ON REPRESENTATIONS OF LIE ALGEBRAS OF A GENERALIZED TAVIS-CUMMINGS MODEL

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Consider the Lie algebras $L^s_{r,t} : [K_1, K_2] = sK_3$, $[K_3, K_1] = rK_1$, $[K_3, K_2] = -rK_2$, $[K_3, K_4] = 0$, $[K_4, K_1] = -tK_1$, and $[K_4, K_2] = tK_2$, subject to the physical conditions, K_3 and K_4 are real diagonal operators representing energy, $K_2 = K_1^\dagger$, and the Hamiltonian $H = \omega_1 K_3 + (\omega_1 + \omega_2)K_4 + \lambda(t)(K_1 e^{-i\phi} + K_2 e^{i\phi})$ is a Hermitian operator. Matrix representations are discussed and faithful representations of least degree for $L^s_{r,t}$ satisfying the physical requirements are given for appropriate values of $r, s, t \in \mathbb{R}$.

1. Introduction

Introducing an algebraic method to solve certain types of linear partial differential equations, Steinberg [6] exploited the Lie-algebraic decomposition formulas of Baker, Campbell, Hausdorff, and Zassenhaus (cf. [7]) and their matrix realization. A faithful matrix representation of low degree is required. In [2, 3, 4], the faithful matrix representations of least degree were discussed for the Lie algebra L^s_r generated by K_+, K_-, and K_0 satisfying the commutation relations: $[K_0, K_\pm] = \pm rK_\pm$ and $[K_+, K_-] = sK_0$ subject to the physical properties $K_- = K_+^\dagger$ († for Hermitian conjugation), K_0 is a real diagonal operator, and $(K_+ + K_-)$ is real. The Lie algebra L^s_r was introduced as a generalization of the coupled quantized harmonic oscillators [5] namely, the model of light amplifier L^{-2}_1, and the model of two-level optical atom L^2_1, whose Hamiltonian model $H = K_0 + \lambda(K_+ + K_-)$, λ is the coupling parameter. Note that, L^1_2 is exactly the Lie algebra $\mathfrak{sl}(2)$.

In this paper, $L^s_{r,t}$ is considered to be the Lie algebra generated by K_1, K_2, K_3, and K_4, satisfying the commutation relations: $[K_1, K_2] = sK_3$,

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$[K_3,K_1]=rK_1$, $[K_3,K_2]=-rK_2$, $[K_3,K_4]=0$, $[K_4,K_1]=-tK_1$, $[K_4,K_2]=tK_2$, subject to the physical conditions, K_3 and K_4 are real diagonal operators representing energy, $K_2=K_1^\dagger$, and the Hamiltonian $H=\omega_1K_3+(\omega_1+\omega_2)K_4+\lambda(t)(K_1e^{-i\phi}+K_2e^{i\phi})$ is a Hermitian operator. The Lie algebra $L_{r,t}^s$ is introduced as a generalization of the Tavis-Cummings model namely, $L_{2,1}^1$ in [1]. Obviously, the subalgebra of $L_{r,t}^s$ generated by K_1, K_2, and K_3 in respective with K_+, K_-, and K_0 is a generalization of L_r^s, when dropping the physical condition $(K_+ + K_-)$ must be real. That condition forced the representation matrices of K_+ and K_- to be real, [2, 3, 4]. Faithful matrix representations of least degree are discussed for $L_{r,t}^s$ for appropriate values of $r,s,t\in\mathbb{R}$.

Unless otherwise stated, I_m is the identity matrix of degree m, O is the zero matrix of appropriate size, $\mathbb{N}=\{1,2,\dots,n\}$ and $A=[a_{ij}]$, $B=[b_{ij}]$, $C=[\delta_{ij}c_{ij}]$, and $D=[\delta_{ij}d_{ij}]$ are $n\times n$ real matrices, where the matrices $X=A+iB$, $Y=A^T-iB^T$, C, and D are representation matrices for K_1, K_2, K_3, and K_4, respectively; $i=\sqrt{-1}$. All representations for $L_{r,t}^s$ under consideration are supposed to satisfy the above-mentioned physical requirements.

Lemma 1.1. *The Lie algebra $L_{r,t}^s$ can be defined by*

$$[K_1,K_2]=sK_3,\qquad [K_3,K_1]=rK_1,\qquad [K_4,K_1]=-tK_1, \tag{1.1}$$

where K_3 and K_4 are real diagonal operators and $K_2=K_1^\dagger$.

