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A REPRESENTATION OF JACOBI FUNCTIONS

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Abstract: Recently, the continuous Jacobi transform and its inverse are defined and studied in [1] and [2]. In the present work, the transform is used to derive a series representation for the Jacobi functions $P_\lambda^{(\alpha,\beta)}(x)$, $-\frac{1}{2} \le \alpha,\ \beta \le \frac{1}{2},\ \alpha+\beta=0$, and $\lambda \ge -\frac{1}{2}$. The case $\alpha=\beta=0$ yields a representation for the Legendre functions and has been dealt with in [3]. When λ is a positive integer n, the representation reduces to a single term, viz., the Jacobi polynomial of degree n.

KEY WORDS AND PHRASES: Jacobi functions, Jacobi transform, representation, special functions.

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1. Introduction.

The continuous Jacobi transform and its inverse were introduced and studied in [1] and [2]. These transforms generalize the work of Butzer, Stens and Wehrens [3] on the continuous Legendre transform and the work of Debnath [4] on the discrete Jacobi transform. In [2] an application to sampling technique was given. In the present work, the continuous Jacobi transform is used to derive a series representation of Jacobi functions $P_\lambda^{(\alpha,\beta)}(x)$. The representation includes that for the Legendre function given in [3]. When λ is a positive integer, the representation reduces to the Jacobi polynomial (see e.g. [5]).

2. Preliminaries. In this section we review material needed in the development of the paper.

For α, $\beta > -1$, $\lambda \in R$, $\lambda + \alpha + \beta \neq 0,-1,-2,\ldots$ and $x \in (-1,1]$, the Jacobi function of the first kind, $P_\lambda^{(\alpha,\beta)}(x)$, is given by

$$P_\lambda^{(\alpha,\beta)}(x) = \frac{\Gamma(\lambda+\alpha+1)}{\Gamma(\lambda+1)\Gamma(\alpha+1)} F(-\lambda,\lambda+\alpha+\beta+1;\quad \alpha+1;\ \frac{1-x}{2}) \tag{2.1}$$

(see [6]) where

$$F(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k k!} z^k, \quad |z| < 1,$$

a,b,c real numbers with $c \neq 0,-1,-2,\ldots$.

Since $P_\lambda^{(\alpha,\beta)}(x) = \frac{\Gamma(\alpha-\lambda+1)\Gamma(\lambda-\alpha-\beta)}{\Gamma(1-\lambda)\Gamma(\lambda-\beta)} P_{\lambda-\alpha-\beta-1}^{(\alpha,\beta)}(x)$, we may restrict ourselves to $\lambda \geq -\frac{\alpha+\beta+1}{2}$. The function $P_\lambda^{(\alpha,\beta)}(x)$ satisfies the following relations:

$$\frac{d}{dx}(w(x)(1-x^2)\frac{d}{dx} P_\lambda^{(\alpha,\beta)}(x)) = -\lambda(\lambda+\alpha+\beta+1)\ w(x)P_\lambda^{(\alpha,\beta)}(x) \tag{2.2}$$

$$P_\lambda^{(\alpha,\beta)}(1) = \frac{\Gamma(\lambda+\alpha+1)}{\Gamma(\lambda+1)\Gamma(\alpha+1)}, \tag{2.3}$$

and

$$(1-x^2)\frac{d}{dx} P_\lambda^{(\alpha,\beta)}(x) = (\frac{\lambda(\alpha-\beta)}{2\lambda+\alpha+\beta} - \lambda x)P_\lambda^{(\alpha,\beta)}(x)$$

$$+ \frac{2(\lambda+\alpha)(\lambda+\beta)}{2\lambda+\alpha+\beta} P_{\lambda-1}^{(\alpha,\beta)}(x). \tag{2.4}$$

For a proof of (2.2), (2.3) and (2.4) see [1]. The term w(x) in (2.2) is the weight function $w(x) = (1-x)^\alpha(1+x)^\beta$ and will be used throughout the paper. Furthermore, it was shown in [1] that for $\lambda \geq -\frac{\alpha+\beta+1}{2}$ and for any $x \in (-1,1]$.

