Research Article

A Strong Limit Theorem for Functions of Continuous Random Variables and an Extension of the Shannon-McMillan Theorem

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By means of the notion of likelihood ratio, the limit properties of the sequences of arbitrary-dependent continuous random variables are studied, and a kind of strong limit theorems represented by inequalities with random bounds for functions of continuous random variables is established. The Shannon-McMillan theorem is extended to the case of arbitrary continuous information sources. In the proof, an analytic technique, the tools of Laplace transform, and moment generating functions to study the strong limit theorems are applied.

1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of arbitrary continuous real random variables on the probability space $(\Omega, \mathcal{F}, P)$ with the joint density function

$$f_n(x_1, \ldots, x_n) > 0, \quad n = 1, 2, \ldots, \tag{1.1}$$

where $x_i \in (-\infty, \infty)$, $1 \leq i \leq n$. Let Q be another probability measure on $\mathcal{F}$, and $\{X_n, n \geq 1\}$ is a sequence of independent random variables on the probability space $(\Omega, \mathcal{F}, Q)$ with the marginal density functions $g_k(x_k)$ $(1 \leq k \leq n)$, and let

$$\pi_n(x_1, \ldots, x_n) = \prod_{k=1}^{n} g_k(x_k). \tag{1.2}$$

In order to indicate the deviation between $\{X_n, n \geq 1\}$ on the probability measure P and Q, we first introduce the following definitions.

Definition 1.1. Let $\{X_n, n \geq 1\}$ be a sequence of random variables with joint distribution (1.1), and let $g_k(x_k)$ $(k = 1, 2, \ldots, n)$ be defined by (1.2). Let

$$Z_n(\omega) = \frac{f_n(X_1, \ldots, X_n)}{\pi_n(X_1, \ldots, X_n)}. \tag{1.3}$$

In statistical terms, $Z_n(\omega)$ is called the likelihood ratio, which is of fundamental importance in the theory of testing the statistical hypotheses (cf. [1, page 388]; [2, page 483]).

The random variable

$$r(\omega) = \limsup_{n \to \infty} \frac{1}{n} \ln Z_n(\omega) = -\liminf_{n \to \infty} \frac{1}{n} \ln \left[\left\{ \prod_{k=1}^{n} g_k(X_k) \right\} \bigg/ f_n(X_1, \ldots, X_n) \right] \tag{1.4}$$

is called asymptotic logarithmic likelihood ratio, relative to the product of marginal distribution of (1.2), of X_n, $n \geq 1$, where ln is the natural logarithm, ω is the sample point. For the sake of brevity, we denote $X_k(\omega)$ by X_k.

Although $r(\omega)$ is not a proper metric between probability measures, we nevertheless think of it as a measure of "dissimilarity" between their joint distribution $f_n(x_1, \ldots, x_n)$ and the product $\pi_n(x_1, \ldots, x_n)$ of their marginals.

Obviously, $r(\omega) = 0$, a.s. if and only if $\{X_n, n \geq 1\}$ are independent.

A stochastic process of fundamental importance in the theory of testing hypotheses is the sequence of likelihood ratio. In view of the above discussion of the asymptotic logarithmic likelihood ratio, it is natural to think of $r(\omega)$ as a measure how far (the random deviation of) X_n is from being independent, how dependent they are. The smaller $r(\omega)$ is, the smaller the deviation is (cf. [3–5]).

In [3], the strong deviation theorems for discrete random variables were discussed by using the generating function method. Later, the approach of Laplace transform to study the strong limit theorems was first proposed by Liu [4]. Yang [6] further studied the limit properties for Markov chains indexed by a homogeneous tree through the analytic technique. Many comprehensive works may be found in Liu [7]. The purpose of this paper is to establish a kind of strong deviation theorems represented by inequalities with random bounds for functions of arbitrary continuous random variables, by combining the analytic technique with the method of Laplace transform, and to extend the strong deviation theorems to the differential entropy for arbitrary-dependent continuous information sources in more general settings.

Definition 1.2. Let $\{h_n(x_n), n \geq 1\}$ be a sequence of nonnegative $\mathcal{B}$ orel measurable functions defined on $\mathcal{R}$, the Laplace transform of random variables $h_n(X_n)$ on the probability space $(\Omega, \mathcal{F}, Q)$ is defined by

$$\tilde{f}_n(s) = \int_{-\infty}^{\infty} e^{-sh_n(x_n)} g_n(x_n)\, dx_n = E_Q e^{-sh_n(X_n)}, \quad n = 1, 2, \ldots, \tag{1.5}$$

where E_Q denotes the expectation under Q.