Proof. Indeed $-rK_2=-(rK_1)^\dagger=-[K_3,K_1]^\dagger=[K_3,K_2]$ and similarly, for the relation $[K_4,K_2]=tK_2$. Since K_3 and K_4 are diagonal, they commute. The Hermiticity of the Hamiltonian follows since $\omega_1,\omega_2,\lambda(t)\in\mathbb{R}$. □

As a necessity of Lemma 1.1 we have the following lemma.

Lemma 1.2. *The matrices A, B, C, and D satisfy the following:*

(i) $[A,B^T]$ *is a symmetric matrix,*
(ii) $[A,A^T]+[B,B^T]=sC$,
(iii) $[C,A]=rA$, $[C,B]=rB$,
(iv) $[D,A]=-tA$, $[D,B]=-tB$.

Lemma 1.3. *Let L, M, and K be $n\times n$ matrices such that $[L,M]=aK$, $a\neq 0$, then* $\operatorname{trace}(K)=0$.

Lemma 1.4. *Let $p,q\in\mathbb{N}$, and $\sigma=(pq)$ be a transposition. The representation obtained by applying σ to the rows as well as to the columns of X, Y, C, and D is a conjugate representation for $L_{r,t}^s$ and satisfies the physical requirements.*

Proof. Let P be the elementary matrix obtained by applying σ to the rows of I_n. Since $P = P^{-1} = P^T = P^\dagger$, then the proof of the lemma follows. □

Since $[C,X] = rX$, then for all $i,j \in \mathbb{N}$ we have,

$$a_{ij}(c_{ii} - c_{jj} - r) = 0, \qquad b_{ij}(c_{ii} - c_{jj} - r) = 0. \tag{1.2}$$

Similarly, from Lemma 1.2(iv),

$$a_{ij}(d_{ii} - d_{jj} + t) = 0, \qquad b_{ij}(d_{ii} - d_{jj} + t) = 0. \tag{1.3}$$

If $x_{ij} \neq 0$, then from (1.2) and (1.3)

$$c_{ii} - c_{jj} = r, \qquad d_{jj} - d_{ii} = t. \tag{1.4}$$

Since $[X,Y] = sC$, then for each $i \in \mathbb{N}$ we have,

$$sc_{ii} = \sum_{l=1}^{n}\left(|x_{il}|^2 - |x_{li}|^2\right) = \sum_{l=1}^{n}(a_{il}^2 - a_{li}^2 + b_{il}^2 - b_{li}^2). \tag{1.5}$$

Lemma 1.5. *If $t^2 + r^2 \neq 0$, then*

(1) $x_{ii} = 0$, *for all* $i \in \mathbb{N}$,
(2) *if* $x_{ij} \neq 0$ *then* $x_{ji} = 0$, *for all* $i,j \in \mathbb{N}$.

Proof. If $r \neq 0$, then from (1.2) we have, for each $i \in \mathbb{N}$, that $x_{ii} = 0$. Also, if $x_{ij} \neq 0$, then $c_{jj} - c_{ii} - r = -2r$, thus $x_{ji} = 0$. Similarly, when $t \neq 0$. □

Lemma 1.6. *If $s \neq 0$, then*

(1) $\mathrm{trace}(C) = 0$,
(2) *if* $x_{ij} \neq 0$ *then, for* $i,j \in \mathbb{N}$

$$r = \frac{1}{s}\left[\sum_{l=1}^{n}\left(|x_{il}|^2 - |x_{li}|^2 - |x_{jl}|^2 + |x_{lj}|^2\right)\right]. \tag{1.6}$$

Proof. Since $[X,Y] = sC$ then from Lemma 1.3, $\mathrm{trace}(C) = 0$. The proof of (2), follows from (1.4) and (1.5). □

We build the representation matrices starting with C.

