Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2012, Article ID 141684, 10 pages
doi:10.1155/2012/141684

Research Article

Some Aspects of d-Units in d/BCK-Algebras

Hee Sik Kim,[1] J. Neggers,[2] and Keum Sook So[3]

[1] *Department of Mathematics, Hanyang University, Seoul 133-791, Republic of Korea*
[2] *Department of Mathematics, University of Alabama, Tuscaloosa, AL 35487-0350, USA*
[3] *Department of Mathematics, Hallym University, Chuncheon 200-702, Republic of Korea*

Correspondence should be addressed to Keum Sook So, ksso@hallym.ac.kr

Received 13 August 2012; Revised 2 November 2012; Accepted 9 November 2012

Academic Editor: Frank Werner

We explore properties of the set of d-units of a d-algebra. A property of interest in the study of d-units in d-algebras is the weak associative property. It is noted that many other d-algebras, especially BCK-algebras, are in fact weakly associative. The existence of d/BCK-algebras which are not weakly associative is demonstrated. Moreover, the notions of a d-integral domain and a left-injectivity are discussed.

1. Introduction

Iséki and Tanaka introduced two classes of abstract algebras: BCK-algebras and BCI-algebras [1, 2]. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. Neggers and Kim introduced the notion of d-algebras which is another useful generalization of BCK-algebras and then investigated several relations between d-algebras and BCK-algebras as well as several other relations between d-algebras and oriented digraphs [3]. After that some further aspects were studied [4–7]. As a generalization of BCK-algebras, d-algebras are obtained by deleting two identities. Thus, one may introduce an additional operation $\odot$ and replace one of the deleted new algebras, for example, $(x * (x * y)) * y = 0$, to obtain new algebras $(X, *, \odot, 0)$ for which the conditions (i) $((x \odot y) * x) * y = 0$ and (ii) $(z * x) * y = 0$ implies $z * (x \odot y) = 0$, yielding a companion d-algebra which shares many properties of BCK-algebras and such that not every d-algebra is one. Allen et al. [4] developed a theory of companion d-algebras in sufficient detail to demonstrate considerable parallelism with the theory of BCK-algebras as well as obtaining a collection of results of a novel type. Recently, Allen et al. [8] introduced the notion of deformation in d/BCK-algebras. Using such deformations they constructed d-algebras from BCK-algebras in such a manner as to maintain control over properties of the deformed BCK-algebras via the nature of the deformation employed and observed that certain BCK-algebras cannot be deformed at all, leading to the notion of a rigid d-algebra and consequently of a rigid BCK-algebra as well.

In this paper we study properties of d-units in d-algebras $(X, *, 0)$, that is, elements x of X such that $x * X = X$. Since $0 * X = \{0\}$, $|X| \geq 2$ implies 0 is not a d-unit of $(X, *, 0)$. Hence, d-algebras $(X, *, 0)$ such that every non-zero element is a d-unit are special in the sense that they are "complete" with respect to this property. They are also not uncommon (see Proposition 3.2). It turns out that the property of weakly associativity in d-algebras is an important property in this context. In addition, we consider the class of d-integral domains and left-injective elements of d-algebras (defined below) in analogy with the usual notions in the theory of rings and their modules, where again the d-units investigated in this paper also play a significant role.

2. Preliminaries

In this section, we introduce some notions and propositions on d-algebras discussed in [3, 8–10] for reader's convenience.

An (ordinary) d-algebra [3] is an algebra $(X, *, 0)$ where $*$ is a binary operation and $0 \in X$ such that the following axioms are satisfied:

(I) $x * x = 0$,

(II) $0 * x = 0$,

(III) $x * y = 0$ and $y * x = 0$ imply $x = y$ for all $x, y \in X$.

For brevity we also call X a d-algebra. In X we can define a binary relation "$\leq$" by $x \leq y$ if and only if $x * y = 0$.

A BCK-algebra [1] is a d-algebra X satisfying the following additional axioms:

(IV) $(x * y) * (x * z)) * (z * y) = 0$,

(V) $(x * (x * y)) * y = 0$ for all $x, y, z \in X$.

