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A SIMULTANEOUS SOLUTION TO TWO PROBLEMS ON DERIVATIVES

F.S. CATER

Department of Mathematics
Portland State University
Portland, Oregon 97207

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ABSTRACT. Let A be a nonvoid countable subset of the unit interval [0,1] and let B be an F_σ-subset of [0,1] disjoint from A. Then there exists a derivative f on [0,1] such that $0 \le f \le 1$, f = 0 on A, f>0 on B, and such that the extended real valued function 1/f is also a derivative.

KEY WORDS AND PHRASES. Derivative, primitive, Lebesgue summable, knot point.
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In this note, we construct a derivative f such that 1/f is also a derivative, and f and 1/f have some curious properties mentioned in [1] and [2]. (By an F_σ-set in the real line, we mean the union of countably many closed subsets of R.) We prove

THEOREM 1. Let A be a nonvoid countable subset of [0,1] and let B be an F_σ-subset of [0,1] disjoint from A. Then there exists a measurable function f on [0,1] such that f = 0 on A, f > 0 on B, $0 \le f \le 1$ on [0,1] and

(1) f is everywhere the derivative of its primitive,

(2) 1/f is Lebesgue summable on [0,1],

(3) 1/f is everywhere the derivative of its primitive.

Here we let $\infty = 1/0$.

When m(B) = 1 and A is dense, we will obtain a simple example of a derivative that vanishes on a dense set of measure 0.

From [2] we infer that there exists a derivative f vanishing on A and positive on B. From [1] we infer that there exists a derivative g infinite on A and finite on B. However, Theorem 1 provides a simultaneous solution to both of these problems. To prove Theorem 1 we will employ some of the methods used in [3].

Finally, we use these methods to construct a concrete example of functions g_1 and g_2 that have finite or infinite derivatives at each point, such that the Dini derivatives of their difference, g_1-g_2, satisfy certain pathological properties.

In all that follows, let $(n(i))_{i=1}^{\infty}$ denote the sequence of integers 1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, 1, ...

Proof of Theorem 1. Let $(a_i)_{i=1}^{\infty}$ be a sequence of points in A such that each point of A occurs at least once in the sequence. (Here we do not exclude the possibility that A is a finite set.) We assume, without loss of generality, that B is nonvoid.

Let $B_1 \subset B_2 \subset B_3 \subset \ldots \subset B_i \ldots$ be an expanding sequence of closed sets such that $B = \cup_{i=1}^{\infty} B_i$. (Here we do not exclude the possibility that B is a closed set.) Let u_i denote the distance from the point $a_{n(i)}$ to the set B_i. As in [3], we put $\phi(x) = (1 + |x|)^{-\frac{1}{2}}$.

For each index j, put

$$g_j(x) = 1 + \textstyle\sum_{i=1}^{j} \phi(2^i u_i^{-1}(x-a_{n(i)})),$$

$$g(x) = 1 + \textstyle\sum_{i=1}^{\infty} \phi(2^i u_i^{-1}(x-a_{n(i)})),$$

$$f_j(x) = 1/g_j(x), \quad f(x) = 1/g(x).$$

Here we let $0 = 1/\infty$. Then $g(a) = \infty$ for $a \varepsilon A$, because there are infinitely many indices i for which $a = a_{n(i)}$. On the other hand, $g(b) < \infty$ for $b \varepsilon B$; note that if $b \varepsilon B_k$, then

$$\phi(2^k u_k^{-1}(b-a_{n(k)})) \leqq \phi(2^k) < 2^{-\frac{1}{2}k},$$

$$\textstyle\sum_{i=k}^{\infty} \phi(2^i u_i^{-1}(b-a_{n(i)})) \leqq \sum_{i=k}^{\infty} 2^{-\frac{1}{2}i} < \infty .$$

We infer from Lemma 4 of [3], that g is Lebesgue summable on [0,1]. Note also that

$$g(x)-g_j(x) = g(x)g_j(x)(f_j(x)-f(x)) > 0,$$

and since $g>1$, $g_j>1$, it follows that $g-g_j > f_j-f > 0$.

