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NOTES ON ALMOST-PERIODICITY IN TOPOLOGICAL VECTOR SPACES

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ABSTRACT. A study is made of almost-periodic functions in topological vector spaces with applications to abstract differential equations.

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1. INTRODUCTION.

In our recent papers [1, 2], we extended the theory of almost-periodic functions from Banach spaces to topological vector spaces and gave a few results concerning its applications to abstract differential equations. The following results are the continuation of discussions begun there. Specifically Theorem 2 is a version of a theorem contained in [1, 2] (see Theorem 5.1 in [2]) which was originally inspired from a result due to A. I. Perov (cf. [3] Theorem 1.1).

Let us first recall some useful facts (see [1, 2] for more details). The reader can also find in [4] the elementary properties of linear topological spaces needed here.

DEFINITION 1. A continuous function $f: R \to E$, where E is a complete locally convex space and R is the set of real numbers, is called almost periodic (a.p.) if for each neighborhood (of the origin in E) U, there exists a real number $\ell = \ell(U) > 0$ such that every interval $[a, a + \ell]$ contains at least a point τ such that

$$f(t + \tau) - f(t) \in U \quad \text{for every } t \in R.$$

τ is then called a U-translation number of the function f.

REMARK: $U = U(\varepsilon; p_i, \Lambda \le i \le n)$

$$= \{x \in E; {}_ip(x) < \varepsilon, 1 \le i \le n\}$$

where each $p_i \in Q$, the set of semi-norms on E.

Finally we recall Bochner's criteria: If E is a Frechet space, then a function $f: R \to E$ is a.p. iff for every real sequence $(s_n')_{n=1}^{\infty}$ there exists a subsequence $(s_n)_{n=1}^{\infty}$ such that $(f(t + s_m))_{n=1}^{\infty}$ converges uniformly in $t \in R$.

DEFINITION 2. A Frechet space E is called a perfect Frechet space if the following property is verified in E: every function $\phi: R \to E$ such that

(i) $\{\phi(t);\ t \in R\}$ is bounded in E

(ii) the derivative $\phi'(t)$ is a.p. in E, is necessarily a.p. in E.

2. MAIN RESULTS.

Now let us state and prove:

THEOREM 1. If $f(t)$ is a.p. in a complete locally convex space E, then for every real sequence $(s_n)_{n=1}^{\infty}$ there exists a subsequence $(s_n')_{n=1}^{\infty}$ such that for every neighborhood (of the origin in E) U,

$$f(t + s_n') - f(t + s_m') \in U$$

for all $t \in R$, m and n.

PROOF. Let $U = U(\varepsilon;\ p_i,\ 1 \le i \le n)$ be a neighborhood and $V = V(\frac{\varepsilon}{4};\ p_i,\ 1 \le i \le n)$ a symmetric neighborhood such that $V + V + V + V \quad U$. By the definition of almost-periodicity, there exists $\ell = \ell(V)$ (therefore ℓ depends on U) such that in every real interval of length ℓ, there exists τ such that

$$f(t + \tau) - f(t) \in V$$

for every $t \in R$.

Now for each s_n, we can find τ_n and σ_n such that $s_n = \tau_n + \sigma_n$ with τ_n a V-translation number of f and $\sigma_n \in [0,\ell]$ (it suffices to take $\tau_n \in [s_n - \ell, s_n]$ and then $\sigma_n = s_n - \tau_n$).

As f is uniformly continuous on R (cf. [1, 2]), there exists $\delta = \delta(\varepsilon)$ such that

$$f(t') - f(t'') \in V \tag{2.1}$$

for all $t', t'', |t' - t''| < 2\delta$.

Also $0 \le \sigma_n \le \ell$ for every n; we can then subtract from $(\sigma_n)_{n=1}^{\infty}$, a convergent subsequence $(\sigma_{n_k})_{k=1}^{\infty}$, by the Bolzano-Weierstrass theorem.

Let $\sigma = \lim_{k\to\infty} \sigma_{n_k}$, with $0 \le \sigma \le \ell$.

