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Research Article

A Constructive Sharp Approach to Functional Quantization of Stochastic Processes

Stefan Junglen and Harald Luschgy

FB4-Department of Mathematics, University of Trier, 54286 Trier, Germany

Correspondence should be addressed to Harald Luschgy, luschgy@uni-trier.de

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We present a constructive approach to the functional quantization problem of stochastic processes, with an emphasis on Gaussian processes. The approach is constructive, since we reduce the infinite-dimensional functional quantization problem to a finite-dimensional quantization problem that can be solved numerically. Our approach achieves the sharp rate of the minimal quantization error and can be used to quantize the path space for Gaussian processes and also, for example, Lévy processes.

1. Introduction

We consider a separable Banach space $(E, \|\cdot\|)$ and a Borel random variable $X: (\Omega, \mathcal{F}, \mathbb{P}) \to (E, \mathcal{B}(E))$ with finite rth moment $\mathbb{E}\|X\|^r$ for some $r \in [1, \infty)$.

For a given natural number $n \in \mathbb{N}$, the *quantization problem* consists in finding a set $\alpha \subset E$ that minimizes

$$e_r(X, (E, \|\cdot\|), \alpha) = e_r(X, E, \alpha) := \left(\mathbb{E} \min_{a \in \alpha} \|X - a\|^r \right)^{1/r} \tag{1.1}$$

over all subsets $\alpha \subset E$ with card $(\alpha) \leq n$. Such sets α are called *n-codebooks* or *n-quantizers*. The corresponding infimum

$$e_{n,r}(X, (E, \|\cdot\|)) = e_{n,r}(X, E) := \inf_{\alpha \subset E, \operatorname{card}(\alpha) \leq n} e_r(X, E, \alpha) \tag{1.2}$$

is called the nth L^r-*quantization error* of X in E, and any n-quantizer α fulfilling

$$e_r(X,E,\alpha) = e_{n,r}(X,E) \tag{1.3}$$

is called *r-optimal n-quantizer*. For a given n-quantizer α one defines the nearest neighbor projection

$$\pi_\alpha : E \longrightarrow \alpha, \qquad x \longrightarrow \sum_{a\in\alpha} a\chi_{C_a(\alpha)}(x), \tag{1.4}$$

where the Voronoi partition $\{C_a(\alpha), a \in \alpha\}$ is defined as a Borel partition of E satisfying

$$C_a(\alpha) \subset \left\{ x \in \mathbb{E} : \|x-a\| = \min_{b\in\alpha}\|x-b\| \right\}. \tag{1.5}$$

The random variable $\pi_\alpha(X)$ is called *α-quantization* of X. One can easily verify that $\pi_\alpha(X)$ is the best quantization of X in $\alpha \subset E$, which means that for every random variable Y with values in α we have

$$e_r(X,E,\alpha) = \left(\mathbb{E}\|X-\pi_\alpha(X)\|^r\right)^{1/r} \leq \left(\mathbb{E}\|X-Y\|^r\right)^{1/r}. \tag{1.6}$$

Applications of quantization go back to the 1940s, where quantization was used for the finite-dimensional setting $E = \mathbb{R}^d$, called optimal *vector quantization*, in signal compression and information processing (see, e.g., [1, 2]). Since the beginning of the 21st century, quantization has been applied for example in finance, especially for pricing path-dependent and American style options. Here, vector quantization [3] as well as *functional quantization* [4, 5] is useful. The terminology functional quantization is used when the Banach space E is a function space, such as $E = (L_p[0,1], \|\cdot\|_p)$ or $E = C([0,1], \|\cdot\|_\infty)$. In this case, the realizations of X can be seen as the paths of a stochastic process.

A question of theoretical as well as practical interest is the issue of high-resolution quantization which concerns the behavior of $e_{n,r}(X,E)$ when n tends to infinity. By separability of $(E, \|\cdot\|)$, we can choose a dense subset $\{c_i, i \in \mathbb{N}\}$ and we can deduce in view of

$$0 \leq \lim_{n\to\infty} \mathbb{E} \min_{1\leq i\leq n} \|X-c_i\|^r = \mathbb{E} \lim_{n\to\infty} \min_{1\leq i\leq n} \|X-c_i\|^r = 0 \tag{1.7}$$

that $e_{n,r}(X,E)$ tends to zero as n tends to infinity.

A natural question is then if it is possible to describe the asymptotic behavior of $e_{n,r}(X,E)$. It will be convenient to write $a_n \sim b_n$ for sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ if $a_n/b_n \xrightarrow{n\to\infty} 1$, $a_n \lesssim b_n$ if $\limsup_{n\to\infty} a_n/b_n \leq 1$ and $a_n \approx b_n$ if $0 < \liminf_{n\to\infty} a_n/b_n \leq \limsup_{n\to\infty} a_n/b_n < \infty$.

In the finite-dimensional setting $(\mathbb{R}^d, \|\cdot\|)$ this behavior can satisfactory be described by the *Zador Theorem* (see [6]) for nonsingular distributions $\mathbb{P}^X$. In the infinite dimensional case, no such global result holds so far, without some additional restrictions. To describe one of the most famous results in this field, we call a measurable function $\rho : (s, \infty) \to (0, \infty)$ for an $s \geq 0$ *regularly varying at infinity* with index $b \in \mathbb{R}$ if for every $c > 0$

$$\lim_{x \to \infty} \frac{\rho(cx)}{\rho(x)} = c^b. \tag{1.8}$$

Theorem 1.1 (see [7]). *Let X be a centered Gaussian random variable with values in the separable Hilbert space $(H, \langle \cdot, \cdot \rangle)$ and $\lambda_n, n \in \mathbb{N}$ the decreasing eigenvalues of the covariance operator $C_X : H \to H$, $u \to \mathbb{E}\langle u, X\rangle X$ (which is a symmetric trace class operator). Assume that $\lambda_n \sim \rho(n)$ for some regularly varying function ρ with index $-b < -1$. Then, the asymptotics of the quantization error is given by*

$$e_{n,2}(X, H) \sim \left(\left(\frac{b}{2} \right)^{b-1} \frac{b}{b-1} \right)^{1/2} \omega(\log(n))^{-1/2}, \quad n \longrightarrow \infty, \tag{1.9}$$

where $\omega(x) := 1/x\rho(x)$.

Note that any change of $\sim$ in the assumption that $\lambda_n \sim \rho(n)$ to either $\lesssim$, $\approx$ or $\gtrsim$ leads to the same change in (1.9). Theorem 1.1 can also be extended to an index $b = 1$ (see [7]). Furthermore, a generalization to an arbitrary moment r (see [8]) as well as similar results for special Gaussian random variables and diffusions in non-Hilbertian function spaces (see, e.g., [9–11]) have been developed. Moreover, several authors established a precise link between the quantization error and the behavior of the small ball function of a Gaussian measure (see, e.g., [12, 13]) which can be used to derive asymptotics of quantization errors. More recently, for several types of Lèvy processes (sharp) optimal rates have been developed by several authors (see, e.g., [14–17]).

Coming back to the practical use of quantizers as a good approximation for a stochastic process, one is strongly interested in a constructive approach that allows to implement the coding strategy and to compute (at least numerically) good codebooks.

Considering again Gaussian random variables in a Hilbert space setting, the proof of Theorem 1.1 shows us how to construct *asymptotically r-optimal n-quantizers* for these processes, which means that sequences of n-quantizers α_n, $n \in \mathbb{N}$ satisfy

$$e_r(X, E, \alpha_n) \sim e_{n,r}(X, E), \quad n \longrightarrow \infty. \tag{1.10}$$

These quantizers can be constructed by reducing the quantization problem to a quantization problem of a finite-dimensional normal distributed random variable. Even if there are almost no explicit formulas known for optimal codebooks in finite dimensions, the existence is guaranteed (see [6, Theorem 4.12]) and there exist a lot of deterministic and stochastic numerical algorithms to compute optimal codebooks (see e.g., [18, 19] or [20]). Unfortunately, one needs to know explicitly the eigenvalues and eigenvectors of the covariance operator C_X to pursue this approach.

If we consider other non-Hilbertian function spaces $(E, \|\cdot\|)$ or non-Gaussian random variables in an infinite-dimensional Hilbert space, there is much less known on how to construct asymptotically optimal quantizers. Most approaches to calculate the asymptotics of the quantization error are either non-constructive (e.g., [12, 13]) or tailored to one specific process type (e.g., [9–11]) or the constructed quantizers do not achieve the sharp rate in the sense of (1.10) (e.g., [17] or [20]) but just the weak rate

$$e_r(X, E, \alpha_n) \approx e_{n,r}(X, E), \quad n \longrightarrow \infty. \tag{1.11}$$

In this paper, we develop a constructive approach to calculate sequences of asymptotically r-optimal n-quantizers (in the sense of (1.10)) for a broad class of random variables in infinite dimensional Banach spaces (Section 2). *Constructive* means in this case that we reduce the quantization problem to the quantization problem of a $\mathbb{R}^d$-valued random variable, that can be solved numerically. This approach can either be used in Hilbert spaces in case the eigenvalues and eigenvectors of the covariance operator of a Gaussian random variable are unknown (Sections 3.1 and 3.2), or for quantization problems in different Banach spaces (Sections 4 and 5).

In Section 4, we discuss Gaussian random variables in $(C(0,1), \|\cdot\|_\infty)$. This part is related to the PhD thesis of Wilbertz [20]. More precisely, we sharpen his constructive results by showing that the quantizers constructed in the thesis also achieve the sharp rate for the asymptotic quantization error (in the sense of (1.10)). Moreover, we can show that the dimensions of the subspaces wherein these quantizers are contained can be lessened without loosing the sharp asymptotics property.

In Section 5, we use some ideas of Luschgy and Pagès [17] and develop for Gaussian random variables and for a broad class of Lévy processes asymptotically optimal quantizers in the Banach space $(L_p([0,1]), \|\cdot\|_p)$.

It is worth mentioning that all these quantizers can be constructed without knowing the true rate of the quantization error. This means precisely that we know a (rough) lower bound for the quantization error, that is, $e_{n,r}(X,E) \gtrsim C_1 \log(n)^{-b_1}$ and the true but unknown rate is $e_{n,r}(X,E) \sim C_2 \log(n)^{-b_2}$ for constants $C_1, C_2, b_1, b_2 \in (0,\infty)$, then, we are able to construct a sequence of n-quantizers α_n, $n \in \mathbb{N}$ that satisfies

$$e_r(X, E, \alpha_n) \sim e_{n,r}(X, E) \sim C_2 \log(n)^{-b_2}, \quad n \longrightarrow \infty \tag{1.12}$$

for the optimal but still unknown constants C_2, b_2.

The crucial factors for the numerical implementation are the dimensions of the subspaces, wherein the asymptotically optimal quantizers are contained. We will calculate the dimensions of the subspaces obtained through our approach, and we will see that for all analyzed Gaussian processes, and also for many Lévy processes we are very close to the known asymptotics of the optimal dimension in the case of Gaussian processes in infinite-dimensional Hilbert spaces.

We will give some important examples of Gaussian and Lévy processes in Section 6, and finally illustrate some of our results in Section 7.

Notations and Definitions

If not explicitly differently defined, the following notations hold throughout the paper.

(i) We denote by X a Borel random variable in the separable Banach space $(E, \|\cdot\|)$ with $\mathrm{card}(\mathrm{supp}(\mathbb{P}^X)) = \infty$.

(ii) $\|\cdot\|$ will always denote the norm in E whereas $\|\cdot\|_{L_r(\mathbb{P})}$ will denote the norm in $L_r(\Omega, \mathcal{F}, \mathbb{P})$.

(iii) The scalar product in a Hilbert space H will be denoted by $\langle \cdot, \cdot \rangle$.

(iv) The smallest integer above a given real number x will be denoted by $\lceil x \rceil$.

