Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2012, Article ID 182040, 24 pages
doi:10.1155/2012/182040

Research Article

Controllability and Observability Criteria for Linear Piecewise Constant Impulsive Systems

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Received 14 May 2012; Accepted 22 July 2012

Academic Editor: Junjie Wei

Impulsive differential systems are an important class of mathematical models for many practical systems in physics, chemistry, biology, engineering, and information science that exhibit impulsive dynamical behaviors due to abrupt changes at certain instants during the dynamical processes. This paper studies the controllability and observability of linear piecewise constant impulsive systems. Necessary and sufficient criteria for reachability and controllability are established, respectively. It is proved that the reachability is equivalent to the controllability under some mild conditions. Then, necessary and sufficient criteria for observability and determinability of such systems are established, respectively. It is also proved that the observability is equivalent to the determinability under some mild conditions. Our criteria are of the geometric type, and they can be transformed into algebraic type conveniently. Finally, a numerical example is given to illustrate the utility of our criteria.

1. Introduction

In recent years, there has been increasing interest in the analysis and synthesis of impulsive systems, or impulsive control systems, due to their significance both in theory and in applications [1–15].

Different from another type of systems associated with the impulses, that is, the singular systems or the descriptor systems, impulsive control systems are described by impulsive ordinary differential equations. Many real systems in physics, chemistry, biology, engineering, and information science exhibit impulsive dynamical behaviors due to abrupt changes at certain instants during the continuous dynamical processes. This kind of impulsive behaviors can be modelled by impulsive systems.

Controllability and observability of impulsive control systems have been studied by a number of papers [4, 6, 12, 13, 15, 16]. Leela et al. [4] investigated the controllability of a class of time-invariant impulsive systems with the assumption that the impulses of impulsive control are regulated at discontinuous points. Lakshmikantham and Deo [12] improved Leela et al.'s [4] results. Then, George et al. [13] extended the results to the linear impulsive systems with time-varying coefficients and nonlinear perturbations. Benzaid and Sznaier [6] studied the null controllability of the linear impulsive systems with the control impulses only acting at the discontinuous points. Guan et al. [15] investigated the controllability and observability of linear time-varying impulsive systems. Sufficient and necessary conditions for controllability and observability are established and their applications to time-invariant impulsive control systems are also discussed. Xie and Wang [16] investigated controllability and observability of a simple class of impulsive systems. Necessary and sufficient conditions are obtained.

Controllability and observability are the two most fundamental concepts in modern control theory [17–19]. They have close connections to pole assignment, structural decomposition, quadratic optimal control and observer design, and so forth. In this paper, we aim to derive necessary and sufficient criteria for controllability and observability of linear piecewise constant impulsive control systems. We first investigate the reachability of such systems and a geometric type necessary and sufficient condition is established. Then, we investigate the controllability and an equivalent condition is established as well. Moreover, it is shown that the controllability is not equivalent to reachability for such systems in general case but is equivalent under some extra conditions. Next, we investigate the observability and determinability of such systems, and get similar results as the controllability and reachability case.

This paper is organized as follows. Section 2 formulates the problem and presents the preliminary results. Sections 3 and 4 investigate reachability and controllability, respectively. Observability and determinability are investigated in Section 5. Section 6 contains a numerical example. Finally, we provide the conclusion in Section 7.

2. Preliminaries

Consider the piecewise linear impulsive system given by

$$\begin{gathered} \dot{x}(t) = A_k x(t) + B_k u(t), \quad t \in [t_{k-1}, t_k), \\ x(t_k^+) = E_k x(t_k^-) + F_k u(t_k), \\ y(t) = C_k x(t) + D_k u(t), \quad t \in [t_{k-1}, t_k), \\ x(t_0^+) = x_0, \quad t_0 \geq 0, \end{gathered} \tag{2.1}$$

where $k = 1, 2, \ldots, A_k, B_k, C_k, D_k, E_k$, and F_k are the known $n \times n$, $n \times p$, $p \times n$, $q \times p$, $n \times n$, and $n \times p$ constant matrices; $x(t) \in \mathbb{R}^n$ is the state vector, and $u(t) \in \mathbb{R}^p$ the input vector, $y(t) \in \mathbb{R}^q$ the output vector; $x(t^+) := \lim_{h \to 0^+} x(t+h)$, $x(t^-) := \lim_{h \to 0^-} x(t-h)$, and the discontinuity points are

$$t_1 < t_2 < \cdots < t_k < \cdots, \quad \lim_{k \to \infty} t_k = \infty, \tag{2.2}$$

where $t_0 < t_1$ and $x(t_k^-) = x(t_k)$, which implies that the solution of (2.1) is left-continuous at t_k.