Example 2.1 (see [3]). (a) Every BCK-algebra is an ordinary d-algebra. (b) Let $\mathbf{R}$ be the set of all real numbers and define $x * y := x \cdot (x - y)$, $x, y \in \mathbf{R}$, where " $\cdot$ " and " $-$ " are the ordinary product and subtraction of real numbers. Then $x * x = 0, 0 * x = 0, x * 0 = x^2$. If $x * y = y * x = 0$, then $x \cdot (x - y) = 0$ and $y \cdot (y - x) = 0$, that is, $(x = 0$ or $x = y)$ and $(y = 0$ or $y = x)$, that is, $(x = 0$ and $y = 0)$ or $(x = 0$ and $y = x)$ or $(x = y$ and $y = 0)$ or $(x = y$ and $y = x)$; all imply $x = y$. Hence, $(\mathbf{R}; *, 0)$ is an ordinary d-algebra. But it is not a BCK-algebra, since axiom (V) fails: $(2 * (2 * 0) * 0) = 16 \neq 0$.

An algebra $(X; *, 0)$, where $*$ is a binary operation and $0 \in X$, is said to be a *strong* d-algebra [9] if it satisfies (I), (II) and (III*) for all $x, y \in X$, where

(III*) $x * y = y * x$ implies $x = y$.

Obviously, every strong d-algebra is a d-algebra, but the converse needs not be true. The d-algebra in Example 2.1 (b) is not a strong d-algebra, since $x * y = y * x$ implies either $x = y$ or $x = -y$.

Example 2.2 (see [9]). Let $\mathbf{R}$ be the set of all real numbers and define $x * y := (x - y) \cdot (x - e) + e$, $x, y, e \in \mathbf{R}$, where "$\cdot$" and "$-$" are the ordinary product and subtraction of real numbers. Then $x * x = e; e * x = e; x * y = y * x = e$ yields $(x - y) \cdot (x - e) = 0$, $(y - x) \cdot (y - e) = 0$ and $x = y$ or $x = e = y$, that is, $x = y$, that is, $(\mathbf{R}; *, e)$ is a d-algebra.

However, $(\mathbf{R}; *, e)$ is not a strong d-algebra. If $x*y = y*x \Leftrightarrow (x-y)\cdot(x-e)+e = (y-x)\cdot(y-e)+e \Leftrightarrow (x-y)\cdot(x-e) = -(x-y)\cdot(y-e) \Leftrightarrow (x-y)\cdot(x-e+y-e) = 0 \Leftrightarrow (x-y)\cdot(x+y-2e) = 0 \Leftrightarrow (x = y$ or $x + y = 2e)$, then there exist $x = e + \alpha$ and $y = e - \alpha$ such that $x + y = 2e$, that is, $x * y = y * x$ and $x \neq y$. Hence, axiom (III*) fails and thus the d-algebra $(\mathbf{R}; *, e)$ is not a strong d-algebra.

Theorem 2.3 (see [11]). *The following properties hold in a BCK-algebra: for all $x, y, z \in X$,*

(B1) $x * 0 = x$,

(B2) $(x * y) * z = (x * z) * y$.

A BCK-algebra $(X, *, 0)$ is said to be bounded if there exists an element $x_0 \in X$ such that $x \leq x_0$ for all $x \in X$. We denote it by $(X, *, 0, x_0)$. Note that the usual notation is 1 rather than x_0 in literatures. We call such an element the greatest element of X. In a bounded BCK-algebra, we denote $x_0 * x$ by Nx. A bounded BCK-algebra $(X, *, 0, x_0)$ is called a BCK$_{(DN)}$-algebra [12] if it verifies condition (DN):

(DN) $NNx = x$ for all $x \in X$.

A BCK-algebra $(X, *, 0)$ is said to be commutative if $x * (x * y) = y * (y * x)$ for any $x, y \in X$. We refer useful textbooks for BCK/BCI-algebra to [11–14].

Theorem 2.4. *If $(X, *, 0)$ is a bounded commutative BCK-algebra, then $NNx = x$ for any $x \in X$.*

It is well known that bounded commutative BCK-algebras, D-posets and MV-algebras are logically equivalent each other (see [13, Page 420]).