(v) A sequence $(g_j)_{j\in\mathbb{N}} \in E^{\mathbb{N}}$ is called admissible for a centered Gaussian random variable X in E if and only if for any sequence $(\xi_i)_{i\in\mathbb{N}}$ of independent $N(0,1)$-distributed random variables it holds that $\sum_{i=1}^{\infty} \xi_i g_i$ converges *a.s.* in $(E, \|\cdot\|)$ and $X \stackrel{d}{=} \sum_{i=1}^{\infty} \xi_i g_i$. An admissible sequence $(g_j)_{j\in\mathbb{N}} \in E^{\mathbb{N}}$ is called rate optimal for X in E if and only if

$$\mathbb{E}\left\|\sum_{i=m}^{\infty}\xi_i g_i\right\|^2 \approx \inf\left\{\mathbb{E}\left\|\sum_{i=m}^{\infty}\xi_i f_i\right\|^2 : (f_i)_{i\in\mathbb{N}} \text{ admissible for } X\right\}, \tag{1.13}$$

as $m \to \infty$. A precise characterization of admissible sequences can be found in [21].

(vi) An orthonormal system (ONS) $(h_i)_{i\in\mathbb{N}}$ is called rate optimal for X in the Hilbert space H if and only if

$$\mathbb{E}\left\|\sum_{i=m}^{\infty}h_i\langle h_i, X\rangle\right\|^2 \approx \inf\left\{\mathbb{E}\left\|\sum_{i=m}^{\infty}f_i\langle f_i, X\rangle\right\|^2 : (f_i)_{i\in\mathbb{N}} \text{ONS in } H\right\}, \tag{1.14}$$

as $m \to \infty$.

2. Asymptotically Optimal Quantizers

The main idea is contained in the subsequent abstract result. The proof is based on the following elementary but very useful properties of quantization errors.

Lemma 2.1 (see [22]). *Let E, F be separable Banach spaces, X a random variable in E, and $T : E \to F$.*

(1) *If T is Lipschitz continuous with Lipschitz constant L, then*

$$e_{n,r}(T(X), F) \le L e_{n,r}(X, E), \tag{2.1}$$

and for every n-quantizer α for X it holds that

$$e_r(T(X), F, T(\alpha)) \le L e_r(X, E, \alpha). \tag{2.2}$$

(2) *Let $T : E \to F$ be linear, surjective, and isometric. Then, for $c \ge 0$ and $f \in F$*

$$e_{n,r}\left(cT(X) + f, F\right) = c e_{n,r}(X, E), \tag{2.3}$$

and for every n-quantizer α for X it holds that

$$e_r\big(cT(X)+f,F,T(\alpha)\big)=cT(e_r(X,E,\alpha))+f. \tag{2.4}$$

To formulate the main result, we need for an infinite subset $J\subset\mathbb{N}$ the following.

Condition 1. There exist linear operators $V_m: E\longrightarrow F_m\subset E$ for $m\in J$ with $\|V_m\|_{\mathrm{op}}\le 1$, for finite dimensional subspaces F_m with $\dim(F_m)=m$, where the norm $\|\cdot\|_{\mathrm{op}}$ is defined as

$$\|V_m\|_{\mathrm{op}}:=\sup_{x\in E,\|x\|\le 1}\|V_m(x)\|. \tag{2.5}$$

Condition 2. There exist linear isometric and surjective operators $\phi_m:(F_m,\|\cdot\|)\longrightarrow(\mathbb{R}^m,|\cdot|_m)$ with suitable norms $|\cdot|_m$ in $\mathbb{R}^m$ for all $m\in J$.

Condition 3. There exist random variables Z_m for $m\in J$ in E with $Z_m\stackrel{d}{=}X$, such that for the approximation error $\|\|X-V_m(Z_m)\|\|_{L_r(\mathbb{P})}$it holds that

$$\|\|X-V_m(Z_m)\|\|_{L_r(\mathbb{P})}\longrightarrow 0, \tag{2.6}$$

as $m\longrightarrow\infty$ along J.

Remark 2.2. The crucial point in Condition 1 is the norm one restriction for the operators V_m. Condition 2 becomes Important when constructing the quantizers in $\mathbb{R}^m$ equipped with, in the best case, some well-known norm. As we will see in the proof of the subsequent theorem, to show asymptotic optimality of a constructed sequence of quantizers one needs to know only a rough lower bound for the asymptotic quantization error. In fact, this lower bound allows us in combination with Condition 3 to choose explicitly a sequence $m(n)\in J$, $n\in\mathbb{N}$ such that

$$\big\|\big\|X-V_{m(n)}\big(Z_{m(n)}\big)\big\|\big\|_{L_r(\mathbb{P})}=o(e_{n,r}(X,E)),\quad n\longrightarrow\infty. \tag{2.7}$$

Theorem 2.3. *Assume that Conditions 1–3 hold for some infinite subset $J\subset\mathbb{N}$. One chooses a sequence $(m(n))_{n\in\mathbb{N}}\in J^{\mathbb{N}}$ such that (2.7) is satisfied. For $n\in\mathbb{N}$, let α_n be an r-optimal n-quantizer for $\xi_n:=\phi_{m(n)}(V_{m(n)}(Z_{m(n)}))$ in $(\mathbb{R}^{m(n)},|\cdot|_{m(n)})$.*

Then, $\big(\phi^{-1}_{m(n)}(\alpha_n)\big)_{n\in\mathbb{N}}$ is an asymptotically r-optimal sequence of n-quantizers for X in E and

$$e_{n,r}(X,E)\sim\Big(\mathbb{E}\Big\|X-\pi_{\phi^{-1}_{m(n)}(\alpha_n)}\big(V_{m(n)}\big(Z_{m(n)}\big)\big)\Big\|^r\Big)^{1/r}\sim e_r\Big(X,E,\phi^{-1}_{m(n)}(\alpha_n)\Big), \tag{2.8}$$

as $n\longrightarrow\infty$.

Remark 2.4. Note, that for $n\in\mathbb{N}$ there always exist r-optimal n-quantizers for ξ_n ([6, Theorem 4.12]).

Proof. Using Condition 3 and the fact that $e_{n,r}(X,E)>0$ for all $n\in\mathbb{N}$ since $\operatorname{card}(\operatorname{supp}(\mathbb{P}^X))=\infty$, we can choose a sequence $(m(n))_{n\in\mathbb{N}}\in\mathbb{N}^{\mathbb{N}}$ fulfilling (2.7). Using Lemma 2.1 and Condition 2, we see that $\phi^{-1}_{m(n)}(\alpha_n)$ is an r-optimal n-quantizer for $V_{m(n)}(Z_{m(n)})$ in $F_{m(n)}$. Then, by using Condition 1, (2.7), and Lemma 2.1 we get

$$\begin{aligned}
e_{n,r}(X,E) &\le \left(\mathbb{E}\left\| X-\pi_{\phi^{-1}_{m(n)}(\alpha_n)}\left(V_{m(n)}\left(Z_{m(n)}\right)\right)\right\|^r\right)^{1/r} \le \left(\mathbb{E}\left\| X-V_{m(n)}\left(Z_{m(n)}\right)\right\|^r\right)^{1/r}\\
&\quad+\left(\mathbb{E}\left\| V_{m(n)}\left(Z_{m(n)}\right)-\pi_{\phi^{-1}_{m(n)}(\alpha_n)}\left(V_{m(n)}\left(Z_{m(n)}\right)\right)\right\|^r\right)^{1/r}\\
&= \left(\mathbb{E}\left\| X-V_{m(n)}\left(Z_{m(n)}\right)\right\|^r\right)^{1/r}+e_{n,r}\left(V_{m(n)}\left(Z_{m(n)}\right),\left(F_{m(n)},\|\cdot\|\right)\right)\\
&\le \left(\mathbb{E}\left\| X-V_{m(n)}\left(Z_{m(n)}\right)\right\|^r\right)^{1/r}+e_{n,r}\left(Z_{m(n)},E\right)\\
&= \left(\mathbb{E}\left\| X-V_{m(n)}\left(Z_{m(n)}\right)\right\|^r\right)^{1/r}+e_{n,r}(X,E)\sim e_{n,r}(X,E),\quad n\longrightarrow\infty.
\end{aligned}\tag{2.9}$$

The last equivalence of the assertion follows from (1.6). □

Remark 2.5. We will usually choose $Z_m=X$ for all $m\in\mathbb{N}$, with an exception in Section 3 and $J=\mathbb{N}$.

Remark 2.6. The crucial factor for the numerical implementation of the procedure is the dimensions $(m(n))_{n\in\mathbb{N}}$ of the subspaces $(F_{m(n)})_{n\in\mathbb{N}}$. For the well-known case of the Brownian motion in the Hilbert space $H=L_2([0,1])$ it is known that this dimension sequence can be chosen as $m(n)\approx\log(\mathrm{n})$, $n\to\infty$. In the following examples we will see that we can often obtain similar orders like $\log(n)^c$ for constants c just slightly higher than one.

We point out that there is a nonasymptotic version of Theorem 2.3 for nearly optimal n-quantizers, that is, for n-quantizers, which are optimal up to $\epsilon>0$. Its proof is analogous to the proof of Theorem 2.3.

Proposition 2.7. *Assume that Conditions 1–3 hold. Let $m(\epsilon):=\inf\{m\in\mathbb{N}:\|X-V_m(Z_m)\|_{L_r(\mathbb{P})}<\epsilon\}$, and for $n\in\mathbb{N}$ one sets $\xi_n:=\phi_{m(\epsilon)}(V_{m(\epsilon)}(Z_{m(\epsilon)}))$. Then, it holds for every $n\in\mathbb{N}$ and for every r-optimal n-quantizer α_n for ξ_n in $(\mathbb{R}^{m(\epsilon)},|\cdot|_{m(\epsilon)})$ that*

$$e_r\left(X,E,\pi_{\phi^{-1}_{m(\epsilon)}(\alpha_n)}\left(V_{m(\epsilon)}\left(Z_{m(\epsilon)}\right)\right)\right)\le e_{n,r}(X,E)+\epsilon.\tag{2.10}$$

3. Gaussian Processes with Hilbertian Path Space

In this chapter, let X be a centered Gaussian random variable in the separable Hilbert space $(H,\langle\cdot,\cdot\rangle)$. Following the approach used in the proof of Theorem 1.1, we have for every sequence $(\xi_i)_{i\in\mathbb{N}}$ of independent $N(0,1)$-distributed random variables

$$X\overset{d}{=}\sum_{i=1}^{\infty}\sqrt{\lambda_i}f_i\xi_i,\tag{3.1}$$

where λ_i denote the eigenvalues and f_i denote the corresponding orthonormal eigenvectors of the covariance operator C_X of X (Karhunen-Loève expansion). If these parameters are known, we can choose a sequence $(d_n)_{n\in\mathbb{N}}$ such that a sequence of optimal quantizer α_n for $X_n = \sum_{i=1}^{d(n)} \sqrt{\lambda_i} f_i \xi_i$ is asymptotically optimal for X in E.

In order to construct asymptotically optimal quantizers for Gaussian random variables with unknown eigenvalues or eigenvectors of the covariance operator, we start with more general expansions. In fact, we just need one of the two orthogonalities, either in $L_2(\mathbb{P})$ or in H.

Before we will use these representations for X to find suitable triples (V_m, F_m, ϕ_m) as in Theorem 2.3, note that for Gaussian random variables in H fulfilling suitable assumptions we know that

(1) Let $(h_i)_{i\in\mathbb{N}}$ be an orthonormal basis of H. Then

$$X = \sum_{i=1}^{\infty} h_i \langle h_i, X\rangle \quad \text{a.s..} \tag{3.2}$$

Compared to (3.1) we see that $\langle h_i, X\rangle$ are still Gaussian but generally not independent.