$$|P_\lambda^{(\alpha,\beta)}(x)| \leq \frac{\Gamma(\lambda+\alpha+1)}{\Gamma(\lambda+1)\Gamma(\alpha+1)} + M(\lambda,\alpha,\beta)\log\frac{2}{1+x} \tag{2.5}$$

where $M(\lambda,\alpha,\beta)$ is some constant depending upon λ, α and β; and for any λ, $\nu \geq -\frac{\alpha+\beta+1}{2}$, $\lambda \neq \nu$, $\lambda \neq -(\nu+\alpha+\beta+1)$, $\alpha > -\frac{1}{2}$, $-\frac{1}{2} < \beta < \frac{1}{2}$ we have the relation

$$\frac{1}{2^{\alpha+\beta+1}} \int_{-1}^{1} w(x)P_\lambda^{(\alpha,\beta)}(x)P_\nu^{(\beta,\alpha)}(-x)dx \tag{2.6}$$

$$= \frac{\Gamma(\lambda+\alpha+1)\Gamma(\nu+\beta+1)}{\Gamma(\lambda-\nu)(\lambda+\nu+\alpha+\beta+1)}\{\frac{\sin\pi\lambda}{\Gamma(\nu+1)\Gamma(\lambda+\alpha+\beta+1)} - \frac{\sin\pi\nu}{\Gamma(\lambda+1)\Gamma(\nu+\alpha+\beta+1)}\}.$$

We shall denote, throughout, the weighted square integrable functions on (-1,1) by $L_w^2(-1,1)$. For $f \in L_w^2(-1,1)$, $\alpha > -\frac{1}{2}$, $-\frac{1}{2} < \beta < \frac{1}{2}$, the continuous Jacobi transform (see [1]) is defined by

$$\hat{f}^{(\alpha,\beta)}(\lambda) = \frac{1}{2^{\alpha+\beta+1}} \int_{-1}^{1} w(x) P_\lambda^{(\alpha,\beta)}(x) f(x)\,dx \tag{2.7}$$

When $\alpha = \beta = 0$, $\hat{f}^{(\alpha,\beta)}$ reduces to the continuous Legendre transform studied in [3] and when $\lambda = n \in P$ (P, the set of non-negative integers), $\hat{f}^{(\alpha,\beta)}$ reduces to the discrete Jacobi transform of Debnath [4].

It was shown in [1] that if $\lambda^{\frac{1}{2}} \hat{f}^{(\alpha,\beta)}(\lambda - \frac{1}{2}) \in L^1(R^+)$ and if $\alpha+\beta = 0$ then for almost every $x \in (-1,1)$, we obtain the inversion formula

$$f(x) = 4\int_0^\infty \hat{f}^{(\alpha,\beta)}(\lambda-\tfrac{1}{2}) P_{\lambda-\frac{1}{2}}^{(\alpha,\beta)}(-x) H_0(\lambda)\, \lambda \sin\pi\lambda\, d\lambda \tag{2.8}$$

where

$$H_0(\lambda) = \frac{\Gamma^2(\lambda+\frac{1}{2})}{\Gamma(\lambda+\alpha+\frac{1}{2})\Gamma(\lambda+\beta+\frac{1}{2})} .$$

Since we needed the condition $\alpha+\beta = 0$ to derive (2.8), we shall, from now on, assume this condition on α and β.

In [2] the second continuous Jacobi transform was studied. For $\lambda^{-\beta+\frac{1}{2}} f \in L^1(R^+)$, it is given by

$$\check{f}^{(\alpha,\beta)}(x) = 4\int_0^\infty f(\lambda) P_{\lambda-\frac{1}{2}}^{(\beta,\alpha)}(-x) \frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\lambda+\beta+\frac{1}{2})} \lambda \sin\pi\lambda\, d\lambda \tag{2.9}$$

and the associated inversion formula is

$$f(\lambda) = \tfrac{1}{2} \frac{\Gamma(\lambda+\alpha+\frac{1}{2})}{\Gamma(\lambda+\frac{1}{2})} \int_{-1}^{1} w(x) P_{\lambda-\frac{1}{2}}^{(\alpha,\beta)}(x)\, \check{f}^{(\alpha,\beta)}(x)\,dx \tag{2.10}$$

The relation between the different transforms (see [2]) is

$$(\check{f}^{(\alpha,\beta)}(\cdot))^{\wedge(\alpha,\beta)}(\lambda) = \frac{2\Gamma(\lambda+\alpha+\frac{1}{2})}{\Gamma(\lambda+\frac{1}{2})} f(\lambda)$$

and

$$\left(\frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\lambda+\alpha+\frac{1}{2})} \hat{f}^{(\alpha,\beta)}(\cdot)\right)^{\vee(\alpha,\beta)}(x) = f(x).$$

As an application of (2.9) and (2.10), it was shown in [2] that if $F \in C(R^+)$ is given by

$$F(\lambda) = \tfrac{1}{2}\int_{-1}^{1} w(x) f(x) P_{\mu\lambda-\frac{1}{2}}^{(\alpha,\beta)}(x)\,dx$$

for some $\mu > 0$, $f \in L^2_w(-1,1)$, then for all $\lambda \in \mathbf{R}^+$, we have

$$F(\lambda) = \sum_{n=0}^{\infty} \frac{(2n+1)\Gamma(n+1)\Gamma(\mu\lambda+\alpha+\frac{1}{2})\sin\pi(\lambda\mu-(n+\frac{1}{2}))}{\pi(\lambda^2\mu^2-(n+\frac{1}{2})^2)\Gamma(n+\alpha+1)\Gamma(\lambda\mu+\frac{1}{2})} F(\frac{n+\frac{1}{2}}{\mu}). \tag{2.11}$$

We will employ (2.7), (2.8), (2.9) and (2.10) to derive the representation formula of the Jacobi functions. Since $\alpha + \beta = 0$, we shall write $P_\lambda^{(\alpha,\beta)}(x)$ as $P_\lambda^{(\alpha,-\alpha)}(x)$.