We have the following assumptions in this paper.

(1) Assume that there exists $s_0 \in (0, \infty)$ such that

$$\tilde{f}_n(s) < \infty, \quad s \in [-s_0, s_0], \ n = 1, 2, \ldots. \tag{1.6}$$

(2) Assume $M > 0$ is a constant, satisfying

$$\sup_n E_Q h_n(X_n) \leq M < \infty. \tag{1.7}$$

In order to prove our main results, we first give a lemma, and it will be shown that it plays a central role in the proofs.

Lemma 1.3. *Let $f_n(x_1, \ldots, x_n)$, $g_n(x_1, \ldots, x_n)$ be two probability functions on $(\Omega, \mathcal{F}, P)$, let*

$$T_n(\omega) = \frac{g_n(X_1, \ldots, X_n)}{f_n(X_1, \ldots, X_n)}, \tag{1.8}$$

then

$$\limsup_{n \to \infty} \frac{1}{n} \ln T_n(\omega) \leq 0 \quad a.s. \tag{1.9}$$

Proof. By [8], $\{T_n, \mathcal{F}, n \geq 1\}$ is a nonnegative martingale and $ET_n = 1$, we have by the Doob martingale convergence theorem, there exists an integral random variable $T_\infty(\omega)$, such that $T_n \to T_\infty$, a.s. and (1.9) follows. □

2. Main results

Theorem 2.1. *Let $\{X_n, n \geq 1\}$, $Z_n(\omega)$, $r(\omega)$, $\tilde{f}_n(s)$ be defined as before, and under the assumptions of (1) and (2), let*

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} E_Q e^{s_0 h_k(X_k)} = \limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \tilde{f}_k(-s_0) = M(s_0) < \infty. \tag{2.1}$$

Then

$$\liminf_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} [h_k(X_k) - E_Q h_k(X_k)] \geq -\beta(r(\omega)), \quad a.s. \tag{2.2}$$

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} [h_k(X_k) - E_Q h_k(X_k)] \leq \beta(r(\omega)), \quad a.s. \tag{2.3}$$

where

$$\beta(x) = \inf\{\phi(s, x), -s_0 < s < 0\}, \quad x \geq 0, \tag{2.4}$$

$$\phi(s, x) = -\frac{2e^{-2}sM(s_0)}{(s_0 - |s|)^2} - \frac{x}{s}, \quad x \geq 0, \tag{2.5}$$

$$0 \leq \beta(x) \leq \frac{\sqrt{x}(1 + \sqrt{x})}{s_0} [2e^{-2}M(s_0) + 1], \quad \lim_{x \to 0+} \beta(x) = \beta(0) = 0. \tag{2.6}$$

Remark 2.2. Let

$$\alpha(x) = \sup\{\phi(s,x),\, 0 < s < s_0\}, \quad x \geq 0, \tag{2.7}$$

then

$$\alpha(x) = -\beta(x). \tag{2.8}$$

Proof. Let s be an arbitrary real number in $(-s_0, s_0)$, let

$$g_n(s, x_n) = e^{-s h_n(x_n)} g_n(x_n) / \widetilde{f}_n(s), \quad x_n \in (-\infty, \infty), \; n = 1, 2, \ldots, \tag{2.9}$$

then $\int_{-\infty}^{\infty} g_n(s, x_n)\, dx_n = 1$, and let

$$q_n(s; x_1, \ldots, x_n) = \prod_{k=1}^{n} g_k(s, x_k). \tag{2.10}$$

Therefore, $q_n(s; x_1, \ldots, x_n)$ is an n multivariate probability density function, let

$$t_n(s, \omega) = \frac{q_n(s; X_1, \ldots, X_n)}{f_n(X_1, \ldots, X_n)}, \quad n = 1, 2, \ldots. \tag{2.11}$$

By Lemma 1.3, there exists a set $A(s)$ such that $P(A(s)) = 1$, so we have

$$\limsup_{n \to \infty} \frac{1}{n} \ln t_n(s, \omega) \leq 0, \quad \omega \in A(s). \tag{2.12}$$