First, we consider the solution of the system (2.1).

Lemma 2.1. *For any $t \in (t_{k-1}, t_k]$, $k = 1, 2, \ldots$, the general solution of the system* (2.1) *is given by*

$$
\begin{aligned}
x(t) = \exp[A_k(t - t_{k-1})]\Bigg\{ &\prod_{i=k-1}^{1} E_i \exp(A_i h_i) x(t_0) \\
&+ \sum_{i=1}^{k-2} \prod_{j=k-1}^{i+1} E_j \exp(A_j h_j) \left(E_i \int_{t_{i-1}}^{t_i} \exp[A_i(t_i - s)] B_i u(s) ds + F_i u(t_i) \right) \\
&+ E_{k-1} \int_{t_{k-2}}^{t_{k-1}} \exp[A_{k-1}(t_{k-1} - s)] B_{k-1} u(s) ds + F_{k-1} u(t_{k-1}) \Bigg\} \\
&+ \int_{t_{k-1}}^{t} \exp[A_k(t - s)] B_k u(s) ds,
\end{aligned}
\tag{2.3}
$$

where $h_k = t_k - t_{k-1}$, $k = 1, 2, \ldots$.

Proof. For $t \in (t_0, t_1]$, we have

$$x(t) = \exp[A_1(t - t_0)] x(t_0) + \int_{t_0}^{t} \exp[A_1(t - s)] B_1 u(s) ds. \tag{2.4}$$

For $t = t_1^+$, we have

$$x(t_1^+) = E_1 \left(\exp(A_1 h_1) x(t_0) + \int_{t_0}^{t_1} \exp[A_1(t_1 - s)] B_1 u(s) ds \right) + F_1 u(t_1). \tag{2.5}$$

Similarly, for $t \in (t_{i-1}, t_i]$, $i = 2, 3, \ldots, k$, we have

$$x(t) = \exp[A_i(t - t_{i-1})] x(t_{i-1}^+) + \int_{t_{i-1}}^{t} \exp[A_i(t - s)] B_i u(s) ds. \tag{2.6}$$

And, for $t = t_i^+$, $i = 2, 3, \ldots, k$, we have

$$x(t_i^+) = E_i \left(\exp(A_i h_i) x(t_{i-1}) + \int_{t_{i-1}}^{t_i} \exp[A_i(t_i - s)] B_i u(s) ds \right) + F_i u(t_i). \tag{2.7}$$

Thus, by (2.4), (2.5), (2.6), and (2.7), it is easy to verify (2.3). □

If $t_f \in (t_0, t_1]$, then we are just concerned with a linear time-invariant system. Controllability and observability criteria can be found in standard text books [18, 19]. Thus, in the remainder of the paper, we will only be concerned with the case $t_f \in (t_{k-1}, t_k]$, $k = 2, 3, \ldots$.

Now, we give some mathematical preliminaries as the basic tools in the following discussion.

Given matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times p}$, denote $\mathcal{Im}(B)$ as the *range* of B, that is, $\mathcal{Im}(B) = \{y \mid y = Bx, \text{for all } x \in \mathbb{R}^{n\times n}\}$, and denote $\langle A \mid B\rangle$ as the *minimal invariant subspace* of A on $\mathcal{Im}(B)$, that is, $\langle A \mid B\rangle = \mathcal{Im}(B) + \mathcal{Im}(AB) + \cdots + \mathcal{Im}(A^{n-1}B)$. Given a linear subspace $\mathcal{W} \subseteq \mathbb{R}^n$, denote $\mathcal{W}_\perp$ as the *orthogonal complement* of $\mathcal{W}$, that is, $\mathcal{W}_\perp = \{x \mid x^T\mathcal{W} = 0\}$.