Definition 2.5 (see [10]). Let $(X, *, 0)$ be a d-algebra and $\emptyset \neq I \subseteq X$. I is called a d-subalgebra of X if $x * y \in I$ whenever $x \in I$ and $y \in I$. I is called a BCK-*ideal* of X if it satisfies:

(D_0) $0 \in I$,

(D_1) $x * y \in I$ and $y \in I$ imply $x \in I$.

I is called a d-*ideal* of X if it satisfies (D_1) and

(D_2) $x \in I$ and $y \in X$ imply $x * y \in I$, that is, $I * X \subseteq I$.

Note that, by axiom (I) and definition of d-subalgebra, $0 \in X$ can be deduced easily.

Example 2.6 (see [10]). Let $X := \{0, 1, 2, 3, 4\}$ be a d-algebra which is not a BCK-algebra with the following table:

$$\begin{array}{c|ccccc} * & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 2 & 2 & 2 & 0 & 3 & 0 \\ 3 & 3 & 3 & 2 & 0 & 3 \\ 4 & 4 & 4 & 1 & 1 & 0 \end{array} \tag{2.1}$$

Then $I := \{0, a\}$ is a d-ideal of X.

In a d-algebra, a BCK-ideal need not be a d-subalgebra, and also a d-subalgebra need not be a BCK-ideal. Clearly, $\{0\}$ is a d-subalgebra of any d-algebra X and every d-ideal of X is a d-subalgebra [10].

Let X be a d-algebra and $x \in X$. Define $x * X := \{x * a \mid a \in X\}$. X is said to be edge if for any $x \in X$, $x * X = \{x, 0\}$. It is known that if X is an edge d-algebra, then $x * 0 = x$ for any $x \in X$ [3].

3. d-Units and Weakly Associativity

Let $(X, *, 0)$ be a d-algebra. An element x of X is said to be a d-unit if $x * X = X$, where $x * X := \{x * y \mid y \in X\}$.

Example 3.1. Let $X := \{0, 1, 2, 3\}$ be a set with the following table:

$$\begin{array}{c|cccc} * & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 3 & 2 \\ 2 & 2 & 1 & 0 & 3 \\ 3 & 3 & 3 & 3 & 0 \end{array} \tag{3.1}$$

Then $(X, *, 0)$ is a d-algebra. It is easy to show that 1, 2 are d-units of X.

Proposition 3.2. *Let $(X, +, \cdot, 0, 1)$ be a field. Define a binary operation "$*$" on X by*

$$x * y := x \cdot (x - y) \tag{3.2}$$

for any $x, y \in X$. Then $(X, *, 0)$ is a d-algebra such that every non-zero element of X is a d-unit.

Proof. It is easy to show that $(X, *, 0)$ is a d-algebra. Given a non-zero element x and any element u in X, the equation $x * y = x^2 - xy = u$ has a solution $y = (x^2 - u)/x$. $\square$

Note that the d-algebra $(X, *, 0)$ in Proposition 3.2 is not a BCK-algebra, since $x * 0 = x^2 \neq x$.

A d-algebra $(X, *, 0)$ is said to be weakly associative if for any $x, y, z \in X$, there exists a $w \in X$ such that $(x * y) * z = x * w$.

It is known that if $(X, *, 0)$ is a BCK-algebra with condition (S), then $(x * y) * z = x * (y \circ z)$ for all $x, y, z \in X$, where $y \circ z$ is the greatest element of the set $A(y, z) := \{x \in X \mid x * y \leq z\}$ [11]. Hence every BCK-algebra with condition (S) is weakly associative.