By (1.3), (2.9), (2.11), and (2.12), we have

$$\limsup_{n \to \infty} \frac{1}{n} \left[-s \sum_{k=1}^{n} h_k(X_k) - \sum_{k=1}^{n} \ln \widetilde{f}_k(s) - \ln Z_n(\omega) \right] \leq 0, \quad \omega \in A(s). \tag{2.13}$$

Therefore,

$$r(\omega) \geq 0, \quad \omega \in A(0). \tag{2.14}$$

By (2.13) and (1.4), the property of the superior limit

$$\limsup_{n \to \infty}(a_n - b_n) \leq 0 \implies \limsup_{n \to \infty} a_n \leq \limsup_{n \to \infty} b_n, \tag{2.15}$$

and the inequality $\ln x \leq x - 1$ $(x > 0)$, we have

$$
\begin{aligned}
\limsup_{n \to \infty} & \frac{1}{n}(-s) \sum_{k=1}^{n} \left[h_k(X_k) - E_Q h_k(X_k) \right] \\
&\leq \limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \left[\ln \widetilde{f}_k(s) + s E_Q h_k(X_k) \right] + r(\omega) \\
&\leq \limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \left[\widetilde{f}_k(s) - 1 + s E_Q h_k(X_k) \right] + r(\omega) \\
&= \limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \left[\widetilde{f}_k(s) - 1 - (-s) E_Q h_k(X_k) \right] + r(\omega), \quad \omega \in A(s).
\end{aligned}
\tag{2.16}
$$

By the inequality $0 \leq e^x - 1 - x \leq (1/2)x^2 e^{|x|}$, which can be found in [9], we have

$$0 \leq e^{-sh_k(X_k)} - 1 - (-s)h_k(X_k) \leq (1/2)(sh_k(X_k))^2 e^{|sh_k(X_k)|}. \tag{2.17}$$

By (2.5) and (2.17), we have

$$\begin{aligned}
\limsup_{n \to \infty} \frac{1}{n}(-s)\sum_{k=1}^{n} [h_k(X_k) - E_Q h_k(X_k)] \\
\leq (1/2)s^2 \limsup_{n \to \infty} \frac{1}{n}\sum_{k=1}^{n} E_Q[(h_k(X_k))^2 e^{|s||h_k(X_k)|}] + r(\omega), \quad \omega \in A(s).
\end{aligned} \tag{2.18}$$

It is easy to see that $\varphi(x) = t^x x^2$ $(t > 1)$ attains its largest value $\varphi(-2/\ln t) = 4e^{-2}/(\ln t)^2$ on the interval $(-\infty, 0]$, and $\varphi(x) = t^x x^2$ $(0 < t < 1)$ attains its largest value $\varphi(-2/\ln t) = 4e^{-2}/(\ln t)^2$ on the interval $[0, \infty)$, we have

$$\sup\{e^{(s-s_0)h_k(X_k)}[h_k(X_k)]^2, k \geq 1\} \leq \frac{4e^{-2}}{(s-s_0)^2} = \frac{4e^{-2}}{(s_0-s)^2}, \quad 0 < s < s_0, \tag{2.19}$$

$$\sup\{e^{(s_0+s)(-h_k(X_k))}[-h_k(X_k)]^2, k \geq 1\} \leq \frac{4e^{-2}}{(s_0+s)^2}, \quad -s_0 < s < 0. \tag{2.20}$$

Let $0 < s < s_0$ in (2.18), by (2.19) and (2.1), we obtain

$$\begin{aligned}
(-s)\liminf_{n \to \infty} \frac{1}{n}\sum_{k=1}^{n} [h_k(X_k) - E_Q h_k(X_k)] \\
\leq (1/2)s^2 \limsup_{n \to \infty} \frac{1}{n}\sum_{k=1}^{n} E_Q[(h_k(X_k))^2 e^{|s||h_k(X_k)|}] + r(\omega) \\
= (1/2)s^2 \limsup_{n \to \infty} \frac{1}{n}\sum_{k=1}^{n} E_Q[e^{s_0 h_k(X_k)} e^{(s-s_0)h_k(X_k)}(h_k(X_k))^2] + r(\omega) \\
\leq \frac{2e^{-2}s^2 M(s_0)}{(s_0-s)^2} + r(\omega), \quad \omega \in A(s).
\end{aligned} \tag{2.21}$$