3. Derivation of the Representation Formula. Again, throughout this section we shall assume $\alpha+\beta = 0$, $-\frac{1}{2} < \alpha$, $\beta < \frac{1}{2}$ and $\alpha \neq 0$. The case $\alpha = 0$ reduces to the representation of the Legendre functions and has been developed in [3].

The series representation that we will develop, in this section, for $P^{(\alpha,-\alpha)}(x)$ is

$$P_\lambda^{(\alpha,-\alpha)}(x) = \frac{\Gamma(\lambda+\alpha+1)}{\Gamma(\lambda+1)\Gamma(\alpha+1)} \frac{\sin\pi\lambda}{\pi} \cdot$$

$$\cdot \{\lambda(\lambda+1)\sum_{n=1}^{\infty} \frac{(2n+1)n!(-1)^n P_n^{(\alpha,-\alpha)}(x)}{n(n+1)(\alpha+1)_n(\lambda-n)(\lambda+n+1)} + 1$$

$$+ \frac{1}{\lambda(\lambda+1)} - \sum_{n=0}^{\infty} \frac{(-1)^n}{\alpha+n+1} (\frac{1-x}{1+x})^{n+1}\}, \quad 0 \le x < 1, \tag{3.1}$$

and

$$P_\lambda^{(\alpha,-\alpha)}(x) = \frac{\Gamma(\lambda+\alpha+1)}{\Gamma(\lambda+1)\Gamma(\alpha+1)} \frac{\sin\pi\lambda}{\pi} \cdot$$

$$\{\lambda(\lambda+1)\sum_{n=1}^{\infty} \frac{(2n+1)n!(-1)^n P_n^{(\alpha,-\alpha)}(x)}{n(n+1)(\alpha+1)_n(\lambda-n)(\lambda+n+1)} + 1 + \frac{1}{\lambda(\lambda+1)}$$

$$- \frac{1}{\alpha} + (\frac{1+x}{1-x})^\alpha \frac{\pi}{\sin\pi\alpha} + \sum_{n=0}^{\infty} \frac{(-1)^n}{\alpha-n-1} (\frac{1+x}{1-x})^{n+1}, \quad -1 < x \le 0. \tag{3.2}$$

In order to derive (3.1) and (3.2), we shall first introduce an auxillary function $k(x;h)$, apply (2.7), (2.9) to $k(x;h)$ and utilize the uniqueness of the Jacobi transform.

Lemma 3.1. For $h \in (-1,1)$, define

$$k(x;h) = \begin{cases} \frac{1}{\alpha}\left[(\frac{1+x}{1-x})^\alpha - (\frac{1+h}{1-h})^\alpha\right], & h \le x < 1, \\ 0 & , -1 < x \le h \end{cases}$$

Then

$$\hat{k}^{(\alpha,-\alpha)}(x;h)(\lambda) = \begin{cases} \frac{1}{\lambda(\lambda+1)}\{\frac{\Gamma(\lambda+\alpha+1)}{\Gamma(\lambda+1)\Gamma(\alpha+1)} - P_\lambda^{(\alpha,-\alpha)}(h)\}, & \lambda\neq 0,\ \lambda\geq -\frac{1}{2} \\ \frac{1}{2\alpha}\{1-h-(\frac{1+h}{1-h})^\alpha \int_h^1 (\frac{1-x}{1+x})^\alpha dx\}, & \lambda=0. \end{cases}$$

Proof. (2.2) together with (2.7) yields for $\lambda\neq 0$ and $\alpha+\beta=0$

$$\hat{k}^{(\alpha,-\alpha)}(\cdot;h)(\lambda) = \tfrac{1}{2}\int_{-1}^{1}(1-x)^\alpha(1+x)^{-\alpha}P_\lambda^{(\alpha,-\alpha)}(x)k(x;h)dx$$

$$= -\tfrac{1}{2}\frac{1}{\lambda(\lambda+1)}\int_{-1}^{1}\frac{d}{dx}((1-x)^{\alpha+1}(1+x)^{-\alpha+1}\frac{d}{dx}P_\lambda^{(\alpha,-\alpha)}(x))k(x;h)dx.$$

