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Research Article

Exponential Attractor for a First-Order Dissipative Lattice Dynamical System

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We construct an exponential attractor for a first-order dissipative lattice dynamical system arising from spatial discretization of reaction-diffusion equations in $\mathbb{R}^k$. And we obtain fractal dimension of the exponential attractor.

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1. Introduction

Lattice systems arise in many applications, for example, in chemical reaction theory, image processing, pattern recognition, material science, biology, electrical engineering, laser systems, and so forth. A lattice dynamical system (LDS) is an infinite system of ordinary differential equations (lattice ODEs) or of difference equations. In some cases, they arise from spatial discretizations of partial differential equations (PDEs), but they possess their own form.

Let $k \in \mathbb{N}$ be a fixed positive integer. Denote

$$\ell^2 = \left\{ u \mid u = (u_i)_{i\in\mathbb{Z}^k},\ i = (i_1, i_2, \ldots, i_k) \in \mathbb{Z}^k,\ u_i \in \mathbb{R},\ \sum_{i\in\mathbb{Z}^k} u_i^2 < +\infty \right\}, \tag{1.1}$$

where $\mathbb{Z}$ is the set of integers. Define a linear operator A acting on ℓ^2 in the following way: for any $u = (u_i)_{i\in\mathbb{Z}^k} \in \ell^2$, $i = (i_1, i_2, \ldots, i_k) \in \mathbb{Z}^k$,

$$\begin{aligned}(Au_i)_{i\in\mathbb{Z}^k} = 2ku_{(i_1,i_2,\ldots,i_k)} - u_{(i_1-1,i_2,\ldots,i_k)} - u_{(i_1,i_2-1,\ldots,i_k)} - \cdots \\ - u_{(i_1,i_2,\ldots,i_k-1)} - u_{(i_1+1,i_2,\ldots,i_k)} - u_{(i_1,i_2+1,\ldots,i_k)} - \cdots - u_{(i_1,i_2,\ldots,i_k+1)}.\end{aligned} \tag{1.2}$$

In this paper, we will consider the following first-order lattice dynamical system:

$$\begin{gathered}\dot{u} + Au + \lambda u + \overline{g}(u) = \overline{q}, \quad t > 0, \\ u(0) = (u_{0i})_{i\in\mathbb{Z}^k} = u_0,\end{gathered} \tag{1.3}$$

where $\lambda > 0$, $u = (u_i)_{i\in\mathbb{Z}^k}$, $Au = (Au_i)_{i\in\mathbb{Z}^k}$, $\dot{u} = (\dot{u}_i)_{i\in\mathbb{Z}^k}$ denote the first-order derivative, and $\overline{g}(u) = (g(u_i))_{i\in\mathbb{Z}^k}$, $\overline{q} = (q_i)_{i\in\mathbb{Z}^k}\ell^2$. Then, problem (1.3) can be regarded as a discrete analogue of the following reaction diffusion equation in $\mathbb{R}^k$:

$$\partial_t u - \Delta u + \lambda u + g(u) = q(x), \quad t \geq 0. \tag{1.4}$$

One example is the Chafee-Infante equation.

Bates [1] and his collaborators made some results on a global attractor for lattice dynamical system (LDS). Zhou [2] applied them to a first-order dissipative lattice dynamical systems analogue to problem (1.3), proved the existence of the global attractor for the LDS, and considered the finite-dimensional approximation of the attractor. Wang [3] and Zhao and Zhou [4] studied asymptotic behavior of nonautonomous lattice systems. In standard definition of exponential attractor, a compact and positively invariant M set is needed for the semigroup $S(t)$, and the system $(S(t), M)$ possesses a global attractor $\mathcal{A}$. More specifically, the semigroup $S(t)$ is not compact for all positive t. So, it is difficult to find a compact and positively invariant M which is not the attractor $\mathcal{A}$. The first-order dissipative lattice dynamical systems analogue to problem (1.3) is such an example. Babin and Nicolaenko [5] consider reaction-diffusion systems in unbounded domains, prove the existence of exponential attractors for such systems, and estimate their fractal dimension. In [5], the compactness assumption plays a relatively minor role in the whole construction. In [6], Eden et al. provide constructions of exponential attractor for a Lipschitz α-contraction S on a closed bounded B that satisfies the discrete squeezing property, where B is not assumed to be compact.