(2) Let $(g_j)_{j\in\mathbb{N}}$ be an admissible sequence for X in H such that

$$X \overset{d}{=} \sum_{i=1}^{\infty} \xi_i g_i. \tag{3.3}$$

Compared to (3.1) the sequence $(g_i)_{i\in\mathbb{N}}$ is generally not orthogonal.

$$e_{n,2}(X, H) \approx e_{n,s}(X, H), \quad n \longrightarrow \infty \tag{3.4}$$

for all $s \geq 1$; see [13]. Thus, we will focus on the case $s = 2$ to search for lower bounds for the quantization errors.

3.1. Orthonormal Basis

Let $(h_m)_{m\in\mathbb{N}}$ be an orthonormal basis of H. For the subsequent subsection we use the following notations.

(1) We set $F_m = \operatorname{span}\{h_1, \ldots, h_m\}$.

(2) We set $V_m := pr_{F_m} : E \rightarrow F_m$, the orthogonal projection on F_m. It is well known that $\|V_m\|_{\text{op}} = 1$.

(3) Define the linear, surjective, and isometric operators ϕ_m by

$$\phi_m : (F_m, \|\cdot\|) \longrightarrow (\mathbb{R}^m, \|\cdot\|_2), \qquad h_i \longrightarrow e_i, \tag{3.5}$$

where e_i denotes the ith unit vector in $\mathbb{R}^m$, $1 \leq i \leq m$.

Theorem 3.1. *Assume that the eigenvalue sequence $(\lambda_j)_{j\in\mathbb{N}}$ of the covariance operator C_X satisfies $\lambda_j \approx j^{-b}$ for $-b < -1$, and let $\epsilon > 0$ be arbitrary. Assume further that $(h_j)_{j\in\mathbb{N}}$ is a rate optimal ONS for X in H. One sets $m(n) = \lceil \log(n)^{1+\epsilon} \rceil$ for $n \in \mathbb{N}$. Then, one gets for every sequence $(\alpha_n)_{n\in\mathbb{N}}$ of r-optimal n-quantizers for $\phi_{m(n)}(V_{m(n)}(X))$ in $(\mathbb{R}^{m(n)}, \|\cdot\|_2)$ the asymptotics*

$$e_{n,r}(X,H) \sim e_r\left(X, H, \phi_{m(n)}^{-1}(\alpha_n)\right) \sim \left(\mathbb{E}\left\| X - \pi_{\phi_{m(n)}^{-1}(\alpha_n)}\left(V_{m(n)}(X)\right)\right\|^r\right)^{1/r}, \tag{3.6}$$

as $n \longrightarrow \infty$.

Proof. Let $(f_j)_{j\in\mathbb{N}}$ be the corresponding orthonormal eigenvector sequence of C_X. Classic eigenvalue theory yields for every $m \in \mathbb{N}$

$$\mathbb{E}\left\| \sum_{i=m}^{\infty} f_i \langle f_i, X\rangle \right\|^2 = \sum_{i=m}^{\infty} \lambda_i \le \sum_{i=m}^{\infty} \mathbb{E}\langle h_i, X\rangle^2 = \mathbb{E}\left\| \sum_{i=m}^{\infty} h_i \langle h_i, X\rangle \right\|^2. \tag{3.7}$$

Combining this with rate optimality of the ONS $(h_j)_{j\in\mathbb{N}}$ for X, we get

$$\begin{aligned} \mathbb{E}\left\| X - V_{m(n)}(X)\right\|^2 = \mathbb{E}\left\| \sum_{i=m(n)+1}^{\infty} h_i\langle h_i, X\rangle \right\|^2 &= \sum_{i=m(n)+1}^{\infty} \mathbb{E}\langle h_i, X\rangle^2 \\ &\approx \sum_{i=m(n)+1}^{\infty} \lambda_j \approx m(n)^{-(b-1)}, \quad n \longrightarrow \infty. \end{aligned} \tag{3.8}$$

Using the equivalence of the r-norms of Gaussian random variables ([23, Corollary 3.2]), and since $X - V_{m(n)}(X)$ is Gaussian, we get for all $r \ge 1$

$$\left\| \left\| X - V_{m(n)}(X)\right\| \right\|_{L_r(\mathbb{P})} \approx m(n)^{-(1/2)(b-1)}, \quad n \longrightarrow \infty. \tag{3.9}$$

With ω as in Theorem 1.1, we get by using (3.4) and Theorem 1.1 the weak asymptotics $e_{n,r}(X,H) \approx (\omega(\log(n)))^{-1/2} \approx (\log(n))^{-(1/2)(b-1)}$, $n \longrightarrow \infty$. Therefore, the sequence $(m(n))_{n\in\mathbb{N}}$ satisfies (2.7) since

$$\left\| \left\| X - V_{m(n)}(X)\right\| \right\|_{L_r(\mathbb{P})} \approx (\log(n))^{-(1/2)(b-1)(1+\epsilon)} = o(e_{n,r}(X,H)), \quad n \longrightarrow \infty, \tag{3.10}$$

and the assertion follows from Theorem 2.3. □

3.2. Admissible Sequences

In order to show that linear operators V_m similar to those used in the subsection above are suitable for the requirements of Theorem 2.3, we need to do some preparations. Since the covariance operator C_X of a Gaussian random variable is symmetric and compact (in fact trace class), we will use a well-known result concerning these operators. This result can be used for quantization in the following way.

Lemma 3.2. *Let X be a centered Gaussian random variable with values in the Hilbert space H and $X = X_1 + X_2$, where X_1 and X_2 are independent centered Gaussians. Then*

$$C_X = C_{X_1} + C_{X_2}. \tag{3.11}$$

Let $\lambda_i, \lambda_i^{(1)}, \lambda_i^{(2)}$, $i \in \mathbb{N}$ be the positive monotone decreasing eigenvalues of C_X, C_{X_1}, and C_{X_2}. Then, for $i \in \mathbb{N}$ it holds that

$$\lambda_i^{(1)}, \lambda_i^{(2)} \leq \lambda_i. \tag{3.12}$$

Proof. Since X_1, X_2 are independent centered Gaussians, we have $\mathbb{E}\langle X_1, u\rangle X_2 = \mathbb{E}\langle X_1, u\rangle \mathbb{E}X_2 = 0$ for all $u \in H$. This easily leads to

$$\begin{aligned} C_X(u) = \mathbb{E}\langle X_1 + X_2, u\rangle (X_1 + X_2) &= \mathbb{E}\langle X_1, u\rangle X_1 \\ &+ \mathbb{E}\langle X_2, u\rangle X_2 = C_{X_1}(u) + C_{X_2}(u). \end{aligned} \tag{3.13}$$

The covariance operator of a centered Gaussian random variable is positive semidefinite. Hence, by using a result on the relation of the eigenvalues of those operators (see, e.g., [24, page 213]), we get inequalities (3.12). □

Let $(g_i)_{i\in\mathbb{N}}$ be an admissible sequence for X, and assume that $\sum_{i=1}^{\infty} \xi_i g_i = X$ a.s. In this subsection, we use the following notations.

(1) We set $F_m := \operatorname{span}\{g_1, \ldots, g_m\}$.

(2) We define $V_m : H \to F_m \subset H$ by

$$V_m(f_j) := f_j^{(m)} \sqrt{\frac{\lambda_j^{(m)}}{\lambda_j}}, \tag{3.14}$$

for $j \leq m$ and $V_m(f_j) := 0$ for $j > m$, where λ_j and f_j denote the eigenvalues and the corresponding eigenvectors of C_X and $\lambda_j^{(m)}$ and $f_j^{(m)}$ the eigenvalues and the corresponding eigenvectors of C_{X_m}, with X_m defined as

$$X_m := \sum_{i=1}^{m} g_i \xi_i. \tag{3.15}$$

Note that V_m maps H onto F_m since

$$\operatorname{span}\{g_1, \ldots, g_m\} = \operatorname{span}\left\{f_1^{(m)}, \ldots, f_m^{(m)}\right\}. \tag{3.16}$$

Furthermore, it is important to mention that one does not need to know λ_j and f_j explicitly to construct the subsequent quantizers, since we can find for any

$m \in \mathbb{N}$ a random variable $Z_m \stackrel{d}{=} X$ such that $V_m(Z_m) = \sum_{i=1}^{m} \xi_i g_i$ (see the proof of Theorem 3.3), which is explicitly known and sufficient to know for the construction.

(3) Define the linear, surjective, and isometric operators ϕ_m by

$$\phi_m : (F_m, \|\cdot\|) \longrightarrow (\mathbb{R}^m, \|\cdot\|_2), \qquad f_i^{(m)} \longrightarrow e_i, \tag{3.17}$$

where e_i denotes the ith unit vector of $\mathbb{R}^m$ for $1 \leq i \leq m$.

Theorem 3.3. *Assume that the eigenvalue sequence $(\lambda_j)_{j\in\mathbb{N}}$ of the covariance operator C_X satisfies $\lambda_j \approx j^{-b}$ for $-b < -1$, and let $\epsilon > 0$ arbitrary. Assume that $(g_j)_{j\in\mathbb{N}}$ is a rate optimal admissible sequence for X in H. One sets $m(n) = \lceil \log(n)^{1+\epsilon} \rceil$ for $n \in \mathbb{N}$. Then, there exist random variables Z_m, $m \in \mathbb{N}$, with $Z_m \stackrel{d}{=} X$ such that for every sequence $(\alpha_n)_{n\in\mathbb{N}}$ of r-optimal n-quantizers for $\phi_{m(n)}(V_{m(n)}(Z_{m(n)}))$ in $(\mathbb{R}^{m(n)}, \|\cdot\|_2)$*

$$e_{n,r}(X,H) \sim e_r\left(X, H, \phi_{m(n)}^{-1}(\alpha_n)\right) \sim \left(\mathbb{E}\left\| X - \pi_{\phi_{m(n)}^{-1}(\alpha_n)}\left(V_{m(n)}\left(Z_{m(n)}\right)\right)\right\|^r\right)^{1/r}, \tag{3.18}$$

as $n \longrightarrow \infty$.

Proof. Linearity of $(V_m)_{m\in\mathbb{N}}$ follows from the orthogonality of the eigenvectors. In view of the inequalities for the eigenvalues in Lemma 3.2 and the orthonormality of the family $(f_i)_{i\in\mathbb{N}}$, we have for every $h = \sum_{i=1}^{\infty} f_i a_i \in H$ with $\|h\|^2 = \sum_{i=1}^{\infty} a_i^2 \leq 1$

$$\|V_m(h)\|^2 = \left\|V_m\left(\sum_{i=1}^{\infty} a_i f_i\right)\right\|^2 = \sum_{i=1}^{m} a_i^2 \frac{\lambda_i^{(m)}}{\lambda_i} \leq \sum_{i=1}^{\infty} a_i^2 \leq 1, \tag{3.19}$$

such that $\|V_m\|_{op} \leq 1$.