Remark 1.7. Using Lemma 1.4, C can be rearranged into k diagonal blocks, the ith diagonal block consists of the k_i scalar matrices, $\{c_iI_{m_{i,0}}, (c_i - r)I_{m_{i,1}}, \ldots, [c_i - r(k_i - 1)]I_{m_{i,(k_i-1)}}\}$, where $m_{i,j}$ is the repetitions of $(c_i - rj)$

in the diagonal of C; for $i=1,2,\dots,k$ and $j=0,1,\dots,k_i-1$. Thus,

$$\begin{aligned} C = \operatorname{diag}\{ & c_1 I_{m_{1,0}}, (c_1-r)I_{m_{1,1}},\dots,[c_1-r(k_1-1)]I_{m_{1,(k_1-1)}},\dots,\\ & c_i I_{m_{i,0}},(c_i-r)I_{m_{i,1}},\dots,[c_i-r(k_i-1)]I_{m_{i,(k_i-1)}},\dots,\\ & c_k I_{m_{k,0}},(c_k-r)I_{m_{k,1}},\dots,[c_k-r(k_k-1)]I_{m_{k,(k_k-1)}}\}, \end{aligned} \tag{1.7}$$

where

$$c_i \neq c_j, \quad \text{whenever } i \neq j, \quad \text{for } i,j = 1,2,\dots,k, \tag{1.8}$$

$$[c_i - rj] - c_{i+1} \neq r, \quad \text{for } j=0,\dots,k_i-1;\ i=1,2,\dots,k-1. \tag{1.9}$$

The ith diagonal block of C is called the c_i-block and k_i is its length. Any diagonal entry c of C such that $c=c_i-rl$, for $l\geq 0$ then $0\leq l\leq k_i-1$ for some $i=1,\dots,k$, that is, c belongs to the c_i-block. If $c_i-l_1r=c_j-l_2r$, $0\leq l_1\leq k_i-1$, $0\leq l_2\leq k_j-1$, then c_i and c_j are in the same block, violating (1.9).

We use the notations given in Remark 1.7.

2. Faithful representations for $L^s_{r,t}$ where $rs\neq 0$

LEMMA 2.1. *The matrices A and B can be partitioned into submatrices of the same size corresponding to those of C. The nonzero submatrices of A and B are all off-diagonal submatrices.*

Proof. From (1.2), the diagonal submatrices of A and B are square zero submatrices of orders $m_{1,0},\dots,m_{k,(k_k-1)}$, in respective to those of C. Let c_{ii}, c_{jj}, and c_{ll}; $i,j,l\in\mathbb{N}$, be from different diagonal submatrices of C, and suppose that $a_{ij}\neq 0$ and $a_{il}\neq 0$, then from (1.2), $c_{ll}=c_{jj}$ contradicting (1.8). Similarly, if a_{ji} and a_{li} are from different submatrices in A they cannot be both nonzero. In view of (1.2), only the off-diagonal submatrices of A may be nonzero. Thus we have, $A=[A_{ij}]$ where $A_{ij}=O$, for $j\neq i+1$. And similarly for B. □

LEMMA 2.2. *For $k>1$, if $k_i=1$, for some $i=1,2,\dots,k$, then $L^s_{r,t}$ has a representation of degree $n-m_{i,0}$. Moreover, if the entries in the ith row and the ith column of X are all zeros, then $L^s_{r,t}$ has a representation of degree $n-1$.*

Proof. We use Lemma 1.4 so that the c_i-block becomes the first block of the main diagonal of C. Since for all $j\in\mathbb{N}$, $1\leq i\leq m_{1,0}$, $|c_{ii}-c_{jj}|\neq r$, otherwise $k_i>1$, then from (1.2) the representation is fully reducible since, $A=\left[\begin{smallmatrix}0&0\\0&A'\end{smallmatrix}\right]$, $B=\left[\begin{smallmatrix}0&0\\0&B'\end{smallmatrix}\right]$, $C=\left[\begin{smallmatrix}C_1'&0\\0&C_2'\end{smallmatrix}\right]$, and $D=\left[\begin{smallmatrix}D_1'&0\\0&D_2'\end{smallmatrix}\right]$. The matrices

$X' = A' + iB'$, $Y' = X'^{\dagger}$, C_2', and D_2' are all of degree $n - m_{i,0}$ and satisfy the lemma. Similar argument holds when the entries in the ith row and the ith column of X are all zeros. □

So, it can be assumed that if $k > 1$ then $k_i > 1$; $i = 1, \dots, k$. And for $X \neq O$, if the entries of the ith row of X are all zeros, then those of the ith column are not all zeros, and vice versa, in such cases, we get from (1.5) that $sc_{ii} \neq 0$.