Dividing the two sides of (2.21) by $-s$, we obtain

$$\begin{aligned}
\liminf_{n \to \infty} \frac{1}{n}\sum_{k=1}^{n} [h_k(X_k) - E_Q h_k(X_k)] \geq -\frac{2e^{-2}s M(s_0)}{(s_0-s)^2} - \frac{r(\omega)}{s} \\
\hat{=} \phi(s, r(\omega)), \quad 0 < s < s_0, \ \omega \in A(s) \cap A(0).
\end{aligned} \tag{2.22}$$

By (2.14) and $0 < s < s_0$, obviously $\phi(s, r(\omega)) \leq 0$, hence $\alpha(r(\omega)) \leq 0$. Let Q^+ be the set of rational numbers in the interval $(0, s_0)$, and let $A^* = \cap_{s \in Q^+} A(s)$, then $P(A^*) = 1$. By (2.22), then we have

$$\liminf_{n \to \infty} \frac{1}{n}\sum_{k=1}^{n} [h_k(X_k) - E_Q h_k(X_k)] \geq \phi(s, r(\omega)), \quad \omega \in A^* \cap A(0), \ \forall s \in Q^+. \tag{2.23}$$

It is easy to see that $\phi(s, x)$ is a continuous function with respect to s on the interval $(0, s_0)$. For each $\omega \in A^* \cap A(0)$ $(0 \le r(\omega) < \infty)$, take $s_n(\omega) \in Q^+$, $n = 1, 2, \ldots$, such that

$$\lim_{n \to \infty} \phi(s_n(\omega), r(\omega)) = \alpha(r(\omega)). \tag{2.24}$$

By (2.23), (2.24), and (2.8), we have

$$\liminf_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} [h_k(X_k) - E_Q h_k(X_k)] \ge -\beta(r(\omega)), \quad \omega \in A^* \cap A(0). \tag{2.25}$$

Since $P(A^* \cap A(0)) = 1$, (2.2) follows from (2.25).

Let $-s_0 < s < 0$ in (2.18), by (2.20) and (2.1), we have

$$\begin{aligned}
\limsup_{n \to \infty} &\frac{1}{n} \sum_{k=1}^{n} [h_k(X_k) - E_Q h_k(X_k)] \\
&\le -(1/2)s \limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} E_Q \left[(h_k(X_k))^2 e^{|s||h_k(X_k)|} \right] - \frac{r(\omega)}{s} \\
&= -(1/2)s \limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} E_Q \left[e^{s_0 h_k(X_k)} e^{(s_0+s)(-h_k(X_k))} (-h_k(X_k))^2 \right] - \frac{r(\omega)}{s} \\
&\le -\frac{2e^{-2} s M(s_0)}{(s_0 + s)^2} - \frac{r(\omega)}{s} \\
&\hat{=} \phi(s, r(\omega)), \quad -s_0 < s < 0, \ \omega \in A(s) \cap A(0).
\end{aligned} \tag{2.26}$$

By (2.14) and $-s_0 < s < 0$, obviously $\phi(s, r(\omega)) \ge 0$, hence $\beta(r(\omega)) \ge 0$. Let Q^- be the set of rational numbers in the interval $(-s_0, 0)$, and let $A_* = \cap_{s \in Q^-} A(s)$, then $P(A_*) = 1$. Then we have by (2.26)

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} [h_k(X_k) - E_Q h_k(X_k)] \le \phi(s, r(\omega)), \quad \omega \in A_* \cap A(0), \ \forall s \in Q^-. \tag{2.27}$$

It is clear that $\phi(s, x)$ is a continuous function with respect to s on the interval $(-s_0, 0)$. For each $\omega \in A_* \cap A(0)$ $(0 \le r(\omega) < \infty)$, take $\lambda_n(\omega) \in Q^-$, $n = 1, 2, \ldots$, such that

$$\lim_{n \to \infty} \phi(\lambda_n(\omega), r(\omega)) = \beta(r(\omega)). \tag{2.28}$$

By (2.27) and (2.28), we have

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} [h_k(X_k) - E_Q h_k(X_k)] \le \beta(r(\omega)), \quad \omega \in A_* \cap A(0). \tag{2.29}$$

Since $P(A_* \cap A(0)) = 1$, (2.3) follows from (2.29).