The following lemma is a generalization of Theorem 7.8.1 in [17], which is the starting point for deriving the criteria of reachability and controllability.

Lemma 2.2. *Given matrices* $A, E \in \mathbb{R}^{n\times n}$, $B, F \in \mathbb{R}^{n\times p}$, *for any* $0 \le t_0 < t_f < +\infty$, *one has*

$$\begin{aligned}&\left\{x \mid x = E\int_{t_0}^{t_f} \exp[A(t_f - s)]Bu(s)ds + Fu(t_f),\ \forall \text{ piecewise continuous } u\right\}\\&\quad = E\langle A \mid B\rangle + \mathcal{Im}(F).\end{aligned} \tag{2.8}$$

Proof. See Appendix A. □

Lemma 2.3. *Given two matrices* $A \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{q\times n}$, *two scalars* $t_0 < t_f$, *and a vector* $x \in \mathbb{R}^n$, *the following two statements are equivalent:*

(a) $C\exp[A(t - t_0)]x = 0$, $t \in [t_0, t_f]$,

(b) $x^T\langle A^T \mid C^T\rangle = 0$.

Proof. See Appendix B. □

Lemma 2.4. *Given a matrix* $A \in \mathbb{R}^{n\times n}$ *and a linear subspace* $\mathcal{W} \subseteq \mathbb{R}^n$, *the following two statements are equivalent:*

(a) $\mathcal{Im}(A) \subseteq \mathcal{W}$,

(b) $A^TW_\perp = 0$.

Proof. See Appendix C. □

3. Reachability

In this section, we first investigate the reachability of system (2.1).

Definition 3.1 (reachability). The system (2.1) is said to be (completely) reachable on $[t_0, t_f]$ ($t_0 < t_f$) if, for any terminal state $x_f \in \mathbb{R}^n$, there exists a piecewise continuous input $u(t) : [t_0, t_f] \to \mathbb{R}^p$ such that the system (2.1) is driven from $x(t_0) = 0$ to $x(t_f) = x_f$. Moreover, the set of all the reachable states on $[t_0, t_f]$ is said to be the *reachable set* on $[t_0, t_f]$, denoted as $\mathcal{R}[t_0, t_f]$.

Theorem 3.2. *For the system* (2.1), *the reachable set on* $[t_0, t_f]$, *where* $t_f \in (t_{k-1}, t_k]$, *is given by*

$$\begin{aligned}\mathcal{R}[t_0, t_f] = {} & \exp[A_k(t_f - t_{k-1})] \sum_{i=1}^{k-2} \prod_{j=k-1}^{i+1} E_j \exp(A_j h_j)(E_i\langle A_i \mid B_i\rangle + \mathcal{I}m(F_i)) \\ & + \exp[A_k(t_f - t_{k-1})](E_{k-1}\langle A_{k-1} \mid B_{k-1}\rangle + \mathcal{I}m(F_{k-1})) + \langle A_k \mid B_k\rangle.\end{aligned} \tag{3.1}$$

Proof. By Lemma 2.1, letting $x(t_0) = 0$, we have

$$\begin{aligned}x(t) = \exp[A_k(t - t_{k-1})] \Bigg\{ & \sum_{i=1}^{k-2} \prod_{j=k-1}^{i+1} E_j \exp(A_j h_j) \\ & \times \left(E_i \int_{t_{i-1}}^{t_i} \exp[A_i(t_i - s)] B_i u(s) ds + F_i u(t_i) \right) \\ & + E_{k-1} \int_{t_{k-2}}^{t_{k-1}} \exp[A_{k-1}(t_{k-1} - s)] B_{k-1} u(s) ds + F_{k-1} u(t_{k-1}) \Bigg\} \\ & + \int_{t_{k-1}}^{t} \exp[A_k(t - s)] B_k u(s) ds.\end{aligned} \tag{3.2}$$