On integrating by parts, we obtain

$$\hat{k}^{(\alpha,-\alpha)}(\cdot;h)(\lambda) = \frac{1}{\lambda(\lambda+1)}\int_h^1 \frac{d}{dx}P_\lambda^{(\alpha,-\alpha)}(x)dx$$

from which it follows that for $\lambda\neq 0$, $\lambda\geq -\frac{1}{2}$

$$\hat{k}^{(\alpha,-\alpha)}(\cdot;h)(\lambda) = \frac{1}{\lambda(\lambda+1)}\{P_\lambda^{(\alpha,-\alpha)}(1)-P_\lambda^{(\alpha,-\alpha)}(h)\}$$

Equivalently,

$$\hat{k}^{(\alpha,-\alpha)}(\cdot;h)(\lambda) = \frac{1}{\lambda(\lambda+1)}\{\frac{\Gamma(\lambda+\alpha+1)}{\Gamma(\lambda+1)\Gamma(\alpha+1)} - P_\lambda^{(\alpha,-\alpha)}(h)\}$$

from (2.3).

When $\lambda=0$, $P_0^{(\alpha,-\alpha)}(x)=1$. This together with (2.7) yields

$$\hat{k}^{(\alpha,-\alpha)}(\cdot;h)(0) = \tfrac{1}{2}\int_{-1}^{1}(1-x)^\alpha(1+x)^{-\alpha}k(x;h)dx$$

$$= \tfrac{1}{2}\int_h^1(1-x)^\alpha(1+x)^{-\alpha}\{\frac{1}{\alpha}[(\frac{1+x}{1-x})^\alpha-(\frac{1+h}{1-h})^\alpha]\}dx$$

$$= \frac{1}{2\alpha}[1-h-(\frac{1+h}{1-h})^\alpha\int_h^1(\frac{1-x}{1+x})^\alpha dx].$$

This completes the proof of Lemma 3.1.

Since $\lambda^{\frac{1}{2}}\hat{k}^{(\alpha,-\alpha)}(\cdot,h)(\lambda-\frac{1}{2})\in L^1(R^+)$ and since $k(x;h)$ is continuous on $(-1,1)$, it follows from (2.8) and Lemma 3.1 that for $\lambda\neq 0$

$$k(x;h) = 4\int_0^\infty \frac{1}{\lambda^2-\frac{1}{4}}\{\frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\lambda-\alpha+\frac{1}{2})\Gamma(\alpha+1)} - \frac{\Gamma^2(\lambda+\frac{1}{2})P_{\lambda-\frac{1}{2}}^{(\alpha,-\alpha)}(h)}{\Gamma(\lambda+\alpha+\frac{1}{2})\Gamma(\lambda-\alpha+\frac{1}{2})}\}.$$

$$\cdot P_{\lambda-\frac{1}{2}}^{(-\alpha,\alpha)}(-x)\lambda\sin\pi\lambda d\lambda. \tag{3.3}$$

From (2.11) with $\mu=1$, $\sigma\geq 0$, $h\in(-1,1)$ and Lemma 3.1, we have

$$\hat{k}^{(\alpha,-\alpha)}(\cdot,h)(\sigma-\tfrac{1}{2}) = \frac{1}{\sigma^2-\frac{1}{4}}\left[\frac{\Gamma(\sigma+\alpha+\frac{1}{2})}{\Gamma(\sigma+\frac{1}{2})\Gamma(\alpha+\frac{1}{2})} - P_{\sigma-\frac{1}{2}}^{(\alpha,-\alpha)}(h)\right]$$

$$= \frac{\Gamma(\sigma+\alpha+\frac{1}{2})\hat{k}^{(\alpha,-\alpha)}(\cdot;h)(0)\sin\pi(\sigma-\frac{1}{2})}{\pi(\sigma^2-\frac{1}{4})\Gamma(\sigma+\frac{1}{2})\Gamma(\alpha+1)} +$$

$$+ \sum_{n=1}^{\infty} \frac{(2n+1)\Gamma(n+1)\Gamma(\sigma+\alpha+\frac{1}{2})\sin\pi(\sigma-n-\frac{1}{2})}{\pi(\sigma^2-(n+\frac{1}{2})^2)\Gamma(n+\alpha+1)\Gamma(\sigma+\frac{1}{2})} \frac{1}{n(n+1)} \cdot$$

$$\cdot\left[\frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)n!} - P_n^{(\alpha,-\alpha)}(x)\right]$$

where $\hat{k}^{(\alpha,-\alpha)}(\cdot;h)(0)$ is as given in Lemma 3.1. Replacing σ by $\lambda+\frac{1}{2}$ in the above expression together with Lemma 3.1 and the uniqueness of the Jacobi transform imply