By (2.4), (2.5), and (2.14), if $x > 0$, we have

$$0 \le \beta(x) \le \phi\left(-\frac{s_0 \sqrt{x}}{1 + \sqrt{x}}, x\right) = \frac{\sqrt{x}(1 + \sqrt{x})}{s_0} \left[2e^{-2} M(s_0) + 1\right]. \tag{2.30}$$

If $x = 0$, we have

$$\beta(0) \le \phi(-n^{-1}, 0) = \frac{2e^{-2} M(s_0)}{n(s_0 - n^{-1})^2}, \quad n \ge 1. \tag{2.31}$$

Noticing that $\beta(x) \ge 0$, $(x \ge 0)$, (2.6) follows from (2.30) and (2.31). □

Corollary 2.3. *If $P = Q$, or $\{X_n, n \geq 1\}$ is a sequence of independent random variables, and under the assumptions of (1) and (2), then*

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} [h_k(X_k) - Eh_k(X_k)] = 0 \quad a.s. \tag{2.32}$$

Proof. In this case, $f_n(x_1, \ldots, x_n) = \prod_{k=1}^{n} g_k(x_k)$, and $r(\omega) = 0$ a.s. Hence, (2.32) follows directly from (2.2) and (2.3). □

3. An extension of the Shannon-McMillan theorem

In order to understand better, we first introduce some definitions in information theory in this section.

Let $\{X_n, n \geq 1\}$ be a sequence produced by an arbitrary continuous information source on the probability space $(\Omega, \mathcal{F}, P)$ with the joint density function

$$f_n(x_1, \ldots, x_n) > 0, \quad x_k \in (-\infty, \infty), \quad 1 \leq k \leq n, \ n = 1, 2, \ldots. \tag{3.1}$$

For the sake of brevity, we denote $f_n > 0$, and X_k stands for $X_k(\omega)$. Let

$$p_n(\omega) = -(1/n) \ln f_n(X_1, \ldots, X_n), \tag{3.2}$$

where ω is the sample point, $p_n(\omega)$ is called the sample entropy or the entropy density of $\{X_k, 1 \leq k \leq n\}$. Also let Q be another probability measure on $\mathcal{F}$ with the density function

$$q_n(x_1, \ldots, x_n) > 0, \quad x_k \in (-\infty, \infty), \quad 1 \leq k \leq n, \ n = 1, 2, \ldots. \tag{3.3}$$

Let

$$
\begin{aligned}
L_n(\omega) &= \ln [f_n(X_1, \ldots, X_n)/q_n(X_1, \ldots, X_n)], \\
L(\omega) &= \limsup_{n \to \infty} (1/n) L_n(\omega) \\
&= -\liminf_{n \to \infty} (1/n) \ln [q_n(X_1, \ldots, X_n)/f_n(X_1, \ldots, X_n)], \\
D(f_n \| q_n) &= E_P L_n \\
&= E_P \ln [f_n(X_1, \ldots, X_n)/q_n(X_1, \ldots, X_n)].
\end{aligned}
\tag{3.4}
$$

$L_n(\omega)$, $L(\omega)$, and $D(f_n \| q_n)$ are called the sample relative entropy, the sample relative entropy rate, and the relative entropy, respectively, relative to the reference density function $q_n(x_1, \ldots, x_n)$. Indeed, they all are the measure of the deviation between the true joint distribution density function $f_n(x_1, \ldots, x_n)$ and the reference distribution density function $q_n(x_1, \ldots, x_n)$ (cf. [10, pages 12, 18]).

A question of importance in information theory is the study of the limit properties of the relative entropy density $f_n(\omega)$. Since Shannon's initial work was published (cf. [11]), there has been a great deal of investigation about this question (e.g., cf. [12–20]).

In this paper, a class of small deviation theorems (i.e., the strong limit theorems represented by inequalities) is established by using the analytical technique, and an extension of the Shannon-McMillan theorem to the arbitrary-dependent continuous information sources is given. Especially, an approach of applying the tool of Laplace transform to the study of the strong deviation theorems on the differential entropy is proposed.

Let $h_k(x_k) = -\ln g_k(x_k)$ $(1 \le k \le n, n = 1,2,\ldots)$ in (1.5), then we give the following definitions.