Note next that for every $m \in \mathbb{N}$ there exist independent $N(0,1)$-distributed random variables $(\zeta_i^{(m)})_{1\leq i\leq m}$ satisfying

$$\sum_{i=1}^{m} \xi_i g_i = \sum_{i=1}^{m} \sqrt{\lambda_i^{(m)}} f_i^{(m)} \zeta_i^{(m)} \quad \text{a.s.} \tag{3.20}$$

Then, we choose random variables $(\zeta_i^{(m)})_{m+1\leq i<\infty}$ such that $(\zeta_i^{(m)})_{1\leq i<\infty}$ is a sequence of independent $N(0,1)$-distributed random variables. We set

$$Z_m := \sum_{i=1}^{\infty} \zeta_i^{(m)} \sqrt{\lambda_i} f_i \tag{3.21}$$

and get by using rate optimality of the admissible sequences $(g_j)_{j\in\mathbb{N}}$ and $(\sqrt{\lambda_j} f_j)_{j\in\mathbb{N}}$

$$\begin{aligned}
\mathbb{E}\|X - V_m(Z_m)\|^2 &= \mathbb{E}\left\| \sum_{i=1}^{\infty} g_i\xi_i - V_m\Big(\sum_{i=1}^{\infty}\sqrt{\lambda_i} f_i \zeta_i^{(m)}\Big)\right\|^2 \\
&= \mathbb{E}\left\| \sum_{i=1}^{\infty} g_i\xi_i - \sum_{i=1}^{m}\sqrt{\lambda_i^{(m)}}\, f_i^{(m)}\zeta_i^{(m)}\right\|^2 = \mathbb{E}\left\| \sum_{i=m+1}^{\infty} g_i\xi_i\right\|^2 \\
&\approx \mathbb{E}\left\| \sum_{i=m+1}^{\infty}\sqrt{\lambda_i} f_i\xi_i\right\|^2 = \sum_{i=m+1}^{\infty}\lambda_i \approx m^{-(b-1)}, \quad m \longrightarrow \infty,
\end{aligned} \tag{3.22}$$

where rate optimality of $(\sqrt{\lambda_j} f_j)_{j\in\mathbb{N}}$ is a consequence of

$$\|\|X\|\|^2_{L_2(\mathbb{P})} - \mathbb{E}\left\| \sum_{i=m+1}^{\infty} g_i\xi_i\right\|^2 = \mathbb{E}\left\| \sum_{i=1}^{m} g_i\xi_i\right\|^2 = \sum_{i=1}^{m}\lambda_i^{(m)} \le \sum_{i=1}^{m}\lambda_i. \tag{3.23}$$

Using the equivalence of the r-norms of Gaussian random variables ([23, Corollary 3.2]), and since $X - V_{m(n)}(X)$ is Gaussian, we get for all $r \ge 1$

$$\|\|X - V_{m(n)}(X)\|\|_{L_r(\mathbb{P})} \approx m(n)^{-(1/2)(b-1)}, \quad n \longrightarrow \infty. \tag{3.24}$$

With ω as in Theorem 1.1, we get by using (3.4) and Theorem 1.1 the weak asymptotics $e_{n,r}(X,H) \approx (\omega(\log(n)))^{-1/2} \approx (\log(n))^{-(1/2)(b-1)}$, $n \to \infty$. Therefore, the sequence $(m(n))_{n\in\mathbb{N}}$ satisfies (2.7) since

$$\|\|X - V_{m(n)}(X)\|\|_{L_r(\mathbb{P})} \approx \big(\log(n)\big)^{-(1/2)(b-1)(1+\epsilon)} = o(e_{n,r}(X,H)), \quad n \longrightarrow \infty, \tag{3.25}$$

and the assertion follows from Theorem 2.3. □

3.3. Comparison of the Different Schemes

At least in the case $r = 2$, we have a strong preference for using the method as described in Section 3.1. We use the notations as in the above subsections including an additional indexation $i = 1, 2$ for $(V_m^{(i)}, \phi_m^{(i)}, \alpha_n^{(i)})$ and $m, n \in \mathbb{N}$, where $\alpha_n^{(i)}$, for $i = 1, 2$, are defined as in Theorems 3.1 and 3.3. Note that for this purpose the size of the codebook n and the size of the subspaces $\dim(F_m) = m$ can be chosen arbitrarily (i.e., m does not depend on n). The ONS $(h_i)_{i\in\mathbb{N}}$ is chosen as the ONS derived with the Gram-Schmidt procedure from the admissible sequence $(g_j)_{j\in\mathbb{N}}$ for the Gaussian random variable X in the Hilbert space H, such that the definition of F_m coincides in the twosubsections.

Proposition 3.4. *It holds for $m, n \in \mathbb{N}$ that*

$$\mathbb{E}\left\| X - \pi_{(\phi_m^{(2)})^{-1}(\alpha_n^{(2)})}\left(V_m^{(2)}(Z_m)\right)\right\|^2 \geq \mathbb{E}\left\| X - \pi_{(\phi_m^{(1)}(\phi_m^{(1)})^{-1}(\alpha_n^{(1)})}(V_m^{(1)}(X))\right\|^2. \tag{3.26}$$

Proof. Consider for X the decomposition $X = pr_{F_m^\perp}(X) + pr_{F_m}(X)$. The key is the orthogonality of $pr_{F_m^\perp}(X)$ to $pr_{F_m}(X)$, $\pi_{(\phi_m^{(2)})^{-1}(\alpha_n^{(2)})}(V_m^{(2)}(Z_m))$, and $\pi_{(\phi_m^{(1)})^{-1}(\alpha_n^{(1)})}(V_m^{(1)}(X))$, which gives the two equalities in the following calculation:

$$\begin{aligned}
&\mathbb{E}\left\| X - \pi_{(\phi_m^{(2)})^{-1}(\alpha_n^{(2)})}\left(V_m^{(2)}(Z_m)\right)\right\|^2 \\
&\quad = \mathbb{E}\left\| pr_{F_m}(X) - \pi_{(\phi_m^{(2)})^{-1}(\alpha_n^{(2)})}\left(V_m^{(2)}\left(Z_m^{(2)}\right)\right)\right\|^2 + \mathbb{E}\left\| pr_{F_m^\perp}(X)\right\|^2 \\
&\quad \overset{(*)}{\geq} \mathbb{E}\left\| pr_{F_m}(X) - \pi_{(\phi_m^{(1)})^{-1}(\alpha_n^{(1)})}\left(V_m^{(1)}(X)\right)\right\|^2 + \mathbb{E}\left\| pr_{F_m^\perp}(X)\right\|^2 \\
&\quad = \mathbb{E}\left\| X - \pi_{(\phi_m^{(1)})^{-1}(\alpha_n^{(1)})}(V_m^{(1)}(X))\right\|^2.
\end{aligned} \tag{3.27}$$

The inequality $(*)$ follows from the optimality of the codebook $(\phi_m^{(1)})^{-1}(\alpha_n^{(1)})$ for $pr_{F_m}(X) = V_m^{(1)}(X)$. □

4. Gaussian Processes with Paths in $(C([0,1]), \|\cdot\|_\infty)$

In the previous section, where we worked with Gaussian random variables in Hilbert spaces, we saw that special Hilbertian subspaces, projections, and other operators linked to the Gaussian random variable were good tools to develop asymptotically optimal quantizers based on Theorem 2.3. Since we now consider the non-Hilbertian separable Banach space $(C([0,1]), \|\cdot\|_\infty)$, we have to find different tools that are suitable to use Theorem 2.3.

The tools used in [20] are B-splines of order $s \in \mathbb{N}$. In the case $s = 2$, that we will consider in the sequel, these splines span the same subspace of $C([0,1], \|\cdot\|_\infty)$ as the classical Schauder basis. We set for $x \in [0,1]$, $m \geq 2$, and $1 \leq i \leq m$ the knots $t_i^{(m)} := (i-1)/(m-1)$ and the hat functions

$$f_i^{(m)}(x) := \chi_{[t_i^{(m)}, t_{i+1}^{(m)}]}(x)\left(1 - \left(x - t_i^{(m)}\right)(m-1)\right) + \chi_{[t_{i-1}^{(m)}, t_i^{(m)})}(x)\left(x - t_{i-1}^{(m)}\right)(m-1). \tag{4.1}$$

For the remainder of this subsection, we will use the following notations.

(1) As subspaces F_m we set $F_m := \operatorname{span}\{f_j^{(m)}, 1 \leq j \leq m\}$.

(2) As linear and continuous operators $V_m : C([0,1]) \to F_m$ we set the quasiinterpolant

$$V_m(f) := \sum_{i=1}^{m} f_i^{(m)} \beta_i^{(m)}(f), \tag{4.2}$$

where $\beta_i^{(m)}(f) := f(t_i^{(m)})$.

(3) The linear and surjective isometric mappings ϕ_m one defines as

$$\begin{aligned} \phi_m &: (F_m, \|\cdot\|_\infty) \longrightarrow (R^m, \|\cdot\|_\infty), \\ &\sum_{i=1}^{m} a_i f_i^{(m)} \longrightarrow (a_1, \dots, a_m). \end{aligned} \tag{4.3}$$

It is easy to see that $\| \sum_{i=1}^{m} a_i f_i^{(m)} \|_\infty = \|(a_1, \dots, a_m)\|_\infty$ holds for every $a \in \mathbb{R}^m$.

For the application of Theorem 2.3, we need to know the error bounds for the approximation of X with the quasiinterpolant $V_m(X)$. For Gaussian random variables, we can provide the following result based on the smoothness of an admissible sequence for X in E.

Proposition 4.1. *Let $(g_j)_{j\in\mathbb{N}}$ be admissible for the centered Gaussian random variable X in $(C([0,1]), \|\cdot\|_\infty)$. Assume that*

(1) $\|g_j\| \le C_1 j^{-\theta}$ *for every* $j \ge 1$, $\theta > 1/2$, *and* $C_1 < \infty$,

(2) $g_j \in C^2([0,1])$ *with* $\|g_j''\| \le C_2 j^{-\theta+2}$ *for every* $j \ge 1$ *and* $C_2 < \infty$.

Then, for any $\epsilon > 0$ and some constant $C < \infty$it holds that

$$\| \|X - V_m(X)\| \|_{L_r(\mathbb{P})} \le C m^{-0,8(\theta-(1/2))+\epsilon}, \tag{4.4}$$

for every $r \ge 1$.

Proof. Using of [25, Theorem 1], we get

$$\left\| \left\| \sum_{i=k}^{\infty} \xi_i g_i \right\| \right\|_{L_r(\mathbb{P})} \le \frac{C_3}{k^{\theta-(1/2)-\epsilon_1}} \tag{4.5}$$

for an arbitrary $\epsilon_1 > 0$, some constant $C_3 < \infty$, and every $k \in \mathbb{N}$. Thus, we have

$$\begin{aligned} \| \|X - V_m(X)\| \|_{L_r(\mathbb{P})} &\le \left\| \left\| \sum_{i=k}^{\infty} \xi_i g_i \right\| \right\|_{L_r(\mathbb{P})} + \left\| \left\| V_m\left(\sum_{i=k}^{\infty} \xi_i g_i \right) \right\| \right\|_{L_r(\mathbb{P})} \\ &\quad + \left\| \left\| \sum_{i=1}^{k-1} \xi_i g_i - V_m\left(\sum_{i=1}^{k-1} \xi_i g_i \right) \right\| \right\|_{L_r(\mathbb{P})} \\ &\le \frac{2C_3}{k^{\theta-(1/2)-\epsilon_1}} + \left\| \left\| \sum_{i=1}^{k-1} \xi_i g_i - V_m\left(\sum_{i=1}^{k-1} \xi_i g_i \right) \right\| \right\|_{L_r(\mathbb{P})}. \end{aligned} \tag{4.6}$$

Using of [26, Chapter 7, Theorem 7.3], we get for some constant $C_4 < \infty$

$$\left\| \sum_{i=1}^{k-1} \xi_i g_i - W_m\left(\sum_{i=1}^{k-1} \xi_i g_i \right) \right\| \le C_4 \omega\left(\sum_{i=1}^{k-1} \xi_i g_i, \frac{1}{m-1} \right), \tag{4.7}$$

where the *module of smoothness* $\omega(f,\delta)$ is defined by

$$\omega(f,\delta) := \sup_{0\le h<\delta} \left\| f(x) - 2f(x+h) + f(x+2h) \right\|_\infty. \tag{4.8}$$

For an arbitrary $f \in C^2([0,1])$ we have by using Taylor expansion

$$\frac{\left\| f(x) - 2f(x+h) + f(x+2h) \right\|_\infty}{h^2} \le 2\left\| f'' \right\|_\infty. \tag{4.9}$$

Combining this, we get for an arbitrary $\epsilon_2 > 0$ and constants $C_5, C_6, C_7 < \infty$, using again the equivalence of Gaussian moments,

$$\begin{aligned}
\left\|\|X - V_m(X)\|\right\|_{L_r(\mathbb{P})} &\le \frac{2C_3}{k^{\theta-(1/2)-\epsilon_1}} + \frac{1}{m^2}\left(\mathbb{E}\left\| 2\sum_{i=1}^{k-1} \xi_i g_i'' \right\|_\infty^r\right)^{1/r} \\
&\le \frac{2C_3}{k^{\theta-(1/2)-\epsilon_1}} + \frac{1}{m^2} C_5 \mathbb{E}\left\| 2\sum_{i=1}^{k-1} \xi_i g_i'' \right\|_\infty \\
&\le \frac{2C_3}{k^{\theta-(1/2)-\epsilon_1}} + \frac{1}{m^2} C_6 \sum_{i=1}^{k-1} i^{\theta+2+\epsilon_2} \mathbb{E}|\xi_i| \\
&\le \frac{2C_3}{k^{\theta-(1/2)-\epsilon_1}} + \frac{1}{m^2} C_7 k^{-\theta+3+\epsilon_2}.
\end{aligned} \tag{4.10}$$

To minimize over k, we choose $k = k(m) = m^{0,8}$. Thus, we get for some constant $C < \infty$ and an arbitrary $\epsilon > 0$

$$\left\|\|X - V_m(X)\|\right\|_{L_r(\mathbb{P})} \le C m^{-0,8(\theta-(1/2))+\epsilon}. \tag{4.11}$$

□

Now, we are able to prove the main result of this section.