The main novelty of this work is that we make an improvement in the constructions of exponential attractors is indicated in [6] that if a map S is asymptotically compact on a closed bounded B that satisfies the discrete squeezing property, then S possesses an exponential attractor. S is not assumed to be α-contraction in the result. We apply the result to study an exponential attractor for a first-order dissipative lattice dynamical system. We not only construct an exponential attractor for the lattice dynamical system and consider its finite-dimensional approximation, but also obtain an upper bound of its fractal dimension.

2. A key theorem

Let E be a separable Hilbert space with the norm $\|\cdot\|$, $B \subseteq E$ be nonempty closed bounded set, and $S : B \to B$ be a Lipschitz continuous map with Lipschitz constant L. In this paper, we will always denote dist the Hausdorff semi-distance of sets as follows:

$$\operatorname{dist}(B, C) = \sup_{x\in B}\inf_{y\in C}\|x - y\|, \quad \text{for any } B,\ C \subset E. \tag{2.1}$$

Definition 2.1. S is *asymptotically compact* on B if for any $\{x_n\}_{n\geq 1} \subseteq B$, there is a convergent subsequence of $\{S^n x_n\}$ in E.

Remark 2.2. If S is an α-contraction on B, then S is asymptotically compact on B.

Definition 2.3. S is said to satisfy the *discrete squeezing property* on B if there exists an orthogonal projection P_N of rank N such that for every u and v in B,

$$\|P_N(Su - Sv)\| \leq \|(I - P_N)(Su - Sv)\| \Longrightarrow \|Su - Sv\| \leq \frac{1}{8}\|u - v\|. \tag{2.2}$$

Definition 2.4. A compact set M is called as an exponential attractor for (S, B) if

(i) $A \subseteq M \subseteq B$, where A is the global attractor;

(ii) $SM \subseteq M$, that M is positively invariant under S;

(iii) M has finite fractal dimension; and

(iv) there exist universal constants c_1, c_2 such that for every $u \in B$, for every natural number n, $\mathrm{dist}(S^n u, M) \leq c_1 e^{-c_2 n}$.

Let $P = P_N$ be the orthogonal projection chosen as in the definition of the squeezing property. Denote

$$\mathcal{F} = \max\left\{F \mid \text{all } u,\ v \in F \text{ satisfying } \|u - v\| \leq \sqrt{2}\|Pu - Pv\|\right\} \tag{2.3}$$

for the inclusion relation. From the definition of $\mathcal{F}$, we know $P|_{\mathcal{F}}$ is one-to-one on $\mathcal{F}$. Clearly, $P\mathcal{F}$ is a bounded closed set of a finite dimensional vector space, and therefore, it is compact. So, $\mathcal{F}$ as the preimage under the continuous map $P|_{\mathcal{F}}$ must also be compact.

Let $E^{(k)}$ be a subset of the set $S^{k+1}B$, which is formed by a finite union of exceptional sets of the form $\mathcal{F}$, which is described above, hence all $E^{(k)}$ are compact.

Lemma 2.5. *If S is asymptotically compact on B, then*

$$M = \bigcup_{j,k=1}^{+\infty} S^j\left(E^{(k)}\right) \tag{2.4}$$

is relatively compact.

Proof. Let $\{y_n\}_{n\geq 1}$ be a sequence in M. Then, two cases will appear as follows.
Case 1. There exists a natural number N_0 such that all y_n are in $\bigcup_{j,k=1}^{N_0} S^j(E^{(k)})$;
Case 2. There exists a subsequence (still denoted by $\{y_n\}_{n\geq 1}$) satisfying for every y_n, there exists $x_n \in B$ such that $y_n = S^n x_n$.

In Case 1, since $E^{(k)}$ is compact and S is continuous, there exists a convergent subsequence of $\{y_n\}_{n\geq 1}$ that converges in M. In Case 2, since S is asymptotically compact on B, it is immediate that we can extract from $\{y_n\}_{n\geq 1}$ a subsequence that converges in H. So, M is relatively compact. □

Theorem 2.6. *Let H be a separable Hilbert space and let B be a nonempty closed bounded subset of E. Assume that*

(i) *S is a Lipschitz continuous map with Lipschitz constant L on B;*

(ii) *S is asymptotically compact on B;*

(iii) *S satisfies the discrete squeezing property on B (with rank N_0), then S has an exponential attractor on B:*

$$\mathcal{M} = \mathcal{A} \cup \bigcup_{j,k=1}^{+\infty} S^j\left(E^{(k)}\right), \tag{2.5}$$

where $\mathcal{A}$ is a global attractor for S on B, $E^{(k)}$ is as the above-mentioned. Moreover, the fractal dimension of $\mathcal{M}$ satisfies

$$d_F(\mathcal{M}) \leq N_0 \max\left\{1, \frac{\operatorname{Log}(16L+1)}{2\operatorname{Log} 2}\right\}. \tag{2.6}$$

