Research Article

Iterative Schemes for a Class of Mixed Trifunction Variational Inequalities

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We use the auxiliary principle technique to suggest and analyze some iterative methods for solving a new class of variational inequalities, which is called the mixed trifunction variational inequality. The mixed trifunction variational inequality includes the trifunction variational inequalities and the classical variational inequalities as special cases. Convergence of these iterative methods is proved under very mild and suitable assumptions. Several special cases are also considered. Results proved in this paper continue to hold for these known and new classes of variational inequalities and its variant forms.

1. Introduction

In recent years, variational inequalities have appeared an interesting and dynamic field of pure and applied sciences. Variational techniques are being used to study a wide class of problem with applications in industry, structural engineering, mathematical finance, economics, optimization, transportation, and optimization problems. This has motivated to introduce and study several classes of variational inequalities. It is well known that the minimum of the differentiable convex functions on the convex set can be characterized by the variational inequalities. This result is due to Stampacchia [1]. However, we remark that if the convex function is directionally differentiable, then its minimum is characterized by a class of variational inequalities, which is called the bifunction variational inequality. For the formulation, applications, numerical results, and other aspects of bifunction variational inequalities, see [2–14]. Noor et al. [13] considered a new class of variational

inequalities, which is called the trifunction variational inequality. It has been shown that the trifunction variational inequality includes the variational inequality and bifunction variational inequality as special cases.

Inspired and motivated by the ongoing research in this dynamic and fascinating field, we consider and analyze a new class of variational inequalities, called the mixed trifunction variational inequality. This new class of trifunction variational inequalities includes the trifunction(bifunction) variational inequality and the classical variational inequality as special cases.

There are a substantial number of numerical methods for solving the variational inequalities and trifunction equilibrium problems. Due to the nature of the trifunction variational inequality problem, projection methods and its variant form such as Wiener-Hopf equations cannot be used for solving the trifunction variational inequality. This fact motivated us to use the auxiliary principle technique of Glowinski et al. [15] as developed by Aslam Noor [16] and, Noor et al. [17]. This technique is quite flexible and general one. We again use this technique to suggest some explicit and proximal-point iterative methods for solving these problems. We also consider the convergence criteria of the proposed methods under suitable mild conditions, which is the main results (Theorems 3.4, 3.5, and 3.13) of this paper. Several special cases of our main results are also considered. Results obtained in this paper may be viewed as an improvement and refinement of the previously known results. The ideas and techniques of this paper stimulate further research in this area of pure and applied sciences.

2. Preliminaries

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let K be a closed and convex set in H.

For given trifunction $F(\cdot, \cdot, \cdot) : H \times H \times H \rightarrow R$ and an operator $T : H \rightarrow H$, we consider the problem of finding $u \in K$ such that

$$F(u, Tu, v - u) + \langle Tu, \ v - u \rangle \geq 0, \quad \forall v \in K, \tag{2.1}$$

which is called the trifunction variational inequality. We note that if $F(\cdot, \cdot, \cdot) \cong F(\cdot, \cdot)$, then problem (2.1) is studied in S. Takahashi and W. Takahashi [18], Yao et al. [19, 20], and Noor et al. [12, 14].

We now discuss some important special cases of the problem (2.1).

2.1. Special Cases

(I) If $F(u, Tu, v - u) = 0$, then problem (2.1) is equivalent to finding $u \in K$ such that

$$\langle Tu, v - u \rangle \geq 0, \quad \forall v \in K, \tag{2.2}$$

which is known as a variational inequality, introduced and studied by Stampacchia [1]. A wide class of problems arising in elasticity fluid flow through porous media, and optimization can be studied in the unified framework of problems (2.1). For the applications,

formulation numerical results and other aspects of the variational inequalities and their generalizations, see [1–27].

(II) If $\langle Tu, v - u \rangle = 0$, then problem (2.1) turns into the problem of finding $u \in K$ such that

$$F(u, Tu, v - u) \geq 0, \quad \forall v \in K, \tag{2.3}$$

which is known as trifunction variational inequalities, considered by Noor et al. [13].