Theorem 2.3. *If $rs < 0$, then $X = Y = C = O$.*

Proof. If $k = 1$ and $k_1 = 1$, then from (1.2) $X = Y = O$. If $X = O$, then from (1.5) $C = O$. Suppose that $X \neq O$, there are only two cases to consider namely, the case where $k = 1$ and $k_1 > 1$, and the case where $k > 1$. In both cases $k_1 > 1$, from Lemma 2.1 the first $m_{1,0}$ columns of X are zero columns, and from Lemma 2.2 there must be an $x_{1,j} \neq 0$ for some $m_{1,0} < j \leq (m_{1,0} + m_{1,1})$. Thus from (1.5),

$$sc_{11} = sc_1 = \sum_{l=1}^{n} \left(|x_{1l}|^2 - 0 \right) > 0. \tag{2.1}$$

Let $\alpha = m_{1,0} + m_{1,1} + \cdots + m_{1,(k_1-2)}$. If $k > 1$, we get from (1.9), $[c_1 - r(k_1 - 1)] - c_2 \neq r$, thus from (1.2), the rows $\alpha + 1, \alpha + 2, \dots, \alpha + m_{1,(k_1-1)}$ are zero rows of X. If $k = 1$ and $k_1 > 1$, we get from Lemma 2.1 that the mentioned rows are zero rows of X, being the last rows of X. In both cases, from Lemma 2.2 there must be an $x_{i,\alpha+1} \neq 0$ for some $[\alpha - m_{1,(k_1-2)}] < i \leq \alpha$. From (1.5),

$$sc_{\alpha+1,\alpha+1} = s\left[c_1 - r\left(k_1 - 1\right)\right] = \sum_{l=1}^{n} \left(0 - |x_{l,\alpha+1}|^2 \right) < 0. \tag{2.2}$$

If $s > 0$, then $c_1 > 0$ by (2.1), since $r < 0$, then $[c_1 - r(k_1 - 1)] > 0$, violating (2.2). Similarly, if $s < 0$, we get from (2.1), $[c_1 - r(k_1 - 1)] < 0$, violating (2.2). □

We conclude this section by introducing the 2×2 representation matrices X, Y, C, and D of K_1, K_2, K_3, and K_4, respectively, for $rs > 0$, $t \in \mathbb{R}$

$$\begin{gathered} X = \begin{bmatrix} 0 & a \pm i\sqrt{rs/2 - a^2} \\ 0 & 0 \end{bmatrix}, \qquad Y = \begin{bmatrix} 0 & 0 \\ a \mp i\sqrt{rs/2 - a^2} & 0 \end{bmatrix}, \\ C = \begin{bmatrix} r/2 & 0 \\ 0 & -r/2 \end{bmatrix}, \qquad D = \begin{bmatrix} b & 0 \\ 0 & b + t \end{bmatrix}, \end{gathered} \tag{2.3}$$

for any $a,b \in \mathbb{R}$ such that $|a| \le \sqrt{rs/2}$ and for the linear independency of C and D, take $b \neq -t/2$. These representations are faithful. The 2×2 representation matrices X, Y, C, and D generalize those given in [1].

Clearly, the vector space spanned by X, Y, and C is $\mathfrak{sl}(2,\mathbb{C})$, as a vector space. The representation matrices of L_r^s, in [2], are for the special cases, $a^2 = rs/2$.

3. Faithful representations for $L_{r,t}^s$ where $rst = 0$

The case where $rs \neq 0$ and $t = 0$ was considered in the previous section. So, if $s \neq 0$ we only need to consider the case where $r = 0$ and t is any real number.

3.1. *For $s \neq 0$, $r = 0$, and $t \in \mathbb{R}$*

Since $r = 0$ then any c_i-block of the matrix C has length $k_i = 1$. So, we have $C = \operatorname{diag}(c_1 I_{m_1}, \dots, c_k I_{m_k})$ where $c_i \neq c_j$ whenever $i \neq j$; $i,j = 1,\dots,k$.

Remark 3.1. If X commutes with $Y = X^\dagger$, then X is a normal matrix, and there exists a unitary matrix U such that $X = U^\dagger Z U$ for some complex diagonal matrix Z. If U commutes with C and D, then the diagonal matrices Z, $\bar{Z}$, C, and D are representation matrices for K_1, K_2, K_3, and K_4, respectively, and satisfy the physical requirements. We take $U = I_n$ when X is diagonal.