$$\frac{1}{\lambda(\lambda+1)}\left[\frac{\Gamma(\lambda+\alpha+1)}{\Gamma(\alpha+1)\Gamma(\lambda+1)} - P_\lambda^{(\alpha,-\alpha)}(h)\right] = \frac{\Gamma(\lambda+\alpha+1)\sin\pi\lambda\ \hat{k}^{(\alpha,-\alpha)}(x;h)(0)}{\pi(\lambda)(\lambda+1)\Gamma(\alpha+1)\Gamma(\lambda+1)} +$$

$$+ \sum_{n=1}^{\infty} \frac{(2n+1)\Gamma(n+1)\Gamma(\lambda+\alpha+1)\sin\pi(\lambda-n)}{\pi(\lambda-n)(\lambda+n+1)\Gamma(n+\alpha+1)\Gamma(\lambda+1)} \frac{1}{n(n+1)}\left[\frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)n!} - P_n^{(\alpha,-\alpha)}(x)\right].$$

Therefore,

$$P_\lambda^{(\alpha,-\alpha)}(h) = \frac{\Gamma(\lambda+\alpha+1)}{\Gamma(\alpha+1)\Gamma(\lambda+1)} - \frac{\Gamma(\lambda+\alpha+1)\sin\pi\lambda\ \hat{k}^{(\alpha,-\alpha)}(\cdot,h)(0)}{\pi\Gamma(\alpha+1)\Gamma(\lambda-1)} -$$

$$- \sum_{n=1}^{\infty} \frac{\lambda(\lambda+1)(2n+1)\Gamma(\lambda+\alpha+1)\sin\pi(\lambda-n)}{\pi(\lambda-n)(\lambda+n+1)\Gamma(\lambda+1)n(n+1)\Gamma(\alpha+1)}$$

$$+ \sum_{n=1}^{\infty} \frac{\lambda(\lambda+1)(2n+1)n!\Gamma(\lambda+\alpha+1)\sin\pi(\lambda-n)P_n^{(\alpha,-\alpha)}(h)}{\pi(\lambda-n)(\lambda+n+1)\Gamma(n+\alpha+1)\Gamma(\lambda+1)n(n+1)} \tag{3.4}$$

From (2.7) we now have

$$\hat{P}_\lambda^{(\alpha,-\alpha)}(0) = \tfrac{1}{2}\int_{-1}^{1}(1-x)^\alpha(1+x)^{-\alpha}P_0^{(\alpha,-\alpha)}(x)P_\lambda^{(\alpha,-\alpha)}(x)dx$$

which together with the above expression for $\hat{P}_\lambda^{(\alpha,-\alpha)}(h)$ yields

$$\hat{P}_\lambda^{(\alpha,-\alpha)}(0) = \tfrac{1}{2}\int_{-1}^{1}(1-x)^\alpha(1+x)^{-\alpha}\left[\frac{\Gamma(\lambda+\alpha+1)}{\Gamma(\alpha+1)\Gamma(\lambda+1)}\right.$$

$$- \frac{\Gamma(\lambda+\alpha+1)\sin\pi\lambda\ \hat{k}^{(\alpha,-\alpha)}(x,h)(0)}{\pi\Gamma(\alpha+1)\Gamma(\lambda+1)}$$

$$- \sum_{n=1}^{\infty} \frac{(2n+1)\lambda(\lambda+1)\Gamma(\lambda+\alpha+1)\sin\pi(\lambda-n)}{\pi(\lambda-n)(\lambda+n+1)\Gamma(\lambda+1)n(n+1)\Gamma(\alpha+1)}$$

$$\left.+ \sum_{n=1}^{\infty} \frac{(2n+1)\lambda(\lambda+1)\Gamma(\lambda+\alpha+1)\sin\pi(\lambda-n)P_\lambda^{(\alpha,-\alpha)}(x)}{\pi(\lambda-n)(\lambda+n+1)\Gamma(n+\alpha+1)\Gamma(\lambda+1)n(2n+1)}\right] dx$$

Using Euler's formula [5]

$$\int_0^x (x-t)^\alpha t^\beta dt = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta-2)} x^{\alpha+\beta+1}, \quad \alpha, \quad \beta > -1 \tag{3.5}$$

with $\alpha+\beta=0$, $t=1+u$, we obtain

$$\int_{-1}^{1} (1-x)^\alpha (1+x)^{-\alpha}\, dx = 2\Gamma(\alpha+1)\Gamma(1-\alpha).$$