Proposition 3.3. *The d-algebra $(X, *, 0)$ defined in Proposition 3.2 is weakly associative.*

Proof. Given $x, y, z \in X$, we let $w := x + (x - y)z - x(x - y)^2$. Then we have

$$\begin{aligned} x * w &= x(x - w) = x(x - y)\big(x(x - y) - z\big) \\ &= (x * y)(x * y - z) = (x * y) * z, \end{aligned} \tag{3.3}$$

proving the proposition. $\square$

Example 3.4. Let $X := \{0, 1, 2, 3\}$ be a set with the following table:

$$\begin{array}{c|cccc} * & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 3 & 2 \\ 2 & 3 & 3 & 0 & 1 \\ 3 & 2 & 1 & 2 & 0 \end{array} \tag{3.4}$$

Then $(X, *, 0)$ is a d-algebra which is not a BCK-algebra. We know that $(3*1)*2 = 3$, but there is no element $w \in X$ such that $3 * w = 3$. Hence $(X, *, 0)$ is not weakly associative.

Proposition 3.5. *Let $(X, *, 0)$ be a weakly associative d-algebra and $x, y \in X$. If $x * y$ is a d-unit, then x is also a d-unit.*

Proof. Let $u \in X$ be an arbitrary element of X. Then $(x * y) * z = u$ for some $z \in X$. Since X is weakly associative, there exists $w \in X$ such that $(x * y) * z = x * w$, proving $u = x * w$, which shows that x is a d-unit. □

Proposition 3.6. *Let $(X, \leq)$ be a poset with minimal element 0. Define a binary operation "$*$" on X by*

$$x * y = \begin{cases} 0 & \text{if } x \leq y, \\ x & \text{otherwise.} \end{cases} \tag{3.5}$$

Then $(X, *, 0)$ is a weakly associative BCK-algebra.

Proof. Given $x, y, z \in X$, we have either $(x * y) * z = 0$ or $(x * y) * z = x$. Now, assume $(x * y) * z = 0$. Since X is a BCK-algebra, $x * y \leq x$ and hence $(x * y) * x = 0$. If we take $w := x$, then we obtain $(x * y) * z = 0 = x * x$. Assume $(x * y) * z = x$. If we take $w := 0$, then $x * w = x * 0 = x = (X * y) * z$, proving X is weakly associative. □

Note that the BCK-algebra $(X, *, 0)$ discussed in Proposition 3.6 is a dual Hilbert algebra (see [12, Page 30]). By routine calculations, we found that there is no weakly associative BCK-algebras with order ≤ 6.

By Propositions 3.3 and 3.6, the class of weakly associative BCK-algebras is a proper subclass of weakly associative d-algebras.

Let $(X, *, 0)$ be a d-algebra and let $N := \{0, 1, 2, 3, \ldots\}$. We define the set of all monomials az^k, $a \neq 0, k \in N$ and $\overline{0} := \{0z^k\}_{k \in N}$ by MX where z is an indeterminate. We may regard $\overline{0} \equiv 0z^k$ for all $k \in N$.

Define a binary operation "$\star$" on MX by

$$az^k \star bz^l := (a * b) z^{\alpha k + \beta l}, \tag{3.6}$$

where α and β are fixed elements of N. Then we obtain the following.

Proposition 3.7. *If $(X, *, 0)$ is a d-algebra and if $\alpha \neq \beta$ in N, then $(MX, \star, \overline{0})$ is a d-algebra.*

Proof. For any $az^k, bz^l \in MX$, we have $az^k \star az^k = (a * a)z^{(\alpha+\beta)k} = 0z^{(\alpha+\beta)k} = \overline{0}$ and $\overline{0} \star bz^l = 0z^k \star bz^l = (0 * b)z^{\alpha k+\beta l} = 0z^{\alpha k+\beta l} = \overline{0}$.

Assume that $az^k \star bz^l = bz^l \star az^k = \overline{0}$. Then $a * b = b * a = 0$ and $\alpha k + \beta l = \alpha l + \beta k$. Since $(X, *, 0)$ is a d-algebra, we obtain $a = b$ and $(\alpha - \beta)k = (\alpha - \beta)l$. It follows that either $\alpha = \beta$ or $k = l$. Since $\alpha \neq \beta$, we have $a = b$ and $k = l$, that is, $az^k = bz^l$. This proves the proposition. □

Using Proposition 3.7, we construct a BCK-algebra which is not weakly associative.