Lemma 3.2. *If $C = \operatorname{diag}(c_1 I_{m_1}, \dots, c_k I_{m_k})$ for different c_i's, then the representation is fully reducible into representations of degrees $m_1, \dots, m_k$.*

Proof. The matrix D is diagonal and from (1.2), $x_{ij} = x_{ji} = y_{ij} = y_{ji} = 0$, whenever $c_{ii} \neq c_{jj}$; $i,j \in \mathbb{N}$. □

Lemma 3.3. *Let $K = [K_{ij}]$ be a partitioned matrix which is normal whose diagonal blocks are k square matrices. If $K_{ij} = O$ whenever $j \neq i+1$ (or $j \neq i-1$); $i,j = 1,\dots,k$. Then $K = O$.*

Proof. Let $K = [k_{ij}]$ be an $n \times n$ matrix, then for each $i \in \mathbb{N}$,

$$\sum_{l=1}^{n} |k_{il}|^2 = \sum_{l=1}^{n} |k_{li}|^2. \tag{3.1}$$

Let the diagonal blocks of K be of degrees $i_1, \dots, i_k$, respectively. If $K_{ij} = O$ whenever $j \neq i+1$; $i,j = 1,\dots,k$, then the first i_1 rows of K are zeros, thus from (3.1) the first i_1 columns of K are zeros. Continuing like that in less than k steps, it can be shown that $K = O$. Hence the proof of the lemma follows. □

Theorem 3.4. *The matrix $C = O$, in any representation of $L^s_{0,t}$. If $st \neq 0$, then $X = Y = O$.*

Proof. Suppose $C \neq O$, we use Lemma 1.4 so that $c_1 \neq 0$, from (1.5) and Lemma 3.2, $m_1 s c_1 = \sum_{i=1}^{m_1} s c_{ii} = \sum_{i=1}^{m_1}\sum_{l=1}^{m_1}(|x_{il}|^2 - |x_{li}|^2) = 0$, but $m_1 s c_1 \neq 0$. Then $C = O$. Thus from Lemma 1.1, X is a normal matrix. If $t \neq 0$, we use Lemma 1.4, so that

$$\begin{aligned} D = \operatorname{diag}\Big\{ & d_1 I_{m'_{1,0}}, (d_1 + t) I_{m'_{1,1}}, \dots, [d_1 + t(k'_1 - 1)] I_{m'_{1,(k'_1-1)}}, \dots, d_i I_{m'_{i,0}}, \\ & (d_i + t) I_{m'_{i,1}}, \dots, [d_i + t(k'_i - 1)] I_{m'_{i,(k'_i-1)}}, \dots, d_{k'} I_{m'_{k',0}}, \\ & (d_{k'} + t) I_{m'_{k',1}}, \dots, [d_{k'} + t(k'_{k'} - 1)] I_{m'_{k',(k'_{k'}-1)}} \Big\}, \end{aligned} \tag{3.2}$$

where $m'_{i,j}$ is the repetitions of $(d_i + tj)$ in the diagonal of D; for $i = 1, \dots, k'$ and $j = 0, \dots, k'_i - 1$ such that

$$\begin{aligned} d_i \neq d_j, \quad & \text{whenever } i \neq j, \quad \text{for } i, j = 1, 2, \dots, k', \\ d_{i+1} - [d_i + tj] \neq t, \quad & \text{for } j = 0, \dots, k'_i - 1;\ i = 1, 2, \dots, k' - 1. \end{aligned} \tag{3.3}$$

From (1.3), X can be partitioned into submatrices of the same sizes corresponding to those of D, whose nonzero submatrices are off-diagonal submatrices. Then by Lemma 3.3 $X = Y = O$. □

If $t = 0$ then from Lemma 1.1, the generators commute and such a case can be considered as a special case of $L^0_{0,0}$ of Section 3.3, with $C = O$.

3.2. For $s = 0$ and $r^2 + t^2 \neq 0$

From (1.5) as $s = 0$, then (3.1) holds. If the ith row (or column) of X consists entirely of zeros, the ith column (or row) also, consists entirely of zeros and both can be omitted by the following lemma whose proof is analogous to that of Lemma 2.2. So, if $X \neq O$, it can be considered that X has no zero row or zero column.