For suitable and appropriate choice of the operator and spaces, one can obtain several new and known problems as special cases of the trifunction variational inequalities problems (2.1). For the applications, formulations, numerical methods, and other aspects of the variational inequalities, see [1–27].

Definition 2.1. An operator $T : H \rightarrow H$ is said to be

(i) monotone, if and only if, $\langle Tu - Tv, u - v \rangle \geq 0$, for all $u, v \in H$,

(ii) partially relaxed strongly monotone, if there exists a constant $\alpha > 0$ such that

$$\langle Tu - Tv, z - v \rangle \geq -\alpha \|z - u\|^2, \quad \forall u, v, z \in H. \tag{2.4}$$

Note that for $z = u$, partially relaxed strong monotonicity reduces to monotonicity of the operator T.

Definition 2.2. A trifunction $F(\cdot, \cdot, \cdot) : H \times H \times H \rightarrow R$ with respect to an operator T is said to be

(a) jointly monotone if and only if

$$F(u, Tu, v - u) + F(v, Tv, u - v) \leq 0, \quad \forall u, v \in H \tag{2.5}$$

(b) partially relaxed strongly jointly monotone if and only if there exists a constant $\alpha > 0$ such that

$$F(u, Tu, v - u) + F(v, Tv, z - v) \leq \mu \|z - u\|^2, \quad \forall u, v, z \in H. \tag{2.6}$$

It is clear that for $z = u$, partially relaxed strongly jointly monotone trifunction is simply jointly monotone.

3. Main Results

In this section, we suggest and analyze an iterative method for solving the trifunction variational-inequality problem (2.1) by using the auxiliary principle technique. This technique is mainly due to Glowinski et al. [15] as developed by Aslam Noor [16] and Noor et al. [8–14, 17, 25, 26].

For a given $u \in K$, consider the problem of finding $w \in K$ such that

$$\rho F(u, Tu, v - w) + \langle \rho Tu, v - w \rangle + \langle w - u, v - w \rangle \geq 0, \quad \forall v \in K, \tag{3.1}$$

where $\rho > 0$ is a constant.

If $w = u$, then $w \in H$ is a solution of (2.1). This observation enables us to suggest and analyze the following iterative method for solving the mixed trifunction variational inequality (2.1).

Algorithm 3.1. For a given $u_0 \in H$, compute $u_{n+1} \in H$ from the iterative scheme

$$\rho F(u_n, Tu_n, v - u_{n+1}) + \langle \rho Tu_n, v - u_{n+1} \rangle + \langle u_{n+1} - u_n, v - u_{n+1} \rangle \geq 0, \quad \forall v \in K. \tag{3.2}$$

We now discuss some special cases of Algorithm 3.1.

(III) If $F(u, Tu, v - w) = 0$, then Algorithm 3.1 reduces to the following scheme for variational inequalities (2.2).

Algorithm 3.2. For a given $u_0 \in H$, compute $u_{n+1} \in H$ from the iterative scheme

$$\langle \rho Tu_n, v - u_{n+1} \rangle + \langle u_{n+1} - u_n, v - u_{n+1} \rangle \geq 0, \quad \forall v \in K. \tag{3.3}$$

(IV) If $\langle Tu, v - u \rangle = 0$, then Algorithm 3.1 reduces to the following.

Algorithm 3.3. For a given $u_0 \in H$, compute $u_{n+1} \in H$ from the iterative scheme

$$F(u_n, Tu_n, v - u_{n+1}) + \langle u_{n+1} - u_n, v - u_{n+1} \rangle \geq 0, \quad \forall v \in K, \tag{3.4}$$

which is used for finding the solution of trifunction variational inequality (2.3).

For suitable and appropriate choice of $F(\cdot, \cdot)$, T and spaces, one can define iterative algorithms for solving the different classes of trifunction variational inequalities and related optimization problems.