Example 3.8. Define a binary operation "$*$" on $X := [0,1]$ by $x * y := \max\{0, x - y\}$ for all $x, y \in X$. Then it is easy to see that $(X, *, 0)$ is a BCK-algebra. If we take $\alpha := 3$, $\beta := 7$, then by Proposition 3.7, $(MX, \star, \overline{0})$ is a d-algebra, where $az^k \star bz^l := (a * b)z^{3k+7l}$ for all $az^k, bz^l \in MX$.

We claim that $(MX, \star, \overline{0})$ is not weakly associative. By routine calculation, we obtain $((1/2)z^3 \star (1/3)z^l) \star (1/12)z^m = (1/12)z^{27+7(3l+m)}$ and $(1/2)z^3 \star Az^n = (1/2 - A)z^{9+7n}$ for any $Az^n \in MX$. If $(MX, \star, \overline{0})$ is weakly associative, then $(1/12)z^{27+7(3l+m)} = (1/2 - A)z^{9+7n}$ for some $Az^n \in MX$. It follows that $A = 5/12$ and $18 = 7(n - m - 3l)$, and hence 7 divides 18, a contradiction.

We claim that $(MX, \star, \overline{0})$ is a BCK-algebra. For any $az^k, bz^l, cz^n \in MX$, we have $[az^k \star (az^k \star bz^l)] \star bz^l = [(a * (a * b)) * b]z^{3(3k+7(3k+7l))+7l} = \overline{0}z^{3(3k+7(3k+7l))+7l} = \overline{0}$ and $[(az^k \star bz^l) \star (az^k \star cz^n)] \star (cz^n \star bz^l) = [((a * b) * (a * c)) * (c * b)]z^q = 0z^q = \overline{0}$ for some $q \in N$. Hence $(MX, \star, \overline{0})$ is a BCK-algebra which is not weakly associative.

4. d-Units in BCK-Algebras

Proposition 4.1. *If $(X, *, 0)$ is a BCK-algebra and x_0 is a d-unit of X, then $(X, *, 0, x_0)$ is a bounded BCK-algebra.*

Proof. Let x_0 be a d-unit of X. Then $x_0 * X = X$, which means that for any $y \in X$, there exists $u \in X$ such that $y = x_0 * u$. Hence we have $y * x_0 = (x_0 * u) * x_0 = (x_0 * x_0) * u = 0 * u = 0$ for any $y \in X$. This proves that $(X, *, 0, x_0)$ is a bounded BCK-algebra. □

The converse of Proposition 4.1 need not be true in general.

Example 4.2. Consider the BCK-algebra $(X, *, 0)$ [11, Page 252]) with the following table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	2	0	0	0
3	3	2	1	0	0
4	4	4	2	2	0

(4.1)

Then $(X, *, 0, 4)$ is a bounded BCK-algebra not verifying condition (DN), but 4 is not a d-unit, since $4 * X = \{0, 2, 4\} \neq X$.

Proposition 4.3. *Let $(X, *, 0, x_0)$ be a $BCK_{(DN)}$-algebra and let $x_0 \in X$. Then x_0 is the greatest element of X if and only if x_0 is a d-unit of X.*

Proof. Let x_0 be the greatest element of X. Since X is a $\mathrm{BCK}_{(\mathrm{DN})}$-algebra, by Theorem 2.4, we have $x = x_0 * (x_0 * x) \in x_0 * X$ for any $x \in X$, that is, $X \subseteq x_0 * X$, proving that x_0 is a d-unit of X. The converse was proved in Proposition 4.1. □

Example 4.4. Consider the BCK-algebra $(X, *, 0)$ with the following table:

$$\begin{array}{c|ccccc} * & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 & 0 \\ 3 & 3 & 3 & 2 & 0 & 0 \\ 4 & 4 & 3 & 2 & 1 & 0 \end{array} \tag{4.2}$$

Then $(X, *, 0, 4)$ is a $\mathrm{BCK}_{(\mathrm{DN})}$-algebra [11, Page 253]), namely a bounded commutative BCK-algebra, and 4 is both the greatest element of X and a d-unit of X.

5. d-Integral Domains

Let $(X, *, 0)$ be a d-algebra. An element $x \in X$ is said to be a *d-zero divisor* of X if there exists an element $y(\neq x)$ in X such that $x * y = 0$.