It follows that

$$\begin{aligned}\mathcal{R}[t_0, t_f] = \Bigg\{ x \mid x = {} & \exp[A_k(t_f - t_{k-1})] \\ & \times \left\{ \sum_{i=1}^{k-2} \prod_{j=k-1}^{i+1} E_j \exp(A_j h_j) \left(E_i \int_{t_{i-1}}^{t_i} \exp[A_i(t_i - s)] B_i u(s) ds + F_i u(t_i) \right) \right. \\ & \left. + E_{k-1} \int_{t_{k-2}}^{t_{k-1}} \exp[A_{k-1}(t_{k-1} - s)] B_{k-1} u(s) ds + F_{k-1} u(t_{k-1}) \right\} \\ & + \int_{t_{k-1}}^{t_f} \exp[A_k(t_f - s)] B_k u(s) ds, \forall \text{ piecewise continuous } u \Bigg\} \\ = {} & \exp[A_k(t_f - t_{k-1})] \Bigg(\sum_{i=1}^{k-2} \prod_{j=k-1}^{i+1} E_j \exp(A_j h_j) \\ & \times \Bigg\{ x \mid x = E_i \int_{t_{i-1}}^{t_i} \exp[A_i(t_i - s)] B_i u(s) ds + F_i u(t_i), \\ & \forall \text{ piecewise continuous } u \Bigg\}\end{aligned}$$

$$
\begin{aligned}
&+\left\{x \mid x = E_{k-1}\int_{t_{k-2}}^{t_{k-1}} \exp[A_{k-1}(t_{k-1}-s)]B_{k-1}u(s)ds + F_{k-1}u(t_{k-1})\right.\\
&\qquad \left.\left.\forall \text{ piecewise continuous } u\right\}\right)\\
&+\left\{x \mid x = \int_{t_{k-1}}^{t_f} \exp\left[A_k\left(t_f - s\right)\right]B_k u(s)ds, \forall \text{ piecewise continuous } u\right\}.
\end{aligned}
\tag{3.3}
$$

By Lemma 2.2, we get

$$
\begin{aligned}
\mathcal{R}\left[t_0, t_f\right] = {} & \exp\left[A_k\left(t_f - t_{k-1}\right)\right]\left(\sum_{i=1}^{k-2}\prod_{j=k-1}^{i+1} E_j \exp\left(A_j h_j\right)\left(E_i\langle A_i \mid B_i\rangle + \mathcal{I}m(F_i)\right)\right.\\
&\left. + E_{k-1}\langle A_{k-1} \mid B_{k-1}\rangle + \mathcal{I}m(F_{k-1})\right)\\
&+ \langle A_k \mid B_k\rangle.
\end{aligned}
\tag{3.4}
$$

This is just (3.1). □

Since we have obtained the geometric form of the reachable set, we can establish a geometric type criterion as follows.

Theorem 3.3. *The system* (2.1) *is reachable on* $[t_0, t_f]$, *where* $t_f \in (t_{k-1}, t_k]$, *if and only if*

$$
\sum_{i=1}^{k-2}\prod_{j=k-1}^{i+1} E_j \exp\left(A_j h_j\right)\left(E_i\langle A_i \mid B_i\rangle + \mathcal{I}m(F_i)\right) + E_{k-1}\langle A_{k-1} \mid B_{k-1}\rangle + \mathcal{I}m(F_{k-1}) + \langle A_k \mid B_k\rangle = \mathbb{R}^n.
\tag{3.5}
$$

Proof. Since

$$
\begin{aligned}
\mathcal{R}\left[t_0, t_f\right] &= \exp\left[A_k\left(t_f - t_{k-1}\right)\right]\\
&\quad \times \left(\sum_{i=1}^{k-2}\prod_{j=k-1}^{i+1} E_j \exp\left(A_j h_j\right)\left(E_i\langle A_i \mid B_i\rangle + \mathcal{I}m(F_i)\right) + E_{k-1}\langle A_{k-1} \mid B_{k-1}\rangle + \mathcal{I}m(F_{k-1})\right)\\
&\quad + \langle A_k \mid B_k\rangle\\
&= \exp\left[A_k\left(t_f - t_{k-1}\right)\right]\left(\sum_{i=1}^{k-2}\prod_{j=k-1}^{i+1} E_j \exp\left(A_j h_j\right)\left(E_i\langle A_i \mid B_i\rangle + \mathcal{I}m(F_i)\right)\right.\\
&\qquad \left. + E_{k-1}\langle A_{k-1} \mid B_{k-1}\rangle + \mathcal{I}m(F_{k-1}) + \langle A_k \mid B_k\rangle\right)
\end{aligned}
\tag{3.6}
$$

and the matrix $\exp[A_k(t_f - t_{k-1})]$ is nonsingular, the proof directly follows from Theorem 3.2. □