We now study the convergence analysis of Algorithm 3.1 using the technique of Noor et al. [8–14, 17, 25, 26], and this is the main motivation of our next result.

Theorem 3.4. *Let $u \in H$ be a solution of (2.1) and $u_{n+1} \in H$ an approximate solution obtained from Algorithm 3.1. If the trifunction $F(\cdot, \cdot, \cdot)$ and the operator $T(\cdot)$ are partially relaxed strongly monotone operators with constants $\mu > 0$ and $\sigma > 0$, respectively, then*

$$\|u - u_{n+1}\|^2 \leq \|u - u_n\|^2 - (1 - 2\rho(\mu + \sigma))\|u_{n+1} - u_n\|^2. \tag{3.5}$$

Proof. Let $u \in K$ is a solution of (2.1), then, replacing v by u_{n+1} in (2.1), we have

$$\rho F(u, Tu, u_{n+1} - u) + \rho \langle Tu, u_{n+1} - u \rangle \geq 0, \quad \rho > 0. \tag{3.6}$$

Let $u_{n+1} \in K$ be the approximate solution obtained from Algorithm 3.1. Taking $v = u$ in (3.2), we have

$$\rho F(u_n, Tu_n, u - u_{n+1}) + \langle \rho Tu_n, u - u_{n+1} \rangle + \langle u_{n+1} - u_n, u - u_{n+1} \rangle \geq 0. \tag{3.7}$$

Adding (3.6) and (3.7), we have

$$\rho[F(u, Tu, u_{n+1} - u) + F(u_n, Tu_n, u - u_{n+1})] + \rho\langle Tu - Tu_n, u_{n+1} - u \rangle$$
$$+ \langle u_{n+1} - u_n, u - u_{n+1} \rangle \geq 0, \tag{3.8}$$

which implies that

$$\langle u_{n+1} - u_n, u - u_{n+1} \rangle \geq -\rho[F(u_n, Tu_n, u - u_{n+1}) + F(u, Tu, u_{n+1} - u)] + \rho\langle Tu_n - Tu, u_{n+1} - u \rangle$$
$$\geq -\rho(\mu + \sigma)\|u_{n+1} - u_n\|^2, \tag{3.9}$$

where we have used the partially relaxed strong monotonicity of the trifunction $F(\cdot, \cdot, \cdot)$ and operator T.

Using the relation $2\langle u, v \rangle = \|u + v\|^2 - \|u\|^2 - \|v\|^2$, for all $u, v \in H$, and from (3.9), one can have

$$\|u - u_{n+1}\|^2 \leq \|u - u_n\|^2 - (1 - 2\rho(\mu + \sigma))\|u_{n+1} - u_n\|^2, \tag{3.10}$$

which is the required result (3.5). $\square$

Theorem 3.5. *Let H be a finite dimensional space. If u_{n+1} is the approximate solution obtained from Algorithm 3.1 and $u \in K$ be a solution of problem (2.1). Then $\lim_{n \to \infty} u_n = u$.*

Proof. Let $u \in H$ be a solution of (2.1). For $0 < \rho < 1/(2(\mu + \sigma))$, we see that the sequence $\{\|u - u_n\|\}$ is nonincreasing and consequently $\{u_n\}$ is bounded. Also from (3.5), we have

$$\sum_{n=0}^{\infty} (1 - 2(\mu + \sigma))\|u_{n+1} - u_n\|^2 \leq \|u - u_0\|^2, \tag{3.11}$$

which implies that

$$\lim_{n \to \infty} \|u_{n+1} - u_n\| = 0. \tag{3.12}$$

Let $\hat{u}$ be the cluster point of $\{u_n\}$ and the subsequence $\{u_{n_j}\}$ of this sequence converges to $\hat{u} \in H$. Replacing u_n by u_{n_j} in (3.2) and taking the limit as $n_j \to \infty$ and using (3.12), we have

$$F(\hat{u}, T\hat{u}, v - \hat{u}) + \langle T\hat{u}, \ v - \hat{u} \rangle \geq 0, \quad \forall v \in K, \tag{3.13}$$

which shows $\hat{u}$ solves the trifunction variational inequality (2.1) and

$$\|u_{n+1} - \hat{u}\|^2 \leq \|u_n - \hat{u}\|^2. \tag{3.14}$$

Thus, it follows from the above inequality that the sequence $\{u_n\}$ has exactly one cluster point and $\lim_{n \to \infty} u_n = \hat{u}$, the required result.   □