Example 5.1. Let $X := \{0, 1, 2, 3\}$ be a set with the following table:

$$\begin{array}{c|cccc} * & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 3 & 0 \\ 2 & 2 & 2 & 0 & 1 \\ 3 & 2 & 3 & 0 & 0 \end{array} \tag{5.1}$$

Then $(X, *, 0)$ is a d-algebra and $0, 1, 3$ are d-zero divisors of X.

Note that if $|X| \geq 2$, then 0 is a d-zero divisor of X.

Let $(X, *, 0)$ be a BCK-algebra. If $x * y = 0$ with $y \neq x$, then $x < y$ in the induced order, that is, x is not a maximal element of X. This shows that every non-maximal element of a BCK-algebra X is a d-zero divisor of X.

A d-algebra $(X, *, 0)$ is said to be a d-integral domain if every non-zero element x is not a d-zero divisor, that is, $x \neq 0, x * y = 0, y \in X$ implies $x = y$.

Proposition 5.2. *The d-algebra $(X, *, 0)$ in Proposition 3.2 is a d-integral domain.*

Proof. If $x * y = x(x - y) = 0$ and $x \neq 0$, then $x - y = 0$, that is, $x = y$. Hence x is not a d-zero divisor and the conclusion follows. □

Example 5.3. Let $X := \{0, 1, 2, 3\}$ be a set with the following table:

$$\begin{array}{c|cccc} * & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 2 & 2 & 2 & 0 & 1 \\ 3 & 2 & 3 & 3 & 0 \end{array} \tag{5.2}$$

Then $(X, *, 0)$ is a d-integral domain which is not a BCK-algebra, since every non-zero element is not a d-zero divisor.

Given two posets X and Y, we construct the ordinal sum $X \oplus Y$ of X and Y if $x \le y$ for all $x \in X$ and $y \in Y$ [15].

Proposition 5.4. *Let $(X, \le, 0)$ be an ordinal sum $X = \{0\} \oplus A$, where A is an anti-chain. If one defines a binary operation "$*$" on X by*

$$x * y = \begin{cases} 0 & \text{if } x \le y, \\ x & \text{otherwise,} \end{cases} \tag{5.3}$$

then the BCK-algebra $(X, *, 0)$ is a d-integral domain.

Proof. If $x \ne 0$ in X, then $x \in A$. Since X is a BCK-algebra, we have $x * 0 = x$, and if $y \in A$, $y \ne x$, then $x * y = x$. Hence $x \ne 0$, $x * y = 0$ implies $y = x$, proving the proposition. □

In the situation of Proposition 5.4, it follows that $x \ne 0$ is a d-unit if $|A| = 1$. Indeed, $x * X = \{0, x\}$ for all $x \in A$, and $x * X = X$ implies $|X| = 2$. If $|X| \ge 3$, then $(X, *, 0)$ is a d-integral domain having no d-units.

6. Left-Injective

Let $(X, *, 0)$ be a d-algebra. A non-zero element $x \in X$ is said to be left-injective if $x * y = x * z$ for all $y, z \in X$ implies $y = z$. A d-algebra $(X, *, 0)$ is said to be left-injective if every non-zero element of X is left-injective.

Proposition 6.1. *Let $(X, *, 0)$ be a d-algebra and $|X| \ge 2$. If $x \in X$ is left-injective, then it is not a d-zero divisor of X.*

Proof. If we assume that x is a d-zero divisor of X, then $x * y = 0$ for some $y(\ne x)$ in X. Since X is a d-algebra, $x * x = 0 = x * y$. It follows from x is left-injective that $x = y$, a contradiction. □

Proposition 6.2. *Let $(X, *, 0)$ be a finite d-algebra and $x \in X$. If x is left-injective, then it is a d-unit.*

Proof. Given a left-injective element x, we define a map $\varphi_x : X \to X$ by $\varphi_x(a) := x * a$. Then it is injective mapping, since x is left-injective. Since X is finite, φ_x is onto, which proves that $x * X = \varphi_x(X) = X$. This proves that x is a d-unit. □

In the infinite case, Proposition 6.2 need not be true. We give an example of d-algebra such that every non-zero element $x \in X$ is a left-injective, but not a d-unit element.