Theorem 4.2. *Let X be a centered Gaussian random variable and $(g_j)_{j\in\mathbb{N}}$ an admissible sequence for X in $C([0,1])$ fulfilling the assumptions of Proposition 4.1 with $\theta = b/2$, where the constant $b > 1$ satisfies $\lambda_j \gtrsim Kj^{-b}$ with $\lambda_j, j \in \mathbb{N}$ denoting the monotone decreasing eigenvalues of the covariance operator C_X of X in $H = L_2([0,1])$ and $K > 0$. One sets $m(n) := \lceil \log(n)^{(5/4)+\epsilon} \rceil$ for some $\epsilon > 0$. Then, for every sequence $(\alpha_n)_{n\in\mathbb{N}}$ of r-optimal n-quantizers for $\phi_{m(n)}(V_{m(n)}(X))$ in $(\mathbb{R}^{m(n)}, \|\cdot\|_\infty)$, it holds that*

$$\begin{aligned}
e_{n,r}(X,(C([0,1]),\|\cdot\|_\infty)) &\sim e_r\left(X, C([0,1]), \phi_{m(n)}^{-1}(\alpha_n)\right) \\
&\sim \left(\mathbb{E}\left\| X - \pi_{\phi_{m(n)}^{-1}(\alpha_n)}(V_{m(n)}(X)) \right\|_\infty^r\right)^{1/r},
\end{aligned} \tag{4.12}$$

as $n \to \infty$.

Proof. For every $h \in C(([0,1]), \|\cdot\|_\infty)$, with $\|h\|_\infty \leq 1$it holds that

$$\|V_m(h)\|_\infty \leq \sup_{x\in[0,1]} \sum_{i=1}^{m} \left|h\left(t_i^{(m)}\right)\right| f_i^{(m)}(x) \leq \|h\|_\infty \sup_{x\in[0,1]} \sum_{i=1}^{m} f_i^{(m)}(x) \leq 1, \tag{4.13}$$

since $\{f_i^{(m)}, 1 \leq i \leq m\}$ are partitions of the one for every $m \in \mathbb{N}$, so that $\|V_m\|_{\mathrm{op}} \leq 1$.

We get a lower bound for the quantization error $e_{n,r}(X, C([0,1]))$ from the inequality

$$\|f\|_{L_2([0,1])} \leq \|f\|_\infty, \tag{4.14}$$

for all $f \in C([0,1]) \subset L_2([0,1])$. Consequently, we have

$$e_{n,r}(X, C([0,1])) \geq e_{n,r}(X, L_2([0,1])). \tag{4.15}$$

From Theorem 1.1 and (3.4) we obtain

$$(\log(n))^{-(1/2)(b-1)} \approx \left(\omega(\log(n))\right)^{-1/2} \lesssim e_{n,r}(X, C([0,1])), \quad n \longrightarrow \infty, \tag{4.16}$$

where ω is given as in Theorem 1.1. Finally, we get by combining (4.16) and Proposition 4.1 for sufficiently small $\delta > 0$

$$\begin{aligned} \big\| \|X - V_{m(n)}(X)\| \big\|_{L_r(\mathbb{P})} &\leq C m(n)^{-0{,}8((1/2)(b-1))+\delta} \\ &= o\left((\log(n))^{-(1/2)(b-1)} \right) = o(e_{n,r}(X, C([0,1]))), \quad n \longrightarrow \infty, \end{aligned} \tag{4.17}$$

and the assertion follows from Theorem 2.3. □

5. Processes with Path Space $L_p([0,1], \|\cdot\|_p)$

Another useful tool for our purposes is the Haar basis in $L_p([0,1])$ for $1 \leq p < \infty$, which is defined by

$$\begin{gathered} e_0 := \chi_{[0,1]} \quad e_1 := \chi_{[0,1/2)} - \chi_{[1/2,1]} \\ e_{2^n+k} := 2^{n/2} e_1(2^n \cdot -k), \quad n \in \mathbb{N},\ k \in \{0, \ldots, 2^n - 1\}. \end{gathered} \tag{5.1}$$

This is an orthonormal basis of $L_2([0,1])$ and a Schauder basis of $L_p([0,1])$ for $p \in [1,\infty)$, that is, $\langle f, e_0\rangle + \sum_{n=0}^{\infty} \sum_{k=1}^{2^n-1} \langle f, e_{2^n+k}\rangle e_{2^n+k}$ converges to f in $L_p([0,1])$ for every $f \in L_p([0,1])$; see [27].

The Haar basis was used in [17] to construct rate optimal sequences of quantizers for mean regular processes. These processes are specified through the property that for all $0 \leq s \leq t \leq 1$

$$\mathbb{E}|X_t - X_s|^p \leq \left(\rho(t-s)\right)^p, \tag{5.2}$$

where $\rho : \mathbb{R}_+ \to [0,\infty)$ is regularly varying with index $b > 0$ at 0, which means that

$$\lim_{x\to 0}\frac{\rho(cx)}{\rho(x)} = c^b, \tag{5.3}$$

for all $c > 0$. Condition (5.2) also guarantees that the paths $t \to X_t$ lie in $L_p([0,1])$.

For our approach, it will be convenient to define for $m \in \mathbb{N}$ and $1 \le i \le m+1$ the knots $t_i^{(m)} := (i-1)/m$ and for $1 \le i \le m-1$ the functions

$$f_i^{(m)}(x) := \chi_{[t_i^{(m)},t_{i+1}^{(m)})}(x)\sqrt{m}, \quad f_m^{(m)}(x) := \chi_{[t_m^{(m)},1]}(x)\sqrt{m} \tag{5.4}$$

and the operators

$$V_m(f) := \sum_{i=1}^{m} f_i^{(m)}\left\langle f_i^{(m)}, f\right\rangle. \tag{5.5}$$

Note that for $f \in L_1([0,1])$, $m = 2^{n+1}$, and $n \in \mathbb{N}_0$

$$\langle e_0, f\rangle e_0 + \sum_{i=0}^{n}\sum_{k=0}^{2^i-1}\langle e_{2^i+k}, f\rangle e_{2^i+k} = \sum_{i=1}^{m} f_i^{(m)}\left\langle f_i^{(m)}, f\right\rangle. \tag{5.6}$$

For the remainder of the subsection, we set the following.

(1) We set for $m \in \mathbb{N}$ the subspaces $F_m := \operatorname{span}\{f_1^{(m)}, \ldots, f_m^{(m)}\}$.

(2) Set the linear and continuous operator V_m to

$$\begin{aligned} V_m : L_p([0,1]) &\longrightarrow F_m \\ f &\longrightarrow \sum_{i=1}^{m}\left\langle f_i^{(m)}, f\right\rangle f_i^{(m)}. \end{aligned} \tag{5.7}$$

(3) For $p \in [1,\infty)$ we set the isometric isomorphisms $\phi_{m,p} : (F_m, \|\cdot\|_{L_p}) \to (\mathbb{R}^m, \|\cdot\|_p)$ as

$$\phi_{m,p}\left(\sum_{i=1}^{m} a_i f_i^{(m)}\right) := m^{(1/2-1/p)}(a_1,\ldots,a_m). \tag{5.8}$$

Theorem 5.1. *Let X be a random variable in the Banach space $(E, \|\cdot\|) = (L_p([0,1]), \|\cdot\|_p)$ for some $p \in [1,\infty)$ fulfilling the mean pathwise regularity property*

$$\|X_t - X_s\|_{L_{r\vee p}} \le C(t-s)^a, \tag{5.9}$$

for constants $C, a > 0$ and $t > s \in [0,1]$. Moreover, assume that $K \log(n)^{-b} \lesssim e_{n,r}(X,E)$ for constants $K, b > 0$. Then, for an arbitrary $\epsilon > 0$ and $m(n) := \lceil (\log(n))^{(b/a)+\epsilon} \rceil$ it holds that every sequence of r-optimal n-quantizers $(\alpha_n)_{n\in\mathbb{N}}$ for $\phi_{m(n),p}(V_m(n)(X))$ in $(\mathbb{R}^{m(n)}, \|\cdot\|_p)$ satisfies

$$\begin{aligned} e_{n,r}(X, L_p([0,1])) &\sim e_r\Big(X, L_p([0,1]), \phi^{-1}_{m(n),p}(\alpha_n)\Big) \\ &\sim \left(\mathbb{E}\Big\| X - \pi_{\phi^{-1}_{m(n),p}(\alpha_n)}\big(V_{m(n)}(X)\big)\Big\|^r_{L_p}\right)^{1/r}, \end{aligned} \tag{5.10}$$

as $n \to \infty$.

Proof. As in the above subsections, we check that the sequences V_m and $\phi_{m,p}$ satisfy Conditions 1–3. Since $V_m(f) = \mathbb{E}_\lambda(f \mid \mathcal{F}_m)$, where $\mathcal{F}_m$ is defined by

$$\mathcal{F}_m := \sigma\Big(f_1^{(m)}, \ldots, f_m^{(m)}\Big), \tag{5.11}$$

we get for $f \in L_p([0,1])$, with $\|f\|_p \le 1$ and $p \in [1,\infty)$ by using Jensen's inequality,

$$\|V_m(f)\|^p_{L_p} = \int_{[0,1]} \left|\mathbb{E}_\lambda(f \mid \mathcal{F}_m)\right|^p d\lambda \le \|f\|^p_{L_p}, \tag{5.12}$$

and thus $\|V_m\|_{\text{op}} \le 1$. The operators $\phi_{m,p}$ satisfy Condition 2 of Theorem 2.3 since

$$\begin{aligned} \left\|\sum_{i=1}^m a_i f_i^{(m)}\right\|^p_{L_p} &= \sum_{i=1}^m |a_i|^p \int_{[0,1]} \left(f_i^{(m)}\right)^p = m^{(p/2-1)} \sum_{i=1}^m |a_i|^p \\ &= \left\| m^{(1/2-1/p)}(a_1, \ldots, a_m)\right\|^p_p. \end{aligned} \tag{5.13}$$

For Condition 3, we note that for $t \in [0,1]$

$$\begin{aligned} X_t &= \sum_{i=1}^m f_i^{(m)}(t)\sqrt{m} \int_{[t_i^{(m)}, t_{i+1}^{(m)}]} X_t \, d\lambda(s), \\ (V_m(X))_t &= \sum_{i=1}^m f_i^{(m)}(t)\sqrt{m} \int_{[t_i^{(m)}, t_{i+1}^{(m)}]} X_s \, d\lambda(s). \end{aligned} \tag{5.14}$$