We again use the auxiliary principle technique to suggest and analyze several proximal point algorithms for solving the trifunction variational inequalities (2.1), and this is another motivation of this paper. Here, we show that the one can suggest several form of the auxiliary mixed trifunction variational inequalities type. Each type gives rise to different type of the inertial proximal point method for solving the mixed trifunction variational inequality (2.1). This is the beauty of the auxiliary principle technique. To convey an idea of the technique, we only consider some special cases. The interested readers are invited to explore the novel applications of this technique in other fields of the mathematical and engineering sciences.

(V) For a given $u \in K$, consider the problem of finding $w \in K$ such that

$$\rho F(w, Tw, v - w) + \langle \rho Tw, v - w \rangle + \langle w - u - \gamma(u - u), v - w \rangle \geq 0, \quad \forall v \in K, \tag{3.15}$$

where $\rho \geq 0$ and $\gamma \geq 0$ are constants.

Note that if $w = u$, then $w \in K$ is a solution of (2.1). This observation enables us to suggest and analyze the following iterative method for solving the mixed trifunction variational inequality (2.1).

Algorithm 3.6. For a given $u_0 \in H$, compute $u_{n+1} \in H$ from the iterative scheme

$$\rho F(u_{n+1}, Tu_{n+1}, v - u_{n+1}) + \langle \rho Tu_{n+1}, v - u_{n+1} \rangle + \langle u_{n+1} - u_n - \gamma(u_n - u_{n-1}), v - u_{n+1} \rangle \geq 0,$$

$$\forall v \in K. \tag{3.16}$$

Algorithm 3.6 is called the inertial proximal point method for solving the mixed trifunction variational inequality (2.1).

Note that if $\gamma = 0$, then Algorithm 3.6 reduces to the following inertial proximal point method for solving (2.1).

Algorithm 3.7. For a given $u_0 \in H$, compute $u_{n+1} \in H$ from the iterative scheme

$$\rho F(u_{n+1}, Tu_{n+1}, v - u_{n+1}) + \langle \rho Tu_{n+1}, v - u_{n+1} \rangle + \langle u_{n+1} - u_n, v - u_{n+1} \rangle \geq 0, \quad \forall v \in K. \tag{3.17}$$

(VI) For a given $u \in K$, consider the problem of finding a $w \in K$ such that

$$\rho F(u, Tu, v - w) + \langle \rho Tw, v - w \rangle + \langle w - u - \gamma(u - u), v - w \rangle \geq 0, \quad \forall v \in K. \tag{3.18}$$

Note that if $w = u$, then $w \in K$ is a solution of (2.1). This observation enables us to suggest and analyze the following proximal iterative method for solving the mixed trifunction variational inequalities (2.1).

Algorithm 3.8. For a given $u_0 \in H$, compute $u_{n+1} \in H$ from the iterative scheme

$$\rho F(u_n, Tu_n, v - u_{n+1}) + \langle \rho Tu_{n+1}, v - u_{n+1} \rangle + \langle u_{n+1} - u_n \gamma(u_n - u_{n-1}), v - u_{n+1} \rangle \geq 0,$$

$$\forall v \in K. \tag{3.19}$$

(VII) For a given $u \in K$, consider the problem of finding a $w \in K$ such that

$$\rho F(w, Tw, v - w) + \langle \rho Tu, v - w \rangle + \langle w - u - \gamma(u - u), v - w \rangle \geq 0, \quad \forall v \in K. \tag{3.20}$$

Note that if $w = u$, then $w \in K$ is a solution of (2.1). This observation enables us to suggest and analyze the following iterative method for solving the mixed trifunction variational inequality (2.1).