Example 6.3. Let $X := \mathbf{R}$ be the set of real numbers and let $x * y := \tan^{-1}(x(x-y))$ for any $x, y \in X$. Then it is easy to show that $(X, *, 0)$ is a d-algebra which is not a BCK-algebra, since $x * 0 = \tan^{-1}(x(x-0)) = \tan^{-1}(x^2) \neq x$ in general. Let $x \neq 0$ in X. Assume $x * y = x * z$. Then $\tan^{-1}(x(x-y)) = \tan^{-1}(x(x-z))$. Since $\tan^{-1}$ is a bijective mapping, we obtain $x(x-y) = x(x-z)$, whence $x \neq 0$ implies $xy = xz$ and $y = z$, that is, if $x \neq 0$, it is a left-injective element. Since $x * y \in [-\pi/2, \pi/2]$ in that case, it follows that $x * y = \pi$ does not have a solution in such a case. Hence $x * X \neq X$, that is, x is not a d-unit.

By Proposition 6.2, the d-algebra described in Example 6.3 is a d-integral domain such that every element is not a d-unit. The following example shows that there is a d-algebra such that every non-zero element of X is a d-unit, but not left-injective.

Example 6.4. Let $X := [0, \infty)$. Define a binary operation "$*$" on X by $x * y := x^2(x-y)^2$ for any $x, y \in X$. Then it is easy to show that $(X, *, 0)$ is a d-algebra. We claim that every non-zero element x of X is a d-unit. Given $u \in X$, if we take y in X as follows:

$$y = \begin{cases} \dfrac{x^2 - \sqrt{u}}{x} & \text{if } y \leq x, \\ \dfrac{x^2 + \sqrt{u}}{x} & \text{otherwise,} \end{cases} \tag{6.1}$$

then $x * y = u$, which proves that x is a d-unit. We claim that X is not left-injective, since $5 * 6 = 25 = 5 * 4$, but $6 \neq 4$.

A non-empty subset I of a d-algebra $(X, *, 0)$ is said to be a left-ideal of X if it satisfies the condition (D_2). Every left-ideal I of a d-algebra $(X, *, 0)$ contains 0, since $0 = x * x \in I * X \subseteq I$ for some $x \in I$. Hence, a left-ideal of X satisfies (D_0). Every d-ideal of a d-algebra $(X, *, 0)$ is a left-ideal of X, but the converse may not be true in general.

Example 6.5. Let $X := \{0, a, b, c\}$ be a d-algebra which is not a BCK-algebra with the following table:

$$\begin{array}{c|cccc} * & 0 & a & b & c \\ \hline 0 & 0 & 0 & 0 & 0 \\ a & a & 0 & 1 & a \\ b & b & b & 0 & a \\ c & c & c & a & 0 \end{array} \tag{6.2}$$

Then $J := \{0, a, c\}$ is a left-ideal, but not a d-ideal of X, since $b * c = 0 \in J$ and $c \in J$, but $b \notin J$.

A d-algebra $(X, *, 0)$ is said to be simple if its only left-ideals are $\{0\}$ and X. A d-algebra $(X, *, 0)$ is said to be *d-proper* if for all $x \in X$, $x * X$ is a left-ideal of X.

Example 6.6. In Example 6.3, for any $x \neq 0$ in X, we have $x * X = (-\pi/2, \pi/2) \subsetneqq X$, that is, $x * X$ is a left-ideal of X. This shows that $(X, *, 0)$ is d-proper. It is not simple, since $L * X = (-\pi/2, \pi/2) \subseteq L$ for any $L \subseteq X$ with $(-\pi/2, \pi/2) \subseteq L$, that is, L is a left-ideal of X.

