= \frac{1}{2\pi^2}\int_{n^{-1}}^{\delta} \frac{dg(t)}{t}\int_{n^{-1}}^{\delta} \sum_1^n \{\cos k(u-t) - \cos k(u+t)\} \frac{dg(u)}{u} + o(n)$$

$$= \frac{1}{2\pi^2}\int_{n^{-1}}^{\delta} \frac{dg(t)}{t}\int_{n^{-1}}^{\delta} \frac{\sin(n+1/2)(u-t)}{2\sin\frac{1}{2}(u-t)}\, \frac{1}{u}\, dg(u)$$

$$- \frac{1}{2\pi^2}\int_{n^{-1}}^{\delta} \frac{dg(t)}{t}\int_{n^{-1}}^{\delta} \frac{\sin(n+1/2)(u-t)}{2\sin\frac{1}{2}(u+t)}\, \frac{1}{u} dg(u) + o(n) .$$

On simplifying and using the first part of the lemma we obtain

$$\sum_{k=1}^{n} \{\sigma_k(x) - f'(x)\}^2 = \frac{1}{2\pi^2}\int_{n^{-1}}^{\delta} \frac{dg(t)}{t}\int_{n^{-1}}^{\delta} \frac{\sin n(u-t)}{u(u-t)}\, dg(u)$$

$$- \frac{1}{2\pi^2}\int_{n^{-1}}^{\delta} \frac{dg(t)}{t}\int_{n^{-1}}^{\delta} \frac{\sin n(u+1)}{(u+t)}\, \frac{dg(u)}{u} + o(n)$$

$$= P_1 + P_2 + o(n), \text{ say.}$$

Now, since

$$\frac{1}{u(u-t)} = \frac{1}{t}\left\{\frac{1}{u-t} - \frac{1}{u}\right\}$$

and

$$\int_{n^{-1}}^{\delta} \frac{dg(t)}{t}\int_{t}^{\delta} \frac{\sin n(u-t)}{u(u-t)}\, dg(u) = \int_{n^{-1}}^{\delta} \frac{dg(u)}{u}\int_{n^{-1}}^{u} \frac{\sin n(u-t)}{t(u-t)}\, dg(t) .$$

Therefore

$$P_1 = \frac{1}{2\pi^2}\int_{n^{-1}}^{\delta} \frac{dg(t)}{t}\int_{n^{-1}}^{t} \frac{\sin n(u-t)}{u(u-t)}\, dg(u) + \frac{1}{2\pi^2}\int_{n^{-1}}^{\delta} \frac{dg(t)}{t}\int_{t}^{\delta} \frac{\sin n(u-t)}{u(u-t)}\, dg(u)$$

$$= \frac{1}{\pi^2}\int_{n^{-1}}^{\delta} \frac{dg(t)}{t}\int_{n^{-1}}^{t} \frac{\sin n(u-t)}{u(u-t)}\, dg(u)$$

$$= \frac{1}{\pi^2} \int_{n^{-1}}^{\delta} \frac{dg(t)}{t^2} \int_{n^{-1}}^{t} \left(\frac{1}{u-t} \frac{1}{u} \right) \sin n(u-1)\, dg(u)$$

$$= \frac{1}{\pi^2} \int_{n^{-1}}^{\delta} \frac{dg(t)}{t^2} \int_{n^{-1}}^{t} \frac{\sin n(u-t)}{(u-t)}\, dg(u) + o\left[\int_{n^{-1}}^{\delta} \frac{|dg(t)|}{t} \int_{n^{-1}}^{t} \frac{|dg(u)|}{u} \right]$$

$$= \frac{1}{\pi^2} \int_{n^{-1}}^{\delta} \frac{dg(t)}{t^2} \int_{n^{-1}}^{t} \frac{\sin n(u-t)}{(u-t)}\, dg(u) + o(n)$$

by virtue of the second part of the lemma.