Algorithm 3.9. For a given $u_0 \in H$, compute $u_{n+1} \in H$ from the iterative scheme

$$\rho F(u_{n+1}, Tu_{n+1}, v - u_{n+1}) + \langle \rho Tu_n, v - u_{n+1} \rangle + \langle u_{n+1} - u_n - \gamma(u_n - u_{n-1}), v - u_{n+1} \rangle \geq 0,$$

$$\forall v \in K. \tag{3.21}$$

Some special cases of these algorithms are as below.

If $F(u, Tu, v - u) = 0$, then Algorithm 3.6 reduces to the following scheme for variational inequalities given as (2.2).

Algorithm 3.10. For a given $u_0 \in H$, compute $u_{n+1} \in H$ from the iterative scheme

$$\langle \rho Tu_{n+1}, v - u_{n+1} \rangle + \langle u_{n+1} - u_n - \gamma(u_n - u_{n-1}), v - u_{n+1} \rangle \geq 0, \quad \forall v \in K. \tag{3.22}$$

If $\langle Tu, v - u \rangle = 0$, then Algorithm 3.6 reduces to the following.

Algorithm 3.11. For a given $u_0 \in H$, compute $u_{n+1} \in H$ from the iterative scheme

$$F(u_{n+1}, Tu_{n+1}, v - u_{n+1}) + \langle u_{n+1} - u_n - \gamma(u_n - u_{n-1}), v - u_{n+1} \rangle \geq 0, \quad \forall v \in K, \tag{3.23}$$

which is used for solving the trifunction variational inequality (2.3).

We now again use the auxiliary mixed trifunction variational inequality (3.15) to suggest the following the implicit iterative method for solving the mixed trifunction variational inequality (2.1).

Algorithm 3.12. For a given $u_0 \in H$, compute $u_{n+1} \in H$ from the iterative scheme

$$\rho F(u_{n+1}, Tu_{n+1}, v - u_{n+1}) + \langle \rho Tu_{n+1}, v - u_{n+1} \rangle + \langle u_{n+1} - u_n - \gamma(u_n - u_{n-1}), v - u_{n+1} \rangle \geq 0,$$

$$\forall v \in K. \tag{3.24}$$

We remark that Algorithm 3.12 is quite different from Algorithms 3.6–3.11. One can easily show that the convergence of Algorithm 3.12 also requires the montonicity of the trifunction $F(\cdot,\cdot,\cdot)$ and the operator T.

For suitable and appropriate choice of $F(\cdot,\cdot,\cdot)$, T and spaces, one can define iterative algorithms as special cases of Algorithms 3.7 and 3.8 for solving the different classes variational inequalities and related optimization problems.

We would like to mention that one can study the convergence analysis of Algorithm 3.7 using the technique of Theorems 3.4 and 3.5. However, for the sake of completeness and to convey the main ideas, we include the main steps of the proof.

Theorem 3.13. *Let $u \in H$ be a solution of* (2.1) *and $u_{n+1} \in H$ an approximate solution obtained from Algorithm 3.7. If trifunction $F(\cdot,\cdot,\cdot)$ and operator T are monotone, then*

$$\|u - u_{n+1}\|^2 \leq \|u - u_n\|^2 - \|u_{n+1} - u_n\|^2. \tag{3.25}$$

Proof. Let $u \in K$ be a solution of (2.1). Then, replacing v by u_{n+1} in (2.1), we have

$$\rho F(u, Tu, u_{n+1} - u) + \rho \langle Tu, u_{n+1} - u \rangle \geq 0. \tag{3.26}$$

Let $u_{n+1} \in H$ be the approximate solution obtained from Algorithm 3.6. Taking $v = u$ in (3.16), we have

$$\rho F(u_{n+1}, Tu_{n+1}, u) + \langle \rho Tu_{n+1}, u - u_{n+1} \rangle + \langle u_{n+1} - u_n, u - u_{n+1} \rangle \geq 0. \tag{3.27}$$