Using the inequalities

$$\|f\|_{L_{p'}} \le \|f\|_{L_p}, \qquad \|X\|_{L_{r'}(\mathbb{P})} \le \|X\|_{L_r(\mathbb{P})}, \tag{5.15}$$

for $r \geq r'$, $p \geq p'$, $f \in L_p$ and $X \in L_r(\mathbb{P})$, we get

$$\begin{aligned}
\left\| \|X - V_m(X)\|_{L_p([0,1])} \right\|_{L_r(\mathbb{P})}^{p\vee r} &\leq \left\| \|X - V_m(X)\|_{L_{p\vee r}([0,1])} \right\|_{L_{p\vee r}(\mathbb{P})}^{p\vee r} \\
&= \left\| \|X - V_m(X)\|_{L_{p\vee r}(\mathbb{P})} \right\|_{L_{p\vee r}([0,1])}^{p\vee r} \\
&= \int_{[0,1]} \mathbb{E} \left| \sum_{i=1}^{m} f_i^{(m)}(t)\sqrt{m} \int_{[t_i^{(m)}, t_{i+1}^{(m)}]} (X_t - X_s)\,d\lambda(s) \right|^{p\vee r} d\mathbb{P}\,d\lambda(t) \\
&\leq \int_{[0,1]} \left| \sum_{i=1}^{m} f_i^{(m)}(t)\sqrt{m} \underbrace{\int_{[t_i^{(m)}, t_{i+1}^{(m)}]} \|X_t - X_s\|_{L_{r\vee p}(\mathbb{P})}\,d\lambda(s)}_{\leq C/m^{a+1}} \right|^{p\vee r} d\lambda(t) \\
&\leq \int_{[0,1]} \left| \chi_{[0,1]}(t) \frac{C}{m^a} \right|^{p\vee r} d\lambda(t) = \frac{C^{p\vee r}}{m^{a(p\vee r)}}.
\end{aligned} \tag{5.16}$$

Therefore, we know that the sequence $(m(n))_{n\in\mathbb{N}}$ satisfies (2.7) since we get with (5.16)

$$\left\| \|X - V_{m(n)}(X)\|_{L_p([0,1])} \right\|_{L_r(\mathbb{P})} \leq \frac{C^{p\vee r}}{m(n)^a} = o\left(\log(n)^{-b}\right) = o(e_{n,r}(X, E)), \tag{5.17}$$

as $n \to \infty$, and the assertion follows from Theorem 2.3. □

6. Examples

In this section, we want to present some processes that fulfill the requirements of the Theorems 3.1, 3.3, 4.2, and 5.1. Firstly, we give some examples for Gaussian processes that can be applied to all of the four Theorems, and secondly we describe how our approach can be applied to Lévy processes in view of Theorem 5.1.

Examples 6.1. Gaussian Processes and Brownian Diffusions

(i) Brownian Motion and Fractional Brownian Motion

Let $(X_t^{(H)})_{t\in[0,1]}$ be a fractional Brownian motion with Hurst parameter $H \in (0,1)$ (in the case $H = 1/2$ we have an ordinary Brownian motion). Its covariance function is given by

$$\mathbb{E}X_s^{(H)} X_t^{(H)} = \frac{1}{2}\left(s^{2H} + t^{2H} - |s-t|^{2H}\right). \tag{6.1}$$

Note that except for the case of an ordinary Brownian motion the eigenvalues and eigenvectors of the fractional Brownian motion are not known explicitly. Nevertheless, the sharp asymptotics of the eigenvalues has been determined (see, e.g., [7]).

In [28] the authors constructed an admissible sequence $(g_j)_{j\in\mathbb{N}}$ in $C([0,1])$ that satisfies the requirements of Proposition 4.1 with $\theta = 1/2 + H$. Furthermore, the eigenvalues λ_j of $C_{X^{(H)}}$ in $L_2([0,1])$ satisfy $\lambda_j \approx j^{-(1+2H)}$, see, for example, [7], such that the requirements for Theorem 4.2 are satisfied. Additionally, this sequence is a rate optimal admissible sequence for $X^{(H)}$ in $L_2([0,1])$, such that the requirements for Theorem 3.3 are also met. Constructing recursively an orthonormal sequence $(h_j)_{j\in\mathbb{N}}$ by applying Gram-Schmidt procedure on the sequence $(g_j)_{j\in\mathbb{N}}$ yields a rate optimal ONS for $X^{(H)}$ in $L_2([0,1])$ that can be used in the application of Theorem 3.1. In Section 7 we will illustrate the quantizers constructed for $X^{(H)}$ with this ONS for several Hurst parameters H. Note that there are several other admissible sequences for the fractional Brownian motion which can be applied similarly as described above; see, for example, [29] or [30]. Moreover, we have for $s,t \in [0,1]$ the mean regularity property

$$\mathbb{E}\left\| X_t^H - X_s^H \right\|^p = C_{H,p}|t-s|^{pH}, \tag{6.2}$$

and the asymptotics of the quantization error is given as

$$e_{n,r}\left(X^H, L_p([0,1])\right) \approx e_{n,2}\left(X^H, L_2([0,1])\right) \approx \left(\log(n)\right)^{-H}, \quad n \longrightarrow \infty \tag{6.3}$$

for all $r,p \geq 1$ (see [13]), such that the requirements of Theorem 5.1 are met with $a = b = H$. Note that in [11] the authors showed the existence of constants $k(H,E)$ for $E = C([0,1])$ and $E = L_p([0,1])$ independent of r such that

$$e_{n,r}\left(X^H, E\right) \sim k(H,E)\left(\log(n)\right)^{-H}, \quad n \longrightarrow \infty. \tag{6.4}$$

Therefore, the quantization errors of the sequences of quantizers constructed via Theorems 3.1, 3.3, 4.2, and 5.1 also fulfill this sharp asymptotics.

(ii) Brownian Bridge

Let $(B_t)_{t\in[0,1]}$ be a Brownian bridge with covariance function

$$\mathbb{E}B_s B_t = \min(s,t) - st. \tag{6.5}$$

Since the eigenvalues and eigenvectors of the Brownian bridge are explicitly known, we do not have to search for any other admissible sequence or ONS for $(B_t)_{t\in[0,1]}$ to be applied in $H = L_2([0,1])$. This (the eigenvalue-eigenvector) admissible sequence also satisfies the requirements of Theorem 4.2. The mean pathwise regularity for the Brownian bridge can be deduced by

$$\begin{aligned}\left(\mathbb{E}|B_t - B_s|^p\right)^{1/p} &\leq C_{p,2}\left(\mathbb{E}|B_t - B_s|^2\right)^{1/2} \\ &= C_{p,2}\left(|t-s| - |t-s|^2\right)^{1/2} \\ &\leq C|t-s|^{1/2},\end{aligned} \tag{6.6}$$

for any $p \geq 1$. Combining [31, Theorem 3.7] and [13, Corollary 1.3] yields

$$e_{n,r}(B, L_p([0,1])) \approx (\log(n))^{-1/2}, \quad n \longrightarrow \infty, \tag{6.7}$$

for all $r, p \geq 1$, such that Theorem 5.1 can be applied with $a = b = 1/2$.

(iii) Stationary Ornstein-Uhlenbeck Process

The stationary Ornstein-Uhlenbeck process $(X_t)_{t\in[0,1]}$ is a Gaussian process given through the covariance function

$$\mathbb{E}X_s X_t = \frac{\sigma^2}{2\alpha}\exp(-\alpha|s-t|), \tag{6.8}$$

with parameters $\alpha, \sigma > 0$. An admissible sequence for the stationary Ornstein-Uhlenbeck process in $C([0,1])$ and $L_2([0,1])$ can be found in [21]. This sequence that can be applied to Theorems 3.3 and 4.2 and also by applying Gram-Schmidt procedure to Theorem 3.1. According to [13] we have

$$e_{n,r}(X, L_p([0,1])) \approx (\log(n))^{-1/2}, \quad n \longrightarrow \infty \tag{6.9}$$

for all $r, p \geq 1$. Furthermore, it holds that

$$\begin{aligned}\left(\mathbb{E}|X_t - X_s|^p\right)^{1/p} &= C_{p,2}\left(\mathbb{E}|X_t - X_s|^2\right)^{1/2} \\ &= C_{p,2}\left(\frac{\sigma^2}{\alpha}(1-\exp(-\alpha|s-t|))\right)^{1/2} \\ &\leq C|s-t|^{1/2},\end{aligned} \tag{6.10}$$

and therefore we can choose $a = b = 1/2$ to apply Theorem 5.1.

(iv) Fractional Ornstein-Uhlenbeck Process

The fractional Ornstein-Uhlenbeck process $(X_t^{(H)})_{t\in[0,1]}$ for $H \in (0,2)$ is a continuous stationary centered Gaussian process with the covariance function

$$\mathbb{E}X_s^{(H)}X_t^{(H)} = e^{-\alpha|t-s|^H}, \quad \alpha > 0. \tag{6.11}$$

In [22] the authors constructed an admissible sequence $(g_j)_{j\in\mathbb{N}}^{(H)}$ for $H \in (0,1]$ that satisfies the requirements of Proposition 4.1 with $\theta = 1/2 + H/2$. Since the eigenvalues λ_j of $C_{X^{(H)}}$ in $L_2([0,1])$ satisfy $\lambda_j \approx j^{-1+H}$, we get again that the assumptions of Theorem 4.2 are satisfied. Similarly, we can use this sequence in Theorems 3.3 and 3.1.

(v) Brownian Diffusions

We consider a 1-dimensional Brownian diffusion $(X_t)_{t\in[0,1]}$ fulfilling the SDE

$$X_t = \int_0^t b(s,X_s)ds + \int_0^t \sigma(s,X_s)dW_s, \tag{6.12}$$

where the deterministic functions $b,\sigma : [0,1]\times\mathbb{R} \to \mathbb{R}$ satisfy the growth assumption

$$|b(t,x)| + |\sigma(t,x)| \le C(1+|x|). \tag{6.13}$$

Under some additional ellipticity assumption on σ, the asymptotics of the quantization error in $(L_p([0,1],\|\cdot\|_p)$ is then given by

$$e_{n,r}(X, L_p([0,1])) \approx (\log(n))^{-1/2}, \tag{6.14}$$

as $n \to \infty$ (see [10] and also [32]). Furthermore, one shows that for $0 \le s \le t \le 1$

$$\left(\mathbb{E}\|X_t - X_s\|^p\right)^{1/p} \le C(t-s)^{1/2} \tag{6.15}$$

(see [17, Examples 3.1]) such that Theorem 5.1 can be applied with $a = b = 1/2$.

Examples 6.2 (*Lévy processes*). Let $(X_t)_{t\in[0,1]}$ be a real Lévy process, that is, X is a càdlàg process with $\mathbb{P}(X_0 = 0) = 1$ and stationary and independent increments. The characteristic exponent $\psi(u)$ given through the equation

$$\int_{\mathbb{R}} \exp(iux)\mathbb{P}^{X_1}(dx) = \exp(-\psi(u)), \quad u \in \mathbb{R}, \tag{6.16}$$

is characterized by the *Lévy-Khintchine formula*

$$\psi(u) = iau + \frac{1}{2}\sigma^2u^2 + \int_{\mathbb{R}}\left(1 - e^{iux} + iux\chi_{(|x|<1)}\right)\Pi(dx), \tag{6.17}$$

where the characteristic triple (a, σ, Π) contains constants $a \in \mathbb{R}$, $\sigma \geq 0$, and a measure Π on $\mathbb{R} \setminus \{0\}$ satisfying $\int_{\mathbb{R}}(1 \wedge x^2)\Pi(dx) < \infty$. By definition, we know that

$$\mathbb{E}|X_t - X_s|^p = \mathbb{E}|X_{t-s}|^p, \tag{6.18}$$

and it is further known that the latter moment is finite if and only if

$$\int_{(|x|\geq 1)} |x|^p \Pi(dx) < \infty. \tag{6.19}$$

Furthermore, by the *Lévy-Ito decomposition*, X can be written as the sum of independent Lévy processes

$$X = X^{(1)} + X^{(2)} + X^{(3)}, \tag{6.20}$$

where $X^{(3)}$ is a Brownian motion with drift, $X^{(2)}$ is a *Compound Poisson process*, and $X^{(1)}$ is a Lévy process with bounded jumps and without Brownian component.