Lemma 3.5. *If X has m zero rows (or columns), where $0 \leq m < n$, then $L^s_{r,t}$ has a representation of degree $n - m$.*

THEOREM 3.6. *If $s = 0$ and $r^2 + t^2 \neq 0$, $L^s_{r,t}$ has no faithful representations. In any representation, $X = Y = O$.*

Proof. If $r \neq 0$, arrange C as in Remark 1.7 otherwise, let D as in the proof of Theorem 3.4. In view of Lemma 1.5, X can be partitioned into submatrices of the same sizes corresponding to those of C when $r \neq 0$ or to those of D otherwise. The nonzero submatrices of X are all off diagonal submatrices. As $s = 0$ then X is normal and from Lemma 3.3, we get $X = Y = O$. □

3.3. For $s = r = t = 0$

Although physically is not applicable, but for the sake of completeness, we consider the case when K_1, K_2, K_3, and K_4 are commutant operators.

THEOREM 3.7. *The representations of $L^0_{0,0}$ are conjugate to representations where K_1, K_2, K_3, and K_4 are represented by diagonal matrices.*

Proof. Let $X = U^\dagger ZU$ for a unitary matrix U and a complex diagonal matrix Z. We claim that U commutes with C and D, then the theorem holds by using Remark 3.1. We induce on n, the degree of the representation and prove the cases when X is not diagonal.

For $n = 2$: if X is not diagonal then from (1.4), both C and D are scalar matrices and both commute with U.

For $n = 3$: if the diagonal elements of C (or D) are all different, then X must be diagonal. If X has two nonzero elements x_{ij} and x_{lm}, from (1.4), both are nondiagonal elements where x_{lm} is not the x_{ji}, then C and D are scalar matrices and both commute with U. Otherwise, we use Lemma 1.4, so that $X = \left[\begin{smallmatrix} X' & O \\ O & g \end{smallmatrix}\right]$, thus from (1.2) and (1.3) $C = \left[\begin{smallmatrix} cI_2 & O \\ O & a \end{smallmatrix}\right]$ and $D = \left[\begin{smallmatrix} dI_2 & O \\ O & b \end{smallmatrix}\right]$, for some $a, b, c, d \in \mathbb{R}$; $g \in \mathbb{C}$, where X' is not a diagonal matrix. That requires X' to be a normal matrix. So, there exists a unitary matrix U' such that $X' = U'^\dagger MU'$, for some complex diagonal matrix M. Obviously, U' commutes with cI_2 and dI_2. Let $U = \left[\begin{smallmatrix} U' & O \\ O & 1 \end{smallmatrix}\right]$, and $Z = \operatorname{diag}(M, g)$ then U commutes with C and D.

Assume that the theorem is true for $n < m$.

For $n = m$: if both C and D are scalar matrices, then U commutes with C and D. If either C or D is not a scalar matrix, C say, then we use Lemma 1.4 to rearrange C so that $C = \operatorname{diag}(c_1 I_{m_1}, \dots, c_k I_{m_k})$ for different $c_i's$, from (1.2) $X = \operatorname{diag}(X_1, \dots, X_k)$ where X_i is a square matrix of order $m_i < m$. Also, D can be considered as $D = \operatorname{diag}(D_1, \dots, D_k)$ where D_i is a diagonal matrix of degree m_i. Hence, the representation is fully reducible into representations of degrees m_i, $i = 1, \dots, k$. Since X is normal then X_i is normal for $i = 1, \dots, k$. Thus there exists a unitary matrix U_i such that

$X_i = U_i^\dagger Z_i U_i$ for some complex diagonal matrix Z_i, $i = 1,\dots,k$. From the induction U_i commutes with $c_i I_{m_i}$ and D_i. Let $U = \operatorname{diag}(U_1,\dots,U_k)$ and $Z = \operatorname{diag}(Z_1,\dots,Z_k)$, then U commutes with C and D. □

THEOREM 3.8. *The Lie algebra $L^0_{0,0}$ has faithful representations of degree 4 as the least degree.*

Proof. Any linearly independent diagonal matrices Z, $\bar{Z}$, C, and D, of degree 4, with C and D are real, are representation matrices for K_1, K_2, K_3, and K_4, respectively, of a faithful representation. □

We conclude the paper by mentioning the cases where $L^s_{r,t}$ has faithful matrix representations satisfying the physical requirements.

SUMMARY 3.9. *It is assumed that all representations of $L^s_{r,t}$ must satisfy the physical requirements.*

(1) *For $rs > 0$, $t \in \mathbb{R}$, $L^s_{r,t}$ has faithful representations of degree 2 as the least degree.*
(2) *For $r = s = t = 0$, $L^0_{0,0}$ has faithful representation of degree 4 as the least degree where the representation matrices are linearly independent diagonal matrices, with C and D are real matrices.*

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