Adding (3.26) and (3.27), we have

$$\rho[F(u, Tu, u_{n+1} - u) + F(u_n, Tu_n, u - u_{n+1})] + \rho \langle Tu - Tu_{n+1}, u_{n+1} - u \rangle \\ + \langle u_{n+1} - u_n, u - u_{n+1} \rangle \geq 0, \tag{3.28}$$

which implies that

$$\langle u_{n+1} - u_n, u - u_{n+1} \rangle \\ \geq -\rho[F(u_n, Tu_n, u - u_{n+1}) + F(u, Tu, u_{n+1} - u)] + \rho \langle Tu_{n+1} - Tu, u_{n+1} - u \rangle \tag{3.29} \\ \geq 0,$$

where we have used the monotonicity of the operator T and the trifunction $F(\cdot,\cdot,\cdot)$.

Using the relation $2\langle u, v \rangle = \|u + v\|^2 - \|u\|^2 - \|v\|^2$, for all $u, v \in H$ and from (3.29), we have

$$\|u - u_{n+1}\|^2 \leq \|u - u_n\|^2 - \|u_{n+1} - u_n\|^2, \tag{3.30}$$

which is the required result (3.25). $\qquad\square$

Theorem 3.14. *Let H be a finite dimensional space. If u_{n+1} is the approximate solution obtained from Algorithm 3.6 and $u \in K$ is a solution of problem* (2.1)*, then* $\lim_{n \to \infty} u_n = u$.

Proof. Its proof is similar to the Proof of Theorem 3.5. □

4. Conclusion

In this paper, we have used the auxiliary principle technique to suggest and analyze several explicit and inertial proximal point algorithms for solving the trifunction equvariational inequality problem. We have also discussed the convergence criteria of the proposed new iterative methods under some suitable weaker conditions. In this sense, our results can be viewed as refinement and improvement of the previously known results. Note this technique does not involve the projection and the resolvent technique. We have also shown that this technique can be used to suggest several iterative methods for solving various classes of equilibrium and variational inequalities problems. Results proved in this paper may inspire further research in variational inequalities and related optimization problems.

Acknowledgments

References

[1] G. Stampacchia, "Formes bilinéaires coercitives sur les ensembles convexes," *Comptes Rendus de l'Académie des Sciences*, vol. 258, pp. 4413–4416, 1964.

[2] G. P. Crespi, I. Ginchev, and M. Rocca, "Minty variational inequalities, increase-along-rays property and optimization," *Journal of Optimization Theory and Applications*, vol. 123, no. 3, pp. 479–496, 2004.

[3] G. P. Crespi, I. Ginchev, and M. Rocca, "Existence of solutions and star-shapedness in Minty variational inequalities," *Journal of Global Optimization*, vol. 32, no. 4, pp. 485–494, 2005.

[4] G. P. Crespi, I. Ginchev, and M. Rocca, "Some remarks on the Minty vector variational principle," *Journal of Mathematical Analysis and Applications*, vol. 345, no. 1, pp. 165–175, 2008.

[5] Y.-P. Fang and R. Hu, "Parametric well-posedness for variational inequalities defined by bifunctions," *Computers & Mathematics with Applications*, vol. 53, no. 8, pp. 1306–1316, 2007.

[6] C. S. Lalitha and M. Mehta, "Vector variational inequalities with cone-pseudomonotone bifunctions," *Optimization*, vol. 54, no. 3, pp. 327–338, 2005.

[7] M. A. Noor, "Some new classes of nonconvex functions," *Nonlinear Functional Analysis and Applications*, vol. 11, no. 1, pp. 165–171, 2006.

[8] M. A. Noor and K. I. Noor, "Iterative schemes for trifunction hemivariational inequalities," *Optimization Letters*, vol. 5, no. 2, pp. 273–282, 2011.

[9] M. A. Noor, K. I. Noor, and E. Al-Said, "Auxiliary principle technique for solving bifunction variational inequalities," *Journal of Optimization Theory and Applications*, vol. 149, no. 2, pp. 441–445, 2011.