Proposition 6.7. *If $(X, *, 0)$ is a weakly associative d-algebra, then it is d-proper.*

Proof. For any $x \in X$, if $\alpha \in (x * X) * X$, then $\alpha = (x * y) * z$ for some $y, z \in X$. Since X is weakly associative, $(x * y) * z = x * w$ for some $w \in X$, that is, $\alpha = x * w \in x * X$, proving that $x * X$ is a left-ideal of X. □

Theorem 6.8. *Let $(X, *, 0)$ be a d-algebra. Then X is simple and d-proper if and only if every non-zero element of X is a d-unit.*

Proof. Since X is d-proper, $x * X$ is a left-ideal of X for any non-zero element x of X. Moreover, $x \neq 0$ implies $x * 0 \neq 0$ and hence $x * X \neq \{0\}$. By the simplicity of $(X, *, 0)$ it follows that $x * X = X$, and thus x is a d-unit.

Assume that every non-zero element of X is a d-unit. We claim that X is d-proper. For any $x \in X$, if $x = 0$, then $0 * X = \{0\}$ is a left-ideal of X. Assume $x \neq 0$. Since x is a d-unit, we have $x * X = X$ and hence $(x * X) * X = X * X \subseteq X = x * X$, proving that $x * X$ is a left-ideal of X. We claim that X is simple. Assume that L is a left-ideal of X such that $L \neq \{0\}$. If we let $x \neq 0$ in L, then $X = x * X \subseteq L * X \subseteq L$ since x is a d-unit. This proves that X is simple. □

By Theorem 6.8, the d-algebra X described in Example 6.4 is simple and d-proper, but not left-injective.

Acknowledgment

The authors would like to express their great thanks to the referee's careful reading and valuable suggestions.

References

[1] K. Iséki and S. Tanaka, "An introduction to the theory of BCK-algebras," *Mathematica Japonica*, vol. 23, no. 1, pp. 1–26, 1978/79.
[2] K. Iséki, "On BCI-algebras," *Mathematics Seminar Notes*, vol. 8, no. 1, pp. 125–130, 1980.
[3] J. Neggers and H. S. Kim, "On d-algebras," *Mathematica Slovaca*, vol. 49, no. 1, pp. 19–26, 1999.
[4] P. J. Allen, H. S. Kim, and J. Neggers, "Companion d-algebras," *Mathematica Slovaca*, vol. 57, no. 2, pp. 93–106, 2007.
[5] Y. B. Jun, J. Neggers, and H. S. Kim, "Fuzzy d-ideals of d-algebras," *Journal of Fuzzy Mathematics*, vol. 8, no. 1, pp. 123–130, 2000.
[6] Y. C. Lee and H. S. Kim, "On d-subalgebras of d-transitive d^*-algebras," *Mathematica Slovaca*, vol. 49, no. 1, pp. 27–33, 1999.
[7] J. Neggers, A. Dvurečenskij, and H. S. Kim, "On d-fuzzy functions in d-algebras," *Foundations of Physics*, vol. 30, no. 10, pp. 1807–1816, 2000.
[8] P. J. Allen, H. S. Kim, and J. Neggers, "Deformations of d/BCK-algebras," *Bulletin of the Korean Mathematical Society*, vol. 48, no. 2, pp. 315–324, 2011.
[9] J. S. Han, H. S. Kim, and J. Neggers, "Strong and ordinary d-algebras," *Journal of Multiple-Valued Logic and Soft Computing*, vol. 16, no. 3–5, pp. 331–339, 2010.
[10] J. Neggers, Y. B. Jun, and H. S. Kim, "On d-ideals in d-algebras," *Mathematica Slovaca*, vol. 49, no. 3, pp. 243–251, 1999.
[11] J. Meng and Y. B. Jun, *BCK-Algebras*, Kyung Moon Sa, Seoul, South Korea, 1994.
[12] A. Iorgulescu, *Algebras of Logic as BCK Algebras*, Editura ASE, Bucharest, 2008.
[13] A. Dvurečenskij and S. Pulmannová, *New Trends in Quantum Structures*, vol. 516 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2000.
[14] H. Yisheng, *BCI-Algebras*, Science Press, Beijing, China, 2006.
[15] J. Neggers and H. S. Kim, *Basic Posets*, World Scientific Publishing, River Edge, NJ, USA, 1998.