Definition 3.1. The Laplace transform of $-\ln g_k(x_k)$ is defined by

$$\widetilde{f}_k(s) = Ee^{-s(-\ln g_k(X_k))} = \int_{-\infty}^{\infty} e^{-s(-\ln g_k(x_k))} g_k(x_k)\, dx_k. \tag{3.5}$$

Definition 3.2. The differential entropy for continuous random variables X_k is defined by

$$h(X_k) = E\left[-\ln g_k(X_k)\right] = -\int_{-\infty}^{\infty} g_k(x_k) \ln g_k(x_k)\, dx_k. \tag{3.6}$$

In the following theorem, let $\{X_n, n \ge 1\}$ be independent random variables with respect to Q, then the reference density function $q_n(x_1,\ldots,x_n) = \prod_{k=1}^{n} g_k(x_k)$, and let $h_k(X_k) = -\ln g_k(X_k)$ $(1 \le k \le n)$ in Theorem 2.1.

Theorem 3.3. *Let $\{X_n, n \ge 1\}$, $L_n(\omega)$, $L(\omega)$, $\widetilde{f}_n(s)$ be given as above, and under the assumptions of (1) and (2), let*

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} E_Q e^{s_0(-\ln g_k(X_k))} = \limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \widetilde{f}_k(-s_0) = M(s_0) < \infty. \tag{3.7}$$

Then

$$\liminf_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \left[-\ln g_k(X_k) - h(X_k)\right] \ge -\beta(L(\omega)), \quad a.s.$$

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \left[-\ln g_k(X_k) - h(X_k)\right] \le \beta(L(\omega)), \quad a.s. \tag{3.8}$$

where

$$\beta(x) = \inf\left\{\phi(s,x),\; -s_0 < s < 0\right\}, \quad x \ge 0, \tag{3.9}$$

$$\phi(s,x) = -\frac{2e^{-2}sM(s_0)}{(s_0 - |s|)^2} - \frac{x}{s}, \quad x \ge 0, \tag{3.10}$$

$$0 \le \beta(x) \le \frac{\sqrt{x}(1 + \sqrt{x})}{s_0}\left[2e^{-2}M(s_0) + 1\right], \qquad \lim_{x \to 0+} \beta(x) = \beta(0) = 0. \tag{3.11}$$

Remark 3.4. Let

$$\alpha(x) = \sup\left\{\phi(s,x),\; 0 < s < s_0\right\}, \quad x \ge 0, \tag{3.12}$$

then

$$\alpha(x) = -\beta(x). \tag{3.13}$$

Corollary 3.5. *Let $p_n(\omega)$ be defined by* (3.2). *Under the condition of Theorem 3.3, then*

$$\liminf_{n\to\infty}\left[p_n(\omega) - (1/n)h(X_1,\ldots,X_n)\right] \geq \alpha\big(L(\omega)\big) - L(\omega) + H_* \quad a.s.$$

$$\limsup_{n\to\infty}\left[p_n(\omega) - (1/n)h(X_1,\ldots,X_n)\right] \leq \beta\big(L(\omega)\big) + H^* \quad a.s.$$

(3.14)

where $h(X_1,\ldots,X_n) = E[-\ln f_n(X_1,\ldots,X_n)]$ is the differential entropy for $(X_1,\ldots,X_n)$, and

$$
\begin{aligned}
H_* &= \liminf_{n\to\infty}(1/n)\left[\sum_{k=1}^{n}h(X_k) - h(X_1,\ldots,X_n)\right]\\
&= \liminf_{n\to\infty}(1/n)D(f_n\|q_n),\\
H^* &= \limsup_{n\to\infty}(1/n)\left[\sum_{k=1}^{n}h(X_k) - h(X_1,\ldots,X_n)\right]\\
&= \limsup_{n\to\infty}(1/n)D(f_n\|q_n),
\end{aligned}
$$

(3.15)

where $\alpha(L(\omega))$, $\beta(L(\omega))$ are denoted by (3.9)–(3.13).

Corollary 3.6. *If $P = Q$, or $\{X_n, n \geq 1\}$ are independent random variables, and there exists $s_0 > 0$, such that* (2.1) *holds, then*

$$\lim_{n\to\infty}\left[p_n(\omega) - (1/n)h(X_1,\ldots,X_n)\right] = 0 \quad a.s.$$

(3.16)

Acknowledgments

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