[10] M. A. Noor, K. I. Noor, and Z. Huang, "Bifunction hemivariational inequalities," *Journal of Applied Mathematics and Computing*, vol. 35, no. 1-2, pp. 595–605, 2011.

[11] S. M. Guu and J. Li, "Vector variational-like inequalities with generalized bifunctions defined on nonconvex sets," *Nonlinear Analysis, Theory, Methods and Applications*, vol. 71, no. 7-8, pp. 2847–2855, 2009.

[12] M. A. Noor, K. I. Noor, S. Zainab, and E. Al-Said, "Proximal algorithms for solving mixed bifunction variational inequalities," *International Journal of Physical Sciences*, vol. 6, no. 17, pp. 4203–4207, 2011.

[13] M. A. Noor, A. K. Khalifa, K. Al-Bani, and S. K. Khattri, "On trifunction variational inequalities," *International Journal of Nonlinear Science*, vol. 11, no. 1, pp. 17–21, 2011.

[14] M. A. Noor, K. I. Noor, and S. Zainab, "On mixed equilibrium variational inequalities," *International Journal of Physical Sciences*, vol. 6, no. 23, pp. 5412–5418, 2011.

[15] R. Glowinski, J.-L. Lions, and R. Trémolières, *Numerical Analysis of Variational Inequalities*, vol. 8, North-Holland, Amsterdam, The Netherlands, 1981.

[16] M. Aslam Noor, "Some developments in general variational inequalities," *Applied Mathematics and Computation*, vol. 152, no. 1, pp. 199–277, 2004.

[17] M. A. Noor, K. I. Noor, and T. M. Rassias, "Some aspects of variational inequalities," *Journal of Computational and Applied Mathematics*, vol. 47, no. 3, pp. 285–312, 1993.

[18] S. Takahashi and W. Takahashi, "Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mapping in a Hilbert space," *Nonlinear Analysis*, vol. 69, no. 3, pp. 1025–1033, 2008.

[19] Y. Yao, Y. J. Cho, and Y.-C. Liou, "Algorithms of common solutions for variational inclusions, mixed equilibrium problems and fixed point problems," *European Journal of Operational Research*, vol. 212, no. 2, pp. 242–250, 2011.

[20] Y. Yao, M. A. Noor, and Y.-C. Liou, "On iterative methods for equilibrium problems," *Nonlinear Analysis*, vol. 70, no. 1, pp. 497–509, 2009.

[21] F. Giannessi and A. Maugeri, *Variational Inequalities and Network Equilibrium Problems*, Plenum Press, New York, NY, USA, 1995.

[22] F. Giannessi, A. Maugeri, and P. M. Pardalos, *Equilibrium Problems: Nonsmooth Optimization and Variational Inequality Methods*, vol. 58, Kluwer Academic, Dordrecht, The Netherlands, 2001.

[23] M. A. Noor, "General variational inequalities," *Applied Mathematics Letters*, vol. 1, no. 2, pp. 119–122, 1988.

[24] M. A. Noor, "Extended general variational inequalities," *Applied Mathematics Letters*, vol. 22, no. 2, pp. 182–186, 2009.

[25] M. A. Noor, K. I. Noor, A. Waheed, and Eisa-Al-Said, "Modified variation of parameters method for solving system of second-order nonlinear boundary value problem," *International Journal of Physical Sciences*, vol. 5, no. 16, pp. 2426–2431, 2010.

[26] M. A. Noor, K. I. Noor, and E. Al-Said, "On New Proximal Point Methods for Solving the Variational Inequalities," *Journal of Applied Mathematics*, vol. 2012, Article ID 412413, 7 pages, 2012.

[27] M. A. Noor, Y. Yao, and Y.-C. Liou, "Extragradient method for equilibrium problems and variational inequalities," *Albanian Journal of Mathematics*, vol. 2, no. 2, pp. 125–138, 2008.