This together with Lemma 3.1 yields

$$\begin{aligned}\hat{P}_\lambda^{(\alpha,-\alpha)}(0) &= \frac{\Gamma(1-\alpha)\Gamma(\lambda+\alpha+1)}{\Gamma(\lambda+1)} - \\ &- \sum_{n=1}^{\infty} \frac{\lambda(\lambda+1)(2n+1)\Gamma(1-\alpha)\Gamma(\lambda+\alpha+1)\sin\pi(\lambda-n)}{n(n+1)\pi(\lambda-n)(\lambda+n+1)\Gamma(\lambda+1)} \\ &- \frac{\Gamma(\lambda+\alpha+1)\sin\pi\lambda}{2\pi\Gamma(\alpha+1)\Gamma(\lambda+1)} \int_{-1}^{1} (1-x)^\alpha (1+x)^{-\alpha} \frac{1}{2\alpha}(1-x)\,dx \\ &+ \frac{\Gamma(\lambda+\alpha+1)\sin\pi\lambda}{2\pi\Gamma(\alpha+1)\Gamma(\lambda+1)} \int_{-1}^{1} \frac{(1-x)^\alpha(1+x)^{-\alpha}(1+x)^\alpha(1-x)^{-\alpha}}{2\alpha} \int_x^1 \left(\frac{1-t}{1+t}\right)^\alpha dt\,dx \\ &+ \sum_{n=1}^{\infty} \frac{\lambda(\lambda+1)(2n+1)n!\Gamma(\lambda+\alpha+1)\sin\pi(\lambda+n)}{\pi(\lambda-n)(\lambda+n+1)\Gamma(n+\alpha+1)\Gamma(\lambda+1)n(n+1)} \cdot \\ &\cdot \tfrac{1}{2}\int_{-1}^{1} (1-x)^\alpha (1+x)^{-\alpha} P_n^{(\alpha,-\alpha)}(x)\,dx.\end{aligned}$$

The last term in the above expression vanishes by the orthogonality of the Jacobi polynomials; that is,

$$\begin{aligned}&\int_{-1}^{1} (1-x)^\alpha (1+x)^{-\alpha} P_n^{(\alpha,-\alpha)}(x)\,dx = \\ &= \int_{-1}^{1} (1-x)^\alpha (1+x)^{-\alpha} P_0^{(\alpha,-\alpha)}(x) P_n^{(\alpha,-\alpha)}(x)\,dx = 0\end{aligned}$$

Moreover, using (3.5), the third term can be written

$$\tfrac{1}{2}\int_{-1}^{1} (1-x)^{\alpha+1}(1+x)^\alpha dx = \Gamma(\alpha+2)\Gamma(1-\alpha).$$

Therefore,

$$\begin{aligned}\hat{P}_\lambda^{(\alpha,-\alpha)}(0) &= \frac{\Gamma(1-\alpha)\Gamma(\lambda+\alpha+1)}{\Gamma(\lambda+1)} \\ &- \sum_{n=1}^{\infty} \frac{\lambda(\lambda+1)(2n+1)\Gamma(1-\alpha)\Gamma(\lambda+\alpha+1)\sin\pi(\lambda-n)}{n(n+1)\pi(\lambda-n)(\lambda+n+1)\Gamma(\lambda+1)} - \\ &- \frac{\Gamma(\lambda+\alpha+1)\sin\pi\lambda}{2\pi\Gamma(\lambda+1)\Gamma(\alpha+1)\alpha} \Gamma(\alpha+2)\Gamma(1-\alpha) + \\ &+ \frac{\Gamma(\lambda+\alpha+1)\sin\pi\lambda}{4\pi\Gamma(\lambda+1)\Gamma(\alpha+1)\alpha} \int_{-1}^{1}\int_x^1 \left(\frac{1-t}{1+t}\right)^\alpha dt\,dx.\end{aligned} \tag{3.6}$$

From (2.6), (2.7) and the identity $P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x)$, it follows that

$$\hat{P}_\lambda^{(\alpha,-\alpha)}(0) = \frac{\Gamma(\lambda+\alpha+1)\Gamma(1-\alpha)\sin\pi\lambda}{\pi\lambda(\lambda+1)\Gamma(\lambda+1)}, \quad \lambda \neq 0, \ \lambda \geq -\tfrac{1}{2} \tag{3.7}$$

Hence by the uniqueness of the Jacobi transform, we have from (3.6) and (3.7),

$$1 - \sum_{n=1}^{\infty} \frac{\lambda(\lambda+1)(2n+1)\sin\pi(\lambda-n)}{n(n+1)\pi(\lambda-n)(\lambda+n+1)} - \frac{\alpha+1}{2\alpha}\frac{\sin\pi\lambda}{\pi} +$$

$$+ \frac{\sin\pi\lambda}{4\pi\alpha\Gamma(1+\alpha)\Gamma(1-\alpha)}\int_{-1}^{1}\int_{x}^{1}\left(\frac{1-t}{1+t}\right)^{\alpha}dt = \frac{\sin\pi\lambda}{\pi\lambda(\lambda+1)}$$