Now choose any x with $g(x) < \infty$. By Lemma 4 of [3], we have

$$\lim_{h\to 0} h^{-1} \int_x^{x+h} g(t)dt = g(x).$$

Take any $\varepsilon>0$. Select an index j so large that $f_j(x)-f(x)<g(x)-g_j(x)<\varepsilon$. Since f_j and g_j are continuous, when $|h|$ is small enough we have

$$|h^{-1} \int_x^{x+h} g(t)dt - g(x)| < \varepsilon,$$

$$|h^{-1} \int_x^{x+h} g_j(t)dt - g_j(x)| < \varepsilon,$$

$$|h^{-1} \int_x^{x+h} f_j(t)dt - f_j(x)| < \varepsilon.$$

For such j and h we obtain

$$h^{-1} \int_x^{x+h} (g(t)-g_j(t))dt \leq g(x)-g_j(x) + |h^{-1}\int_x^{x+h} g(t)dt - g(x)| + |h^{-1}\int_x^{x+h} g_j(t)dt - g_j(x)| < 3\varepsilon.$$

From $0 < f_j-f < g-g_j$ we obtain

$$\begin{aligned} |h^{-1} \int_x^{x+h} f(t)dt - f(x)| &\leqq |h^{-1} \int_x^{x+h} f_j(t)dt - f_j(x)| + f_j(x) - f(x) + h^{-1} \int_x^{x+h} (f_j(t) - f(t))dt \\ &\leqq 2\varepsilon + h^{-1} \int_x^{x+h} (f_j(t) - f(t))dt \\ &\leqq 2\varepsilon + h^{-1} \int_x^{x+h} (g(t) - g_j(t))dt < 5\varepsilon. \end{aligned}$$

It follows that $\lim_{h\to 0} h^{-1} \int_x^{x+h} f(t)dt = f(x)$.

Choose any x with $g(x) = \infty$. Take any $N>0$. Select j so large that $g_j(x)>N$. Since g_j is continuous, there is a $d>0$ such that $|t-x| < d$ implies $g_j(t) > N$. For such t, $g(t)>g_j(t)>N$ and $f(t)<f_j(t)<N^{-1}$. It follows that for $|h| < d$,

$$h^{-1} \int_x^{x+h} g(t)dt > N, \quad 0 < h^{-1} \int_x^{x+h} f(t)dt < N^{-1}.$$

Finally,

$$\lim_{h\to 0} h^{-1} \int_x^{x+h} g(t)dt = \infty = g(x).$$

$$\lim_{h\to 0} h^{-1} \int_x^{x+h} f(t)dt = 0 = f(x).$$

This completes the proof.

When $m(B) = 1$, we do not know if our argument can be modified to make $f = 0$ on $[0,1]\setminus B$ as in [2]. Perhaps this requires an approach altogether different from ours.

We say that x is a <u>knot point</u> of the function F if its Dini derivatives satisfy

$$D^+F(x) = D^-F(x) = \infty \quad \text{and } D_+F(x) = D_-F(x) = -\infty.$$

We conclude by presenting a simple and direct example of functions g_1 and g_2 having derivatives (finite or infinite) at every point such that g_1-g_2 has knot points in every interval. (Consult [4] for analogous examples.)

Let $\{a_i\}_{i=1}^{\infty}$ and $\{b_i\}_{i=1}^{\infty}$ be countable dense subsets of (0,1) that are disjoint. Let $Z(c,d,x) = \int_0^x \phi(c(t-d))dt$ for $c>0$, $d>0$, $x>0$. We integrate to obtain

$$Z(c,d,x) = \begin{cases} 2c^{-1}[(1+cd)^{\frac{1}{2}} - (1+cd-cx)^{\frac{1}{2}}] & \text{if } x \leq d, \\ 2c^{-1}[(1+cd)^{\frac{1}{2}} + (1+cx-cd)^{\frac{1}{2}} - 2] & \text{if } x > d. \end{cases}$$

Let u_i denote the distance from $a_{n(i)}$ to the set $\{b_1,\ldots,b_i\}$, and let v_i denote the distance from $b_{n(i)}$ to the set $\{a_1,\ldots,a_i\}$. Put

$$g_1(x) = \sum_{i=1}^{\infty} Z(2^i u_i^{-1}, a_{n(i)}, x), \quad g_2(x) = \sum_{i=1}^{\infty} Z(2^i v_i^{-1}, b_{n(i)}, x)$$

for $0<x<1$. By the argument in the proof of Theorem 1 we prove that g_1 and g_2 are absolutely continuous functions on (0,1) with $g_1' = \infty$ on A, $g_2' = \infty$ on B, g_1' finite on B, and g_2' finite on A. Put $g = g_1-g_2$. Then g is absolutely continuous on (0,1), $g' = \infty$ on A and $g' = -\infty$ on B. Each of the sets

$E_1 = \{x: D^+g(x) = \infty\}$, $E_2 = \{x: D^-g(x) = \infty\}$, $E_3 = \{x: D_+g(x) = -\infty\}$ and $E_4 = \{x: D_-g(x) = -\infty\}$ is a dense G_δ-subset of (0,1), i.e., is the intersection of countably many open dense subsets of (0,1). It follows that $E_1 \cap E_2 \cap E_3 \cap E_4$ is also a dense G_δ-subset of (0,1). But each point in this intersection is a knot point of g, even though g_1 and g_2 have derivatives (finite or infinite) everywhere by the proof of Theorem 1.

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