Proof. Note that all the limits point of $\bigcup_{j,k=1}^{+\infty} S^j(E^{(k)})$ belong to $\mathcal{A}$. Together with Lemma 2.5, the proof follows exactly in the same way as the proof of Theorem 2.1 in [7]. □

Remark 2.7. In Theorem 2.6, there are two advantages than all the previous results on the existence of exponential attractor for S:

(i) B is not assumed to be compact;

(ii) if S possesses a global attractor, then S is at least asymptotically compact. So, we only check that if S satisfies the Lipschitz property and the discrete squeezing property to obtain the existence of an exponential attractor for S.

3. Exponential attractor

For $i = (i_1, i_2, \ldots, i_k) \in \mathbb{Z}^k$, we will always denote $\|i\| = \max_{1\leq j\leq k}|i_j|$ in the following discussion. For any $u = (u_i)_{i\in\mathbb{Z}^k} \in \ell^2$, $i = (i_1, i_2, \ldots, i_k) \in \mathbb{Z}^k$, define the operators B_j, $\overline{B}_j$, and A_j, $j \in \{1, 2, \ldots, k\}$ from ℓ^2 to itself as follows: $u_i = u_{(i_1,i_2,\ldots,i_k)} \in \ell^2$, $j = 1, 2, \ldots, k$,

$$\begin{aligned}
(B_j u)_i &= u_{(i_1,\ldots,i_j+1,\ldots,i_k)} - u_{(i_1,\ldots,i_j,\ldots,i_k)}, \\
(\overline{B}_j u)_i &= u_{(i_1,\ldots,i_j,\ldots,i_k)} - u_{(i_1,\ldots,i_j-1,\ldots,i_k)}, \\
(A_j u)_i &= 2u_{(i_1,\ldots,i_j,\ldots,i_k)} - u_{(i_1,\ldots,i_j+1,\ldots,i_k)} - u_{(i_1,\ldots,i_j-1,\ldots,i_k)}.
\end{aligned} \tag{3.1}$$

Then, we have

$$A = A_1 + A_2 + \cdots + A_k, \qquad A_j = B_j\overline{B}_j = \overline{B}_j B_j, \quad j = 1, 2, \ldots, k. \tag{3.2}$$

For any $u = (u_i)_{i\in\mathbb{Z}^k}, v = (v_i)_{i\in\mathbb{Z}^k} \in \ell^2$, we define inner product and norm of ℓ^2 as follows:

$$(u, v) = \sum_{i\in\mathbb{Z}^k} u_i v_i, \qquad \|u\| = \left(\sum_{i\in\mathbb{Z}^k} |u_i|^2\right)^{1/2}, \tag{3.3}$$

then $\ell^2 = (\ell^2, (\cdot,\cdot), \|\cdot\|)$ is a Hilbert space. It is obvious that any $u = (u_i)_{i\in\mathbb{Z}^k}, v = (v_i)_{i\in\mathbb{Z}^k} \in \ell^2$,

$$(Au, v) = \sum_{j=1}^{k} (B_j u, B_j v) = \sum_{j=1}^{k} (\overline{B}_j u, \overline{B}_j v), \qquad \sum_{j=1}^{k} \|B_j u\|^2 \leq 4k\|u\|^2. \tag{3.4}$$

We always make the following assumptions on $g(s) \in C(\mathbb{R})$:

(H1) $g(0) \equiv 0$ and $g(s)s \geq 0$.

(H2) There exists an increasing continuous function $K(r) : \mathbb{R}^+ \to \mathbb{R}^+$ with $K(0) = 0$ such that

$$\sup_{|s|\leq r} |g'(s)| \leq K(r^2), \tag{3.5}$$

where $\mathbb{R}^+ = [0, +\infty)$.

Similar to [2, Theorem 1], we have.

Theorem 3.1. *For any initial data $u_0 \in \ell^2$, there exists a unique local solution $u(t)$ of problem* (1.3) *with $u(0) = u_0$ such that $u(t) \in C^1([0,T], \ell^2)$ for any $T > 0$.*

In fact, it will be showed in Lemma 3.2 below that the local solution $u(t)$ of problem (1.3) exists globally, that is, $u(t) \in C^1([0,+\infty), \ell^2)$. It implies that the map

$$S(t) : u(0) = u_0 \longmapsto u(t), \qquad \ell^2 \longmapsto \ell^2, \tag{3.6}$$

generates a continuous semigroup from ℓ^2 to itself.