Remark 3.4. Theorem 3.3 is a geometric type condition. By simple transformation, we can get an algebraic type condition. In fact, for $i = 1, 2, \ldots,$ denote

$$Q_i = \left[B_i, A_i B_i, \ldots, A_i^{n-1} B_i\right], \tag{3.7}$$

for $i = 1, 2, \ldots, k-2$, denote

$$\begin{gathered} H_i = \left[\prod_{j=k-1}^{i+1} E_j \exp(A_j h_j) E_i Q_i, \ \prod_{j=k-1}^{i+1} E_j \exp(A_j h_j) F_i\right], \\ H_{k-1} = [E_{k-1} Q_{k-1}, F_{k-1}], \end{gathered} \tag{3.8}$$

and, finally, denote

$$Q_{[t_0,t_f]} = [H_1, H_2, \ldots, H_{k-1}, Q_k]. \tag{3.9}$$

Then, it is easy to verify that

$$\exp[A_k(t_f - t_{k-1})]\mathcal{I}m\left(Q_{[t_0,t_f]}\right) = \mathcal{R}[t_0, t_f]. \tag{3.10}$$

Thus, we get the following algebraic type criterion.

Corollary 3.5. *The system* (2.1) *is reachable on* $[t_0, t_f]$, *where* $t_f \in (t_{k-1}, t_k]$, *if and only if*

$$\operatorname{rank}\left(Q_{[t_0,t_f]}\right) = n. \tag{3.11}$$

4. Controllability

In this section, we investigate the controllability of system (2.1).

Definition 4.1 (controllability). The system (2.1) is said to be (completely) controllable on $[t_0, t_f]$ $(t_0 < t_f)$ if, for any initial state $x_0 \in \mathbb{R}^n$, there exists a piecewise continuous input $u(t) : [t_0, t_f] \to \mathbb{R}^p$ such that the system (2.1) is driven from $x(t_0) = x_0$ to $x(t_f) = 0$. Moreover, the set of all the controllable states on $[t_0, t_f]$ is said to be the *controllable set* on $[t_0, t_f]$, denoted as $\mathcal{C}[t_0, t_f]$.

First, we show the relationship between the controllable set and the reachable set.

Theorem 4.2. *For the system* (2.1)*, if E_i is nonsingular, for $i = 1, \dots, k-1$, then the controllable set on $[t_0, t_f]$, where $t_f \in (t_{k-1}, t_k]$, satisfies*

$$\left(\exp[A_k(t-t_{k-1})]\prod_{i=k-1}^{1}E_i\exp(A_ih_i)\right)\mathcal{C}[t_0,t_f] \subseteq \mathcal{R}[t_0,t_f]. \tag{4.1}$$

Proof. By Lemma 2.1, letting $x(t_f) = 0$, we have

$$\begin{aligned}
0 = {} & \exp[A_k(t_f-t_{k-1})]\prod_{i=k-1}^{1}E_i\exp(A_ih_i)x(t_0)\exp[A_k(t_f-t_{k-1})] \\
& \times\Bigg\{\sum_{i=1}^{k-2}\prod_{j=k-1}^{i+1}E_j\exp(A_jh_j) \\
& \qquad\times\left(E_i\int_{t_{i-1}}^{t_i}\exp[A_i(t_i-s)]B_iu(s)ds + F_iu(t_i)\right) \\
& \qquad + E_{k-1}\int_{t_{k-2}}^{t_{k-1}}\exp[A_{k-1}(t_{k-1}-s)]B_{k-1}u(s)ds + F_{k-1}u(t_{k-1})\Bigg\} \\
& + \int_{t_{k-1}}^{t_f}\exp[A_k(t_f-s)]B_ku(s)ds.
\end{aligned} \tag{4.2}$$