Now consider the subsequence $(\sigma_{n_k})_{k=1}^{\infty}$ with

$$\sigma - \delta < \sigma_{n_k} < \sigma + \delta,\ k = 1, 2, \ldots$$

and let $(s_{n_k})_{k=1}^{\infty}$ be the corresponding subsequence where

$$s_{n_k} = \tau_{n_k} + \sigma_{n_k},\ k = 1, 2, \ldots$$

Let us prove the relation

$$f(t + s_{n_k}) - f(t + s_{n_j}) \in U \tag{2.2}$$

for all $t \in R$.

For this, write

$$\begin{aligned} f(t + s_{n_k}) - f(t + s_{n_j}) = {} & f(t + \tau_{n_k} + \sigma_{n_k}) - f(t + \sigma_{n_k}) \\ & + f(t + \sigma_{n_k}) - f(t + \sigma_{n_j}) \\ & + f(t + \sigma_{n_j}) - f(t + \tau_{n_j} + \sigma_{n_j}) . \end{aligned}$$

Because τ_{n_k} and τ_{n_j} are V-translation numbers of f, we shall get

$$f(t + \tau_{n_k} + \sigma_{n_k}) - f(t + \sigma_{n_k}) \in V, \quad \text{for every} \quad t \in R$$
$$f(t + \tau_{n_j} + \sigma_{n_j}) - f(t + \sigma_{n_j}) \in V, \quad \text{for every} \quad t \in R . \tag{2.3}$$

On the other hand

$$|(t + \sigma_{n_k}) - (t + \sigma_{n_j})| = |\sigma_{n_k} - \sigma_{n_j}| < 2\delta ;$$

therefore, by using relation (2.1), we get

$$f(t + \sigma_{n_k}) - f(t + \sigma_{n_j}) \in V, \quad \text{for every} \quad t \in R . \tag{2.4}$$

Finally we can deduce (2.2) from (2.3) and (2.4). The theorem is proved by taking

$$s'_n = s_{n_k}, \quad k = 1, 2, \ldots \qquad \Box$$

3. APPLICATIONS

Let E be a perfect Frechet space and A a closed linear operator with domain $D(A)$ dense in E. Suppose A generates a strongly continuous one-parameter group $T(t)$, $t \in R$.

Consider in such E the differential equation

$$x'(t) = Ax(t), \quad t \in R . \tag{3.1}$$

THEOREM 2. Assume for every semi-norm $p \in Q$, there exists a semi-norm $q \in Q$ such that

$$p(T(t)u) \le q(u)$$

for every $u \in E$ and $t \in R$.

Then every solution $x(t)$ of (3.1) such that $\{x'(t); t \in R\}$ is relatively compact in E is a.p.

PROOF. Let $x(t)$ be such a solution; we can write $x(t) = T(t)x(0)$, $t \in R$; by the property on $T(t)$, $x(t)$ is obviously bounded.

Consider a given real sequence $(s'_n)_{n=1}^{\infty}$; we can extract a subsequence $(s_n)_{n=1}^{\infty}$ such that $(x'(s_n))_{n=1}^{\infty}$ is a Cauchy sequence in E, for $\{x'(t); t \in R\}$ is relatively compact in E. We have

$$\begin{aligned} x'(t + s_n) &= Ax(t + s_n) \\ &= AT(t + s_n)x(0) \\ &= AT(t)T(s_n)x(0) \\ &= AT(t)x(s_n) \\ &= T(t)Ax(s_n) \\ &= T(t)x'(s_n) \end{aligned}$$

for every n and every $t \in R$. Therefore

$$x'(t + s_n) - x'(t + s_m) = T(t)[x'(s_n) - x'(s_m)]$$

for every n, m and $t \in R$.

Take now any $p \in Q$; then there exists $q \in Q$ such that

$$p[x'(t + s_n) - x'(t + s_m)] \le q[x'(s_n) - x'(s_m)]$$

for every $t \in R$; which shows $x'(t)$ is a.p. by Bochner's criteria. As E is a perfect Frechet space, the conclusion is inmediate.

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