Now (3.4) can be expressed as

$$\frac{\Gamma(\lambda+1)\lambda(\alpha+1)}{\Gamma(\lambda+\alpha+1)} P_\lambda^{(\alpha,-\alpha)}(x) =$$

$$= \lambda(\lambda+1)\sum_{n=1}^{\infty} \frac{(2n+1)n!\sin\pi(\lambda-n)P_n^{(\alpha,-\alpha)}(x)}{n(n+1)(\alpha+1)_n\pi(\lambda-n)(\lambda+n+1)} +$$

$$+ \frac{\sin\pi\lambda}{\pi}\left[\tfrac{1}{2} + \frac{1}{\lambda(\lambda+1)} - \frac{\int_{-1}^{1}\int_{x}^{1}(1-t)^{\alpha}(1+t)^{-\alpha}dtdx}{4\alpha\Gamma(\alpha+1)\Gamma(1-\alpha)} + \right.$$

$$\left. + \frac{x}{2\alpha} + \frac{1}{2\alpha}\left(\frac{1+x}{1-x}\right)^{\alpha}\int_{x}^{1}\left(\frac{1-t}{1+t}\right)^{\alpha}dt\right].$$

By interchanging the order of integration and by (3.5) we obtain

$$\int_{-1}^{1}\int_{x}^{1}(1-t)^{\alpha}(1+t)^{-\alpha}dtdx = 2\Gamma(1+\alpha)\Gamma(2-\alpha)$$

Thus,

$$\frac{\Gamma(\lambda+1)\Gamma(\alpha+1)}{\Gamma(\lambda+\alpha+1)} P_\lambda^{(\alpha,-\alpha)}(x) =$$

$$= \lambda(\lambda+1)\sum_{n=1}^{\infty} \frac{(2n+1)n!(-1)^n\sin\pi\lambda P_n^{(\alpha,-\alpha)}(x)}{n(n+1)(\alpha+1)_n\pi(\lambda-n)(\lambda+n+1)}$$

$$+ \frac{\sin\pi\lambda}{\pi}\left[1 + \frac{1}{\lambda(\lambda+1)} - \frac{1}{2\alpha} + \frac{x}{2\alpha} + \frac{1}{2\alpha}\left(\frac{1+x}{1-x}\right)^{\alpha}\int_{x}^{1}\left(\frac{1-t}{1+t}\right)^{\alpha}dt\right]. \tag{3.8}$$

The series representation of the Jacobi function $P_\lambda^{(\alpha,-\alpha)}(x)$ will be completed once we obtain an equivalent expression for the integral.

$$f(x;\alpha) = \int_{x}^{1}\left(\frac{1-t}{1+t}\right)^{\alpha}dt.$$

Lemma 3.2. For $-\tfrac{1}{2} < \alpha < \tfrac{1}{2}$, $(\alpha \neq 0)$, we have

a) $f(x;\alpha) = \frac{(1-x)^{\alpha+1}}{(1+x)^{\alpha}} \left(1 - \frac{2\alpha}{1+x} \sum_{n=1}^{\infty} \frac{(-1)^n}{\alpha+n+1} \left(\frac{1-x}{1+x}\right)^n\right), \quad 0 \le x < 1$

b) $f(x;\alpha) = \frac{2\pi\alpha}{\sin\pi\alpha} - \frac{(1-x)^{\alpha}}{(1+x)^{\alpha-1}} \left(1 - \frac{2\alpha}{1-x} \sum_{n=1}^{\infty} \frac{(-1)^n}{\alpha-n-1} \left(\frac{1+x}{1-x}\right)^n\right), \quad -1 < x < 0$

<u>Proof</u>: a) Integration by parts yields the recursive relation

$$f(x;\alpha) = \frac{1}{\alpha+1} \frac{(1-x)^{\alpha+1}}{(1+x)^{\alpha}} - \frac{\alpha}{\alpha+1} f(x;\alpha+1).$$

By employing this relation and after simplification, we obtain

$$f(x;\alpha) = \frac{(1-x)^{\alpha+1}}{(1+x)^{\alpha}} \left(1 - \frac{2\alpha}{1+} \sum_{n=0}^{\infty} \frac{(-1)^n}{n+\alpha+1} \left(\frac{1-x}{1+x}\right)^n\right)$$