Similarly it can be proved that $P_2 = o(n)$. Thus we get

$$\sum_{k=1}^{n} \{\sigma_k(x) - f'(x)\}^2 = \frac{1}{\pi^2} \int_{n^{-1}}^{\delta} \frac{dg(t)}{t^2} \int_{n^{-1}}^{t} \frac{\sin n(u-t)}{u(u-t)}\, dg(u) + o(n) .$$

Integration by parts gives

$$\int_{n^{-1}}^{t} dg(u) \frac{\sin n\,(u-t)}{(u-t)} = \left[\frac{\sin n\,(u-t)}{(u-t)} \int_{n^{-1}}^{t} dg(u) \right]_{n^{-1}}^{t}$$

$$- \int_{n^{-1}}^{t} \left[\left\{ \frac{n \cos n(u-t)}{(u-t)} - \frac{\sin n\,(u-t)}{(u-t)^2} \right\} dg(u) \right] du .$$

Using (2.5) this is equal to

$$\left[\frac{\sin n(u-t)}{(u-t)}\, o\,\{t\, \lambda^{\beta}\,(t)\} \right]_{n^{-1}}^{t} - o\left[\int_{n^{-1}}^{t} \{n\, t^{\beta} \lambda\,(t)\} \quad \frac{\cos n(u-t)}{(u-t)}\, du \right]$$

$$+ o\left[\int_{n^{-1}}^{t} \frac{\sin n(u-t)}{(u-t)^2} \{t\, \lambda^{\beta}(t)\}\, du \right]$$

$$= o\,[\, n\, t\, \lambda^{\beta}(t)\,] .$$

Therefore

$$\sum_{k=1}^{n} \{\sigma_k(x) - f'(x)\}^2 = o\left[n \int_{n^{-1}}^{\delta} \frac{dg(t)}{t} \lambda^{\beta}(g)\right] + o(n)$$

$$= o(n)\,[G(t)\, \lambda^{\beta}(t)]_{n^{-1}}^{\delta} + o(n)\left[\int_{n^{-1}}^{\delta} dg(t) \frac{\lambda^{\beta}(t)}{t^2}\, dt\right]$$

$$+ o(n)\left[\int_{n^{-1}}^{\delta} \frac{G(t)}{t} \{ \beta\, \lambda^{\beta-1}(t)\, \lambda'(t)\}\, dt\right]$$

$$= o(n) + o(n)\left[\int_{n^{-1}}^{\delta} \frac{\lambda^{2\beta}(t)}{t}\, dt\right]$$

$$+ o(n)\left[\int_{n^{-1}}^{\delta} \beta\, \lambda^{\beta}(t)\, \lambda^{\beta-1}(t)\, \lambda'(t)\, dt\right]$$

$$= o(n) + o(n) \left[\int_{n^{-1}}^{\delta} \frac{1}{2} \frac{d}{dt} \{ \lambda^{2\beta}(t) \} dt \right]$$

$$= o(n) \text{ by the hypothesis (2.7).}$$

Since $\lambda^{\beta}(t)$ is monotonic, hence its differential coefficient is of constant sign. Thus we get

$$\sum_{k=1}^{n} |\sigma_k(x) - f'(x)|^2 = o(n)$$

and therefore

$$\sum_{k=1}^{n} a_{n,k} |\sigma_k(x) - f'(x)|^2 = o(n) .$$

This completes the proof of the theorem.

4. SPECIAL CASES.

By way of an application of our theorem, we take $\beta = 1$, $\lambda(t) = 1/\log(1/t)$ and $a_{n,k} = 1$ then the following result follows, [4]:

THEOREM (Sharma). At a point for which $f'(x)$ exists and

$$G(t) = o[t/\log \tfrac{1}{t}] \text{ as } t \to o ,$$

then

$$\sum_{k=1}^{n} |\sigma_k(x) - f'(x)|^2 = o(n \log\log n) .$$

Since the above theorem is an extension of the result from [C, 1] summability to the case of [C, 2] summability, (Prasad and Singh [3]), our theorem further extends that result under a general type of criterion.

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