It is equivalent to

$$\begin{aligned}
& -\exp[A_k(t_f-t_{k-1})]\prod_{i=k-1}^{1}E_i\exp(A_ih_i)x(t_0) \\
& \quad = \exp[A_k(t_f-t_{k-1})]\Bigg\{\sum_{i=1}^{k-2}\prod_{j=k-1}^{i+1}E_j\exp(A_jh_j) \\
& \qquad\qquad \times\left(E_i\int_{t_{i-1}}^{t_i}\exp[A_i(t_i-s)]B_iu(s)ds + F_iu(t_i)\right) \\
& \qquad\qquad + E_{k-1}\int_{t_{k-2}}^{t_{k-1}}\exp[A_{k-1}(t_{k-1}-s)]B_{k-1}u(s)ds + F_{k-1}u(t_{k-1})\Bigg\} \\
& \qquad + \int_{t_{k-1}}^{t_f}\exp[A_k(t_f-s)]B_ku(s)ds.
\end{aligned} \tag{4.3}$$

This implies that

$$\left(\exp[A_k(t_f - t_{k-1})]\prod_{i=k-1}^{1} E_i \exp(A_i h_i)\right)x(t_0) \in \mathcal{R}[t_0, t_f]. \tag{4.4}$$

Hence,

$$\left(\exp[A_k(t_f - t_{k-1})]\prod_{i=k-1}^{1} E_i \exp(A_i h_i)\right)\mathcal{C}[t_0, t_f] \subseteq \mathcal{R}[t_0, t_f]. \tag{4.5}$$

□

Based on Theorem 4.2, we can establish a criterion for controllability of the system (2.1) as follows.

Theorem 4.3. *The system* (2.1) *is controllable on* $[t_0, t_f]$*, where* $t_f \in (t_{k-1}, t_k]$*, if and only if*

$$\begin{aligned} &\mathcal{Im}\left(\prod_{i=k-1}^{1} E_i \exp(A_i h_i)\right) \\ &\quad\subseteq \sum_{i=1}^{k-2}\prod_{j=k-1}^{i+1} E_j \exp(A_j h_j)(E_i\langle A_i \mid B_i\rangle + \mathcal{Im}(F_i)) + E_{k-1}\langle A_{k-1} \mid B_{k-1}\rangle + \mathcal{Im}(F_{k-1}) + \langle A_k \mid B_k\rangle. \end{aligned} \tag{4.6}$$

Proof. First, it is easy to prove that (4.6) is equivalent to

$$\mathcal{Im}\left(\exp[A_k(t_f - t_{k-1})]\prod_{i=k-1}^{1} E_i \exp(A_i h_i)\right) \subseteq \mathcal{R}[t_0, t_f]. \tag{4.7}$$

Necessity: since the system is controllable, we have

$$\mathcal{C}[t_0, t_f] = \mathbb{R}^n. \tag{4.8}$$

Then, by Theorem 4.2, we get

$$\begin{aligned} \mathcal{R}[t_0, t_f] &\supseteq \left(\exp[A_k(t_f - t_{k-1})]\prod_{i=k-1}^{1} E_i \exp(A_i h_i)\right)\mathbb{R}^n \\ &= \mathcal{Im}\left(\exp[A_k(t_f - t_{k-1})]\prod_{i=k-1}^{1} E_i \exp(A_i h_i)\right). \end{aligned} \tag{4.9}$$

Sufficiency: suppose that (4.7) holds. For any $x \in \mathbb{R}^n$, we have

$$\left(\exp[A_k(t_f - t_{k-1})]\prod_{i=k-1}^{1} E_i \exp(A_i h_i)\right)x \in \mathcal{R}[t_0, t_f]. \tag{4.10}$$