Firstly, we will analyze the mean pathwise regularity of these three types of *Lévy* processes to combine these results with lower bounds for the asymptotical quantization error.

(1) *Mean Pathwise Regularity of the 3 Components of the Lévy-Ito Decomposition:*

(i) According to an extended *Millar's Lemma* [17, Lemma 5], we have, for all Lévy processes with bounded jumps and without Brownian component, that there is for every $p \geq 2$ a constant $C < \infty$ such that for every $t \in [0,1]$

$$\mathbb{E}|X_t|^p \leq Ct = C\left(t^{1/p}\right)^p. \tag{6.21}$$

Combining (6.18) and (6.21), we can choose ρ in (5.2) as $\rho_{1,p}(x) = x^{1/p}$. For $p \in [1,2)$ we have by using (6.21) with $p = 2$

$$\mathbb{E}|X_t|^p \leq \left(\mathbb{E}|X_t|^2\right)^{p/2} \leq (Ct)^{p/2} = \left(C^{1/2}t^{1/2}\right)^p, \tag{6.22}$$

and thus we can choose $\rho_{1,p}(x) = Cx^{1/2}$. Combining these facts, we get $\rho_{1,p}(x) = Cx^{1/2 \vee p}$ for $p \geq 1$.

(ii) We consider the Compound Poisson process

$$X_t = \sum_{k=1}^{K_t} Z_k, \tag{6.23}$$

where K denotes a standard Poisson process with intensity $\lambda = 1$ and $(U_k)_{k\in\mathbb{N}}$ is an i.i.d sequence of random variables with $\|Z_1\|_{L_p(\mathbb{P})} < \infty$. Then, one shows that

$$\mathbb{E}\left|\sum_{k=1}^{K_t} Z_k\right|^p \leq t\|Z_1\|_{L_p(\mathbb{P})}^p \exp(-t)\sum_{k=1}^{\infty}\frac{t^{k-1}k^p}{k!} \leq C\left(t^{1/p}\right)^p, \tag{6.24}$$

so that (5.2) is satisfied with $\phi_{2,p}(x) = x^{1/p}$.

(iii) We consider a Brownian motion with drift. Using Examples 6.1 (i) and Lemma 2.1 we can choose ρ in (5.2) as $\rho_{3,p}(x) = \rho_3(x) = x^{1/2}$ for all $p \geq 1$.

(2) *Lévy Processes with Nonvanishing Brownian Component*
Let X be a Lévy process with non vanishing Brownian component, which means that σ in the characteristic triple satisfies $\sigma > 0$. in [17, Proposition 4] for $r, p \geq 1$, it holds that

$$(\log(n))^{-1/2} \approx C e_{n,r}(W, L_p) \lesssim e_{n,r}(X, L_p), \quad n \longrightarrow \infty \tag{6.25}$$

for some constant $C \in (0,\infty)$, and W denotes a Brownian motion. We consider the Lévy-Ito decomposition $X = X^{(1)} + X^{(2)} + X^{(3)}$ and assume that for $X_t^{(2)} = \sum_{k=1}^{K_t} Z_k$it holds that $\|Z_1\|_{L_{p\vee r}(\mathbb{P})} < \infty$. Therefore, we receive the mean pathwise regularity for X, all $p, r \geq 1$, and some constant $C < \infty$

$$\rho_p(x) := Cx^{1/2\vee r\vee p}. \tag{6.26}$$

Thus, we can apply Theorem 5.1 with $a = 1/2 \vee p \vee r$ and $b = 1/2$.

(3) *Compound Poisson Processes*
For a Compound Poisson process X we know that the rate for the asymptotic quantization error under suitable assumptions is given by

$$e_{n,r}(X, L_p) \approx \exp\left(-\kappa\sqrt{\log(n)\log(\log(n))}\right), \quad n \longrightarrow \infty; \tag{6.27}$$

see [16, Theorems 13, 14] and [17, Proposition 3] for a constant $\kappa \in (0,\infty)$. Thus, the sequence $(m(n))_{n\in\mathbb{N}}$ has to grow faster than in the examples above. To fulfill

$$\left\| \|X - V_m(X)\|_{L_p([0,1])} \right\|_{L_r(\mathbb{P})} = o\left(\exp\left(-\kappa\sqrt{\log(n)\log(\log(n))}\right)\right), \tag{6.28}$$

as $n \to \infty$ (see the proof of Theorem 5.1), we need to choose $m(n) = \lceil (p \vee r)\exp(\kappa\sqrt{\log(n)\log(\log(n))}(1+\epsilon))\rceil$ for an arbitrary $\epsilon > 0$.

(4) *α-stable Lévy Processes with $\alpha \in (0,2)$*
These are Lévy processes satisfying the self-similarity property

$$X_t \overset{d}{=} t^{1/\alpha} X_1, \tag{6.29}$$

and furthermore

$$\mathbb{E}|X_1|^{\alpha} = \infty, \qquad \sup_r \{\mathbb{E}|X_1|^r < \infty\} = \alpha. \tag{6.30}$$

Thus, we can choose $\rho(x) = Cx^{1/\alpha}$ for any $p \geq 1$ and constants $C_p < \infty$. The asymptotics of the quantization error for X is given by

$$e_{n,r}(X, L_p) \approx \log(n)^{-1/\alpha}, \quad n \longrightarrow \infty \tag{6.31}$$

for $r, p \geq 1$ [14], such that we meet the requirements of Theorem 5.1 by setting $a = b = \alpha$.

7. Numerical Illustrations

In this section, we want to highlight the steps needed for a numerical implementation of our approach and also give some illustrating results. For this purpose, it is useful to regard an n-quantizer α_n as an element of E^n (again denoted by α_n) instead of being a subset of E. Then, r-optimality of an n-quantizer α_n for the random variable X in the separable Banach space E reads

$$\alpha_n = \underset{\alpha=(a_1,\ldots,a_n)\in E^n}{\arg\min} D_{n,r}^X(\alpha), \qquad D_{n,r}^X(\alpha) := \mathbb{E}\min_{1\leq i\leq n}\|X - a_i\|^r, \tag{7.1}$$

with $D_{n,r}^X(\alpha)$ also called distortion function for X. The differentiability of the distortion function was treated in [6] for finite-dimensional Banach spaces (what is sufficient for our purpose) and later in [33] for the general case.

Proposition 7.1 (see [6, Lemma 4.10]). *Assume that the norm $\|\cdot\|$ of $\mathbb{R}^d$ is smooth. Let $r > 1$, and assume that any Voronoi diagram $\{V_{a_i}(\alpha), a_i \in \alpha, 1 \leq i \leq n\}$ with $V_{a_i}(\alpha) := \{x \in \mathbb{R}^d : \|x - a_i\| = \min_{a\in\alpha}\|x - a\|\}$ satisfies $\mathbb{P}^X(V_{a_i}(\alpha) \cap V_{a_j}(\alpha)) = 0$ for $i \neq j$. Then, the distortion function is differentiable at every admissible n-tuple $\alpha = (a_1, \ldots, a_n)$ (i.e., $a_i \neq a_j$ for $i \neq j$) with*

$$\nabla D_{n,r}^X(\alpha) = r\mathbb{E}\Big(\chi_{C_{a_i}(\alpha)\setminus\{a_i\}}(X)\|X - a_i\|^{r-1}\nabla\|\cdot\|(X - a_i)\Big) \in \left(\mathbb{R}^d\right)^n, \tag{7.2}$$

where $\{C_{a_i}(\alpha) : 1 \leq i \leq n\}$ denotes any Voronoi partition induced by $\alpha = \{a_1, \ldots, a_n\}$.

Remark 7.2. When $r = 1$, the above result extends to admissible n-tuples with $\mathbb{P}^X(\{a_1, \ldots, a_n\}) = 0$. Furthermore, if the norm is just smooth on a set $A \in \mathcal{B}(E)$ with $\mathbb{P}^X(A) = 1$, then the result still holds true. This is, for example, the case for $(E, \|\cdot\|) = (\mathbb{R}^d, \|\cdot\|_\infty)$ and random variables X with $\mathbb{P}^X(H) = 0$ for all hyperplanes H, which includes the case of normal distributed random variables.

Classic optimization theories now yield that any local minimum is contained in the set of stationary points. So let $n \in \mathbb{N}$, $m = m(n) \in \mathbb{N}$, $r \geq 1, X, V_m$, and ϕ_m be given. The procedure looks as follows.

Step 1. Calculation of the Distribution of the $\mathbb{R}^m$-Valued Random Variable $\zeta := \phi_m(V_m(X))$. This step strongly depends on the shape of the random variable X and the operators V_m.

Being in the setting of Section 3.1 one starts with an orthonormal system $(h_i)_{i\in\mathbb{N}}$ in H providing

$$\phi_m(V_m(X)) = \phi_m\left(V_m\left(\sum_{i=1}^{\infty} h_i\langle h_i, X\rangle\right)\right) = \phi_m\left(\sum_{i=1}^{m} h_i\langle h_i, X\rangle\right) = \sum_{i=1}^{m} e_i\langle h_i, X\rangle, \tag{7.3}$$

where $(e_i)_{1\leq i\leq m}$ denote the unit vectors in $\mathbb{R}^m$. Thus, the covariance matrix of the random variable ζ admits the representation

$$\mathbb{E}\zeta\zeta^{\perp} = \left(\mathbb{E}\langle h_i, X\rangle\langle h_j, X\rangle\right)_{1\leq i,j,\leq m} = \left\langle C_X(h_i), h_j\right\rangle_{1\leq i,j,\leq m}, \tag{7.4}$$

with C_X being the covariance operator of X.

Similarly, we get for Gaussian random variables in the framework of Section 3.2

$$\mathbb{E}\zeta\zeta^{\perp} = \left(\sqrt{\lambda_i^{(m)}}\sqrt{\lambda_j^{(m)}}\right)_{1\leq i,j,\leq m}, \tag{7.5}$$

in the setting of Section 4

$$\mathbb{E}\zeta\zeta^{\perp} = \left(\mathbb{E}X_{t_i^{(m)}}X_{t_j^{(m)}}\right)_{1\leq i,j,\leq m} = \left(\delta_{t_i}\left(C_X\left(\delta_{t_j}\right)\right)\right)_{1\leq i,j,\leq m}, \tag{7.6}$$

and in the setting of Section 5

$$\mathbb{E}\zeta\zeta^{\perp} = \left(m^{1-2/p} f_j^{(m)}\left(C_X\left(f_i^{(m)}\right)\right)\right)_{1\leq i,j,\leq m}, \tag{7.7}$$

with $f_j^{(m)}$ associated with $\int_{[0,1]} \cdot f_j^{(m)}(s)ds$. If one considers in the latter framework a non-Brownian Lévy process, for example, and a compound Poisson process (we use the notations as in Examples 6.2 (1) (ii)), the simulation of the gradient leads to the problem of simulating

$$\int_{t_i}^{t_{i-1}} \sum_{j=1}^{K_t} Z_k dt, \tag{7.8}$$

which is still possible.

Step 2. Use a (stochastic) optimization algorithm to solve the stationarity equation

$$\nabla D^{\zeta}_{n,r}(\alpha) = 0 \in \mathbb{R}^m, \tag{7.9}$$

for $\zeta = \phi_{\mathrm{m}}(V_m(X))$. For this purpose, the computability of the gradient (7.2) is of enormous importance. One may either apply a deterministic gradient-based optimization algorithm (e.g., BFGS) combined with a (Quasi) Monte-Carlo approximation for the gradient, such as the one used in [20], or use a stochastic gradient algorithm, which is in the Hilbert space setting also known as CLVQ (competitive learning vector quantization) algorithm (see, e.g., [19] for more details). In both cases, the random variable $\mathbb{P}^{\zeta}$ needs to be simulated, which is the case for the above described examples.