Lemma 3.2. *Let $B_0 = B(0, r_0)$ be a closed bounded ball of ℓ^2, centered at* 0 *with radius r_0 where*

$$r_0^2 = \frac{1}{\lambda^2} \|\overline{q}\|^2. \tag{3.7}$$

For any bounded set B of ℓ^2, there exists $T(B) \geq 0$ such that

$$S(t)B \subseteq B_0, \quad \forall t \geq T(B). \tag{3.8}$$

Proof. The proof is easily obtained. □

Corollary 3.3. *For any $t \geq 0$, $S(t)B_0 \subseteq B_0$.*

We obtain the following lemma after some simple computation.

Lemma 3.4. *Let $u(t) = (u_i)_{i\in\mathbb{Z}^k} \in \ell^2$ be a solution of problem* (1.3) *with initial data $u_0 = (u_{0i})_{i\in\mathbb{Z}^k} \in B_0$. Then, for any $t \geq 0$,*

$$\sum_{\|i\|>N} |u_i(t)|^2 \leq r_0^2 e^{-\lambda t} + \frac{8C_0 k r_0^2}{\lambda N} + \frac{1}{\lambda^2} \sum_{\|i\|\geq N/2} |q_i|^2. \tag{3.9}$$

From Lemmas 3.2 and 3.4, we have the following.

Theorem 3.5. *The semigroup $\{S(t)\}_{t\geq 0}$ is asymptotically compact in ℓ^2 and possesses a nonempty compact global attractor $\mathcal{A}$. Furthermore, $\mathcal{A} \subseteq B_0$.*

Let $S(t)u_0 = U(t)$ and $S(t)w_0 = W(t)$. Since $u_0, w_0 \in B_0$, by Corollary 3.3, $U(t), W(t) \in B_0$, for $t \geq 0$. Let $Z(t) = S(t)u_0 - S(t)w_0 = U(t) - W(t)$. Then, $Z(t)$ satisfies

$$\dot{Z} + AZ + \lambda Z + \overline{g}(U) - \overline{g}(W) = 0, \qquad Z(0) = u_0 - w_0. \tag{3.10}$$

After some simple computation, we obtain the following.

Lemma 3.6 (Lipschitz property). *For any u_0, $w_0 \in B_0$ and any $T > 0$,*

$$\|S(T)u_0 - S(T)w_0\| \leq e^{(K(r_0^2)-\lambda)T}\|u_0 - w_0\|. \tag{3.11}$$

Let $n \in \mathbb{N}$ be a positive integer. Set

$$\omega = \begin{pmatrix} \omega_{(-n,-n,\ldots,-n,-n)} & \omega_{(-n,-n,\ldots,-n,-n+1)} & \cdots & \omega_{(-n,-n,\ldots,-n,n)} \\ \omega_{(-n,-n,\ldots,-n+1,-n)} & \omega_{(-n,-n,\ldots,-n+1,-n+1)} & \cdots & \omega_{(-n,-n,\ldots,-n+1,n)} \\ \cdots & \cdots & \cdots & \cdots \\ \omega_{(n,n,\ldots,n,-n)} & \omega_{(n,n,\ldots,n,-n+1)} & \cdots & \omega_{(n,n,\ldots,n,n)} \end{pmatrix}. \tag{3.12}$$

For convenience, we always denote

$$\begin{aligned} E_n = \{\omega = (\omega_i)_{i\in\mathbb{Z}^k} \in \ell^2 \mid \omega_i \text{ with subscripts of components} \\ \text{of } \omega \text{ which are ordered as in (3.12) and } \omega_i = 0,\ \|i\| > n\}, \end{aligned} \tag{3.13}$$

with the same inner product and norm as those of ℓ^2.

Let $\mathcal{K}(x)$ be the inverse function of $K(x)$ in (H2). Set

$$T_0 = \max\left\{\frac{4}{\lambda}\operatorname{Log} 2, \frac{1}{\lambda}\left(\operatorname{Log} 2 + 2\operatorname{Log}\|\overline{q}\| - 2\operatorname{Log}\lambda - \operatorname{Log}\mathcal{K}\left(\frac{\lambda}{2}\right)\right)\right\}, \tag{3.14}$$

$$N_0 = \min\left\{N \in \mathbb{N} : \frac{8C_0k\|\overline{q}\|^2}{\lambda N} + \frac{1}{\lambda^2}\sum_{\|i\|\geq N/2}|q_i|^2 \leq \frac{1}{2}\mathcal{K}\left(\frac{\lambda}{2}\right)\right\}. \tag{3.15}$$

Suppose P_N is an orthogonal projection of rank $(2N+1)^k$ on ℓ^2 such that $P_N\ell^2 = E_N$.