This implies that there exists a piecewise continuous function $u(t)$, $t \in [t_0, t_f]$, such that

$$\begin{aligned}
0 = &\exp\left[A_k\left(t_f - t_{k-1}\right)\right] \prod_{i=k-1}^{1} E_i \exp(A_i h_i) x \\
&\times \exp\left[A_k\left(t_f - t_{k-1}\right)\right] \left\{ \sum_{i=1}^{k-2} \prod_{j=k-1}^{i+1} E_j \exp\left(A_j h_j\right) \right. \\
&\qquad \times \left(E_i \int_{t_{i-1}}^{t_i} \exp[A_i(t_i - s)] B_i u(s) ds + F_i u(t_i) \right) \\
&\qquad \left. + E_{k-1} \int_{t_{k-2}}^{t_{k-1}} \exp[A_{k-1}(t_{k-1} - s)] B_{k-1} u(s) ds + F_{k-1} u(t_{k-1}) \right\} \\
&+ \int_{t_{k-1}}^{t_f} \exp\left[A_k\left(t_f - s\right)\right] B_k u(s) ds.
\end{aligned} \tag{4.11}$$

Then, we know that $x \in \mathcal{C}[t_0, t_f]$. Hence, the system (2.1) is controllable. □

In the general case, for system (2.1), controllability is not equivalent to reachability. But under some mild conditions, we can show that they are equivalent.

Corollary 4.4. *For the system* (2.1)*, if E_i is nonsingular, $i = 1, 2, \ldots, k-1$, then the following statements are equivalent:*

(a) *the system is reachable,*

(b) *the system is controllable,*

(c) $\sum_{i=1}^{k-2} \prod_{j=k-1}^{i+1} E_j \exp(A_j h_j)(E_i \langle A_i \mid B_i \rangle + \mathcal{I}m(F_i)) + E_{k-1} \langle A_{k-1} \mid B_{k-1} \rangle + \mathcal{I}m(F_{k-1}) + \langle A_k \mid B_k \rangle = \mathbb{R}^n$.

Proof. Since E_i is nonsingular, $i = 1, 2, \ldots, k-1$, we have that

$$\exp[A_k(t - t_{k-1})] \prod_{i=k-1}^{1} E_i \exp(A_i h_i) \tag{4.12}$$

is nonsingular. It follows that

$$\left(\exp[A_k(t - t_{k-1})] \prod_{i=k-1}^{1} E_i \exp(A_i h_i) \right) \mathcal{C}\left[t_0, t_f\right] = \mathcal{R}\left[t_0, t_f\right]. \tag{4.13}$$

It is easy to see that $\mathcal{C}[t_0, t_f] = \mathbb{R}^n \Leftrightarrow \mathcal{R}[t_0, t_f] = \mathbb{R}^n$. □

Remark 4.5. For system (2.1), assume that $A_i = A$, $B_i = B$, $i = 1, \ldots, k$. Then, it is easy to see that Theorem 4.2 concludes the results of Theorem 3.4 in [15].

Remark 4.6. For system (2.1), assume that $E_i = I$, $F_i = 0$, $i = 1, \ldots, k$. Then, it is easy to see that Theorem 5 in [20] is a special case of Corollary 4.4.

5. Observability and Determinability

In the above analysis, reference is made to reachability and controllability only. It should be noticed that the observability and determinability counterparts can be addressed dualistically. In this section, we outline the relevant concepts and the corresponding criteria.

Definition 5.1 (observability). The system (2.1) is said to be (completely) observable on $[t_0, t_f]$ $(t_0 < t_f)$ if any initial state $x_0 \in \mathbb{R}^n$ can be uniquely determined by the corresponding system input $u(t)$ and the system output $y(t)$, for $t \in [t_0, t_f]$.

Definition 5.2 (determinability). The system (2.1) is said to be (completely) determinable on $[t_0, t_f]$ $(t_0 < t_f)$ if any terminal state $x_f \in \mathbb{R}^n$ can be uniquely determined by the corresponding system input $u(t)$ and the system output $y(t)$, for $t \in [t_0, t_f]$.