The series converges for all x such that $\left|\frac{1-x}{1+x}\right| < 1$; that is, if $0 < x < 1$. When $x = 0$,

$$f(0;\alpha) = 1-2\alpha \sum_{n=0}^{\infty} \frac{(-1)^n}{n+\alpha+1}.$$

b) We rewrite $f(x; \ )$ as

$$f(x;\alpha) = \int_x^0 \left(\frac{1-t}{1+t}\right)^{\alpha} dt + \int_0^1 \left(\frac{1-t}{1+t}\right)^{\alpha} dt = J(x;\alpha) + f(0,\alpha), \text{ say.}$$

By introducing

$$J^*(x;\alpha) = \int_{-1}^{x} \left(\frac{1-t}{1+t}\right)^{\alpha} dt$$

$J(x;\alpha)$ can be written as

$$J(x;\alpha) = J^*(0;\alpha) - J^*(x;\alpha).$$

Upon an integration by parts, we obtain

$$J^*(x;\alpha) = \frac{1}{1-\alpha} \frac{(1-x)^{\alpha}}{(1+x)^{\alpha-1}} + \frac{\alpha}{1-\alpha} J^*(x;\alpha-1)$$

Repeating the above formula, recursively, results in the series.

$$J^*(x;\alpha) = \frac{(1-x)^{\alpha}}{(1+x)^{\alpha-1}} \left(1 - \frac{2\alpha}{1-x} \sum_{n=0}^{\infty} \frac{(-1)^n}{\alpha-n-1} \left(\frac{1+x}{1-x}\right)^n\right)$$

which converges for all x such that $\left|\frac{1+x}{1-x}\right| < 1$; that is, for $-1 < x < 0$. When $x = 0$,

$$J^*(0;\alpha) = 1 - 2\alpha \sum_{n=0}^{\infty} \frac{(-1)^n}{\alpha-n-1}.$$

Thus,

$$f(x;\alpha) = J^*(0,\alpha) + f(0,\alpha) - J^*(x;\alpha)$$

$$= \frac{2\pi\alpha}{\sin\pi\alpha} - \frac{(1-x)^{\alpha}}{(1+x)^{\alpha-1}} \left(1 - \frac{2\alpha}{1-x} \sum_{n=0}^{\infty} \frac{(-1)^n}{\alpha-n-1} \left(\frac{1+x}{1-x}\right)^n\right)$$

which completes the verification of Lemma 3.2.

From (3.8) and Lemma 3.2, the representation of the Jacobi function $P_{\lambda}^{(\alpha,-\alpha)}(x)$ will follow. In particular, for $\lambda \geq -\frac{1}{2}$ $(\lambda \neq 0)$

$$P_{\lambda}^{(\alpha,-\alpha)}(x) = \frac{\Gamma(\lambda+\alpha+1)}{\Gamma(\lambda+1)\Gamma(\alpha+1)} \frac{\sin\pi\lambda}{\pi} \{\lambda(\lambda+1) \cdot$$

$$\cdot \sum_{n=1}^{\infty} \frac{(2n+1)n!(-1)^n P_n^{(\alpha,-\alpha)}(x)}{n(n+1)(\alpha+1)_n(\lambda-n)(\lambda+n+1)} + 1 + \frac{1}{\lambda(\lambda+1)} -$$

$$- \sum_{n=0}^{\infty} \frac{(-1)^n}{n+\alpha+1} \left(\frac{1-x}{1+x}\right)^{n+1}\}, \quad 0 \leq x < 1;$$

and

$$P_{\lambda}^{(\alpha,-\alpha)}(x) = \frac{\Gamma(\lambda+\alpha+1)}{\Gamma(\lambda+1)\Gamma(\alpha+1)} \frac{\sin\pi\lambda}{\pi} \{\lambda(\lambda+1) \cdot$$

$$\cdot \sum_{n=1}^{\infty} \frac{(2n+1)n!(-1)^n P_n^{(\alpha,-\alpha)}(x)}{n(n+1)(\alpha+1)_n(\lambda-n)(\lambda+n+1)} + 1 + \frac{1}{\lambda(\lambda+1)} - \frac{1}{\alpha} +$$

$$+ \frac{(1+x)^{\alpha}}{(1-x)^{\alpha}} \frac{\pi}{\sin\pi\alpha} + \sum_{n=0}^{\infty} \frac{(-1)^n}{\alpha-n-1} \left(\frac{1+x}{1-x}\right)^{n+1}\}, \quad -1 < x < 0.$$

The above representations will hold for $\lambda = 0$ provided that $\frac{\sin\pi\lambda}{\pi\lambda}$ is interpreted to be equal to 1 for $\lambda = 0$. When $\alpha = 0$, the formula reduces to that for Legendre functions derived in [3], provided that $-\frac{1}{\alpha} + \frac{(1+x)^{\alpha}}{(1-x)^{\alpha}} \frac{\pi}{\sin\pi\alpha}$ is given its limiting value of 0 as $\alpha \to 0$.

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