Step 3. Reconstruct the quantizer $\beta = (b_1, \ldots, b_n)$ for the random variable X by setting

$$b_i := \phi_m^{-1}(a_i) \in F_m \subset E, \tag{7.10}$$

for $1 \le i \le n$ with $\alpha = (a_1, \ldots, a_n)$ being some solution of the stationarity (7.9).

Illustration

For illustration purposes, we will concentrate on the case described in Section 3.1 for $r = 2$. Examples for quantizers as constructed in Section 4 can be found in [20]. The quantizers shown in the sequel were calculated numerically, by using the widely used CLVQ-algorithm as described in [19]. To achieve a better accuracy, we finally performed a few steps of a gradient algorithm by approximating the gradient with a Monte Carlo simulation.Let $X^{(H)}$ be a fractional Brownian motion with Hurst parameter H. We used the admissible sequence as described in [28]:

$$X_t^{(H)} \stackrel{d}{=} \sum_{n=1}^{\infty} \frac{\sqrt{2}c_H}{|J_{1-H}(x_n)|} \frac{\sin(x_n t)}{x_n^{1+H}} \zeta_n^1 + \sum_{n=1}^{\infty} \frac{\sqrt{2}c_H}{|J_{-H}(y_n)|} \frac{1-\cos(y_n t)}{y_n^{1+H}} \zeta_n^2, \tag{7.11}$$

where c_H is given as

$$c_H^2 := \frac{\sin(\pi H)\Gamma(1+2H)}{\pi}, \tag{7.12}$$

J_{1-H} and J_{-H} are Bessel functions with corresponding parameters, and x_n and y_n are the ordered roots of the Bessel functions with parameters $-H$ and $1-H$. After ordering the elements of the two parts of the expansion in an alternating manner and applying Gram-Schmidt's procedure for orthogonalization to construct a rate optimal ONS, we used the method as described in Section 3.1. We show the results we obtained for $n = 10$, $m = 4$ and the Hurst parameters $H = 0.3,\ 0.5$, and 0.7 (Figures 1, 2, and 3). To show the effects of changing parameters, we also present the quantizers obtained after increasing the size of the containing subspace ($m = 8$) (Figures 4, 5, and 6) and in addition the effect of increasing the quantizer size ($n = 30$) (Figures 7, 8, and 9). Since $X^{(H)}$ is for $H = 0.5$ an ordinary Brownian motion, one can compare the results with the results obtained for the Brownian motion by using the Karhunen-Loève expansion (see, e.g., [18]).

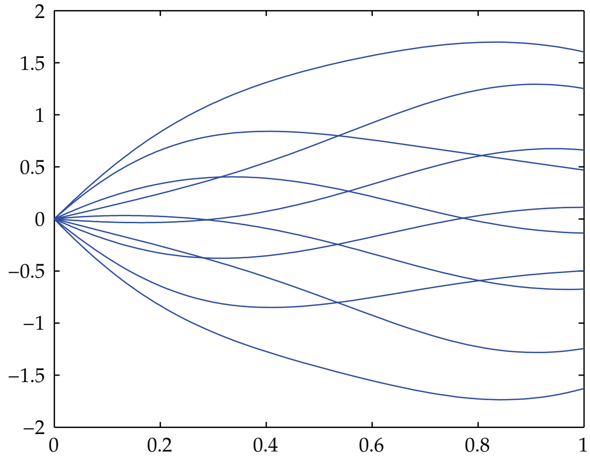

Figure 1: The 10-quantizer for the fractional Brownian motion with Hurst parameter $H = 0.3$ in a 4-dimensional subspace.

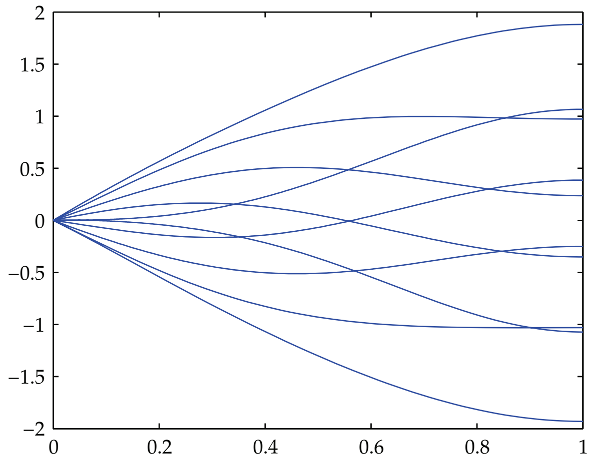

Figure 2: The 10-quantizer for the fractional Brownian motion with Hurst parameter $H = 0.5$ in a 4-dimensional subspace.

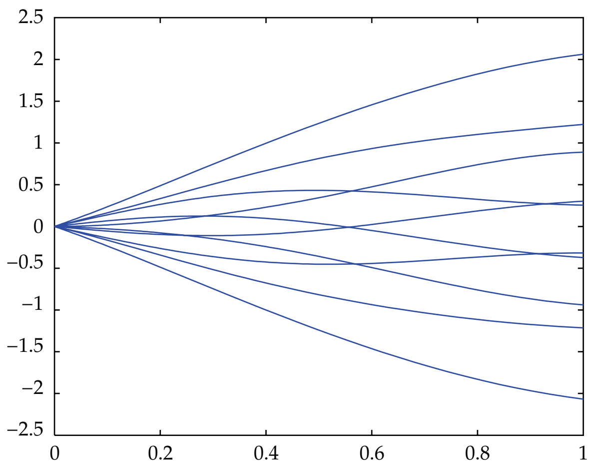

Figure 3: The 10-quantizer for the fractional Brownian motion with Hurst parameter $H = 0.7$ in a 4-dimensional subspace.

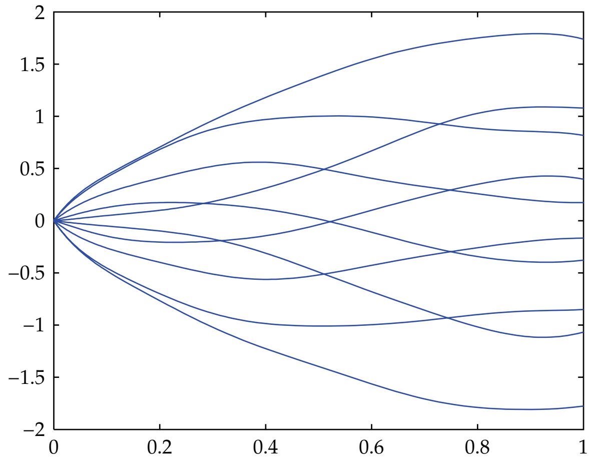

Figure 4: The 10-quantizer for the fractional Brownian motion with Hurst parameter $H = 0.3$ in an 8-dimensional subspace.

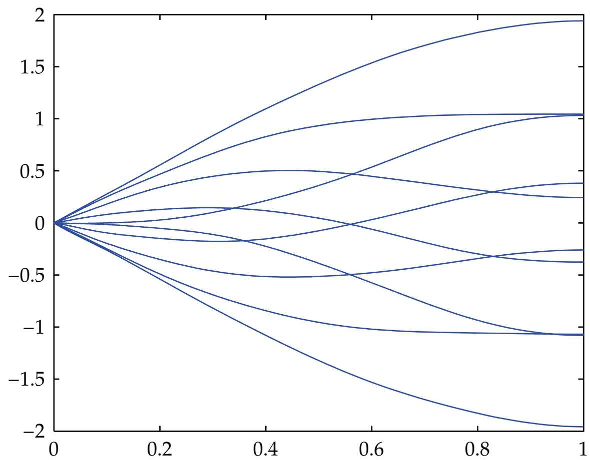

Figure 5: The 10-quantizer for the fractional Brownian motion with Hurst parameter $H = 0.5$ in an 8-dimensional subspace.

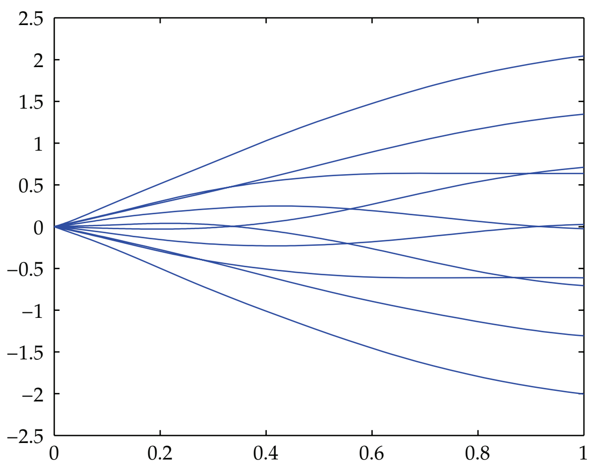

Figure 6: The 10-quantizer for the fractional Brownian motion with Hurst parameter $H = 0.7$ in an 8-dimensional subspace.

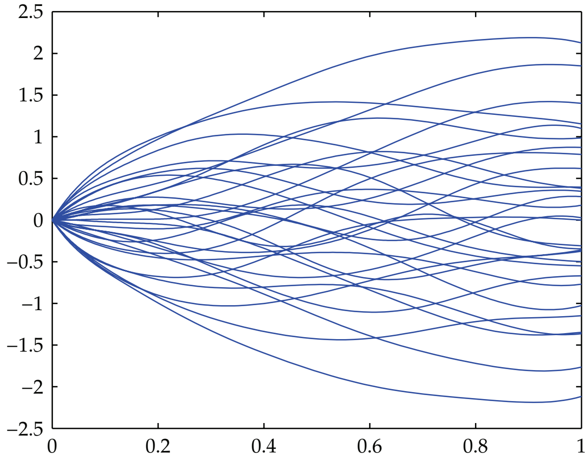

Figure 7: The 30-quantizer for the fractional Brownian motion with Hurst parameter $H = 0.3$ in an 8-dimensional subspace.

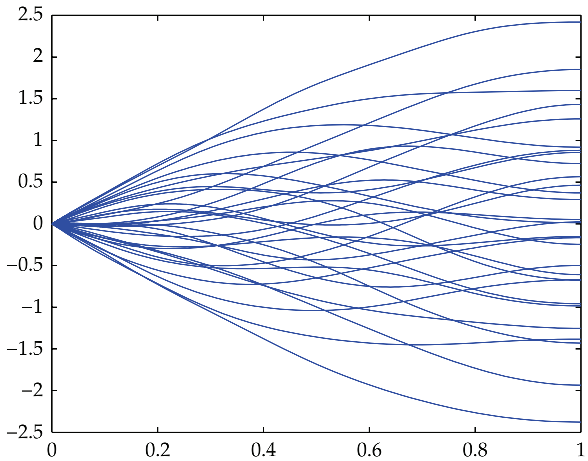

Figure 8: The 30-quantizer for the fractional Brownian motion with Hurst parameter $H = 0.5$ in an 8-dimensional subspace.

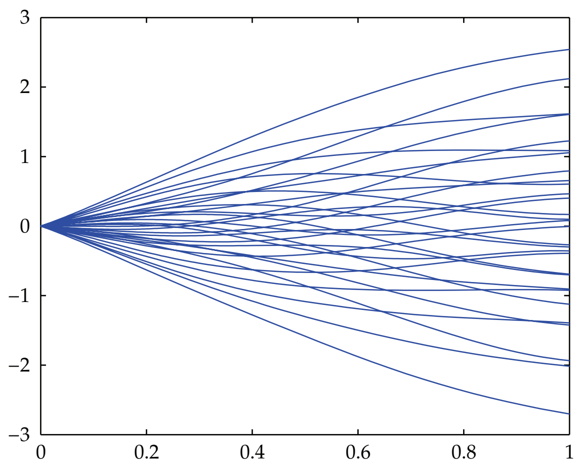

Figure 9: The 30-quantizer for the fractional Brownian motion with Hurst parameter $H = 0.7$ in an 8-dimensional subspace.

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