Lemma 3.7 (Discrete squeezing property). *For any u_0, $w_0 \in B_0$, if*

$$\|P_{N_0}(S(T_0)u_0 - S(T_0)w_0)\| \leq \|(I - P_{N_0})(S(T_0)u_0 - S(T_0)w_0)\|, \tag{3.16}$$

then

$$\|S(T_0)u_0 - S(T_0)w_0\| \leq \frac{1}{8}\|u_0 - w_0\|. \tag{3.17}$$

Proof. Denote $U_{N_0}(t) = (I-P_{N_0})u(t)$, $W_{N_0}(t) = (I-P_{N_0})w(t)$ and $Z_{N_0}(t) = (I-P_{N_0})(u(t)-w(t)) = (I-P_{N_0})Z(t)$. Taking the inner product $(\cdot,\cdot)$ in (3.10) with Z_{N_0}, we have

$$\frac{d}{dt}\|Z_{N_0}\|^2 + 2\lambda\|Z_{N_0}\|^2 + 2(\overline{g}(U) - \overline{g}(W), Z_{N_0}) \leq 0, \tag{3.18}$$

where $\|Z_{N_0}\|^2 = \sum_{\|i\|>N_0}|Z_i|^2$. By the mean value theorem,

$$|(\overline{g}(U) - \overline{g}(W), Z_{N_0})| \leq \sum_{\|i\|>N_0}|g'(U_i + \theta_i(W_i - U_i))||Z_i|^2, \tag{3.19}$$

where $\theta_i \in (0,1)$, $\|i\| > N_0$. By (H2) and Lemma 3.4, for $t \geq T_0$,

$$\left|g'\left(U_i + \theta_i(W_i - U_i)\right)\right| \leq \frac{\lambda}{2}, \tag{3.20}$$

which implies

$$\left|\left(\overline{g}(U) - \overline{g}(W), Z_{N_0}\right)\right| \leq \frac{\lambda}{2} \sum_{\|i\|>N_0} |Z_i|^2. \tag{3.21}$$

By (3.18), (3.21), and the Gronwall inequality, we have

$$\left\|Z_{N_0}(t)\right\|^2 \leq e^{-\lambda(t-T_0)}\left\|u_0 - w_0\right\|^2, \tag{3.22}$$

for all $t \geq T_0$. So, for any $u_0, w_0 \in B_0$, if

$$\left\|P_{N_0}\left(S(T_0)u_0 - S(T_0)w_0\right)\right\| \leq \left\|\left(I - P_{N_0}\right)\left(S(T_0)u_0 - S(T_0)w_0\right)\right\|, \tag{3.23}$$

then

$$\left\|S(T_0)u_0 - S(T_0)w_0\right\| \leq 2\left\|\left(I - P_{N_0}\right)\left(S(T_0)u_0 - S(T_0)w_0\right)\right\| \leq \frac{1}{8}\left\|u_0 - w_0\right\|. \tag{3.24}$$

□

From Theorems 2.6 and 3.5, Lemmas 3.6 and 3.7 in this article, and [7, Theorem 3.1], we obtain.

Theorem 3.8. *The semigroup $S(t)$ determined by problem* (1.3) *with (H1)-(H2) possesses an exponential attractor on B_0:*

$$\mathfrak{M} = \bigcup_{0\leq t\leq T_0} S(t)\left(\mathcal{A} \cup \bigcup_{j,k=1}^{+\infty} S_0^j\left(E^{(k)}\right)\right), \tag{3.25}$$

whose fractal dimension satisfies

$$d_F(\mathfrak{M}) \leq c_0\left(2N_0 + 1\right)^k + 1, \tag{3.26}$$

where T_0 is as (3.14), *$S_0 = S(T_0)$, $E^{(k)}$ is defined as in Section 2 and N_0 is as* (3.15), $c_0 = \max\{1, \operatorname{Log}(16e^{(K(r_0)-\lambda)T_0} + 1)/2\operatorname{Log} 2\}$.

Remark 3.9. Indeed, when $K(r_0^2) < \lambda$, by Lemma 3.6, we easily know that $S(t)$ has an exponential attractor of dimension zero on B_0, which is an equilibrium point of problem (1.3) (the global attractor for $S(t)$).

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