In order to investigate observability and determinability for the system (2.1), we first investigate those of the following zero input system:

$$\begin{gathered}
\dot{x}(t) = A_k x(t), \quad t \in [t_{k-1}, t_k), \\
x\left(t_k^+\right) = E_k x\left(t_k^-\right), \\
y(t) = C_k x(t), \quad t \in [t_{k-1}, t_k), \\
x\left(t_0^+\right) = x_0, \quad t_0 \geq 0.
\end{gathered} \tag{5.1}$$

It is obvious that observability and determinability of the system (2.1) are equivalent to those of the system (5.1), respectively.

For the system (5.1), by Lemma 2.1, the output is given by

$$y(t) = \begin{cases} C_1 \exp[A_1(t - t_0)] x(t_0), & t \in (t_0, t_1], \\ C_i \exp[A_i(t - t_{i-1})] \prod_{j=i-1}^{1} E_j \exp\left(A_j h_j\right) x(t_0), & t \in (t_{i-1}, t_i], \ i = 2, \ldots, k. \end{cases} \tag{5.2}$$

Theorem 5.3. *The system* (5.1) *is observable on* $[t_0, t_f]$, *where* $t_f \in (t_{k-1}, t_k]$, *if and only if*

$$\sum_{i=k}^{2} \prod_{j=1}^{i-1} \exp\left(A_j^T h_j\right) E_j^T \left\langle A_i^T \mid C_i^T \right\rangle + \left\langle A_1^T \mid C_1^T \right\rangle = \mathbb{R}^n. \tag{5.3}$$

Proof. We prove the complementary proposition of Theorem 5.3, that is, the system (5.1) is not observable on $[t_0, t_f]$, where $t_f \in (t_{k-1}, t_k]$, if and only if

$$\sum_{i=k}^{2} \prod_{j=1}^{i-1} \exp\left(A_j^T h_j\right) E_j^T \left\langle A_i^T \mid C_i^T \right\rangle + \left\langle A_1^T \mid C_1^T \right\rangle \subsetneq \mathbb{R}^n. \tag{5.4}$$

Necessity: if the system (5.1) is not observable on $[t_0, t_f]$, where $t_f \in (t_{k-1}, t_k]$, then there exists $x_0 \in \mathbb{R}^n$, nonzero, such that $y(t) \equiv 0$, $t \in [t_0, t_f]$. This means that

$$\begin{gathered} C_1 \exp[A_1(t-t_0)]x_0 = 0, \quad t \in (t_0, t_1], \\ C_i \exp[A_i(t-t_{i-1})] \prod_{j=i-1}^{1} E_j \exp(A_j h_j) x_0 = 0, \quad t \in (t_{i-1}, t_i],\ i = 2, \dots, k-1, \\ C_k \exp[A_k(t-t_{k-1})] \prod_{j=k-1}^{1} E_j \exp(A_j h_j) x_0 = 0, \quad t \in (t_{k-1}, t_f]. \end{gathered} \tag{5.5}$$

By Lemma 2.3, we get

$$\begin{gathered} x_0^T \left\langle A_1^T \mid C_1^T \right\rangle = 0, \\ x_0^T \prod_{j=1}^{i-1} \exp\left(A_j^T h_j\right) E_j^T \left\langle A_i^T \mid C_i^T \right\rangle = 0, \quad i = 2, \dots, k. \end{gathered} \tag{5.6}$$

It follows that

$$x_0^T \left(\left\langle A_1^T \mid C_1^T \right\rangle + \sum_{i=2}^{k} \prod_{j=1}^{i-1} \exp\left(A_j^T h_j\right) E_j^T \left\langle A_i^T \mid C_i^T \right\rangle \right) = 0. \tag{5.7}$$

Then, we know that

$$x_0 \notin \left\langle A_1^T \mid C_1^T \right\rangle + \sum_{i=2}^{k} \prod_{j=1}^{i-1} \exp\left(A_j^T h_j\right) E_j^T \left\langle A_i^T \mid C_i^T \right\rangle. \tag{5.8}$$

This implies (5.4).

Sufficiency: on the contrary, if (5.4) holds, there exists $x_0 \in \mathbb{R}^n$, nonzero, such that

$$x_0^T \left(\left\langle A_1^T \mid C_1^T \right\rangle + \sum_{i=2}^{k} \prod_{j=1}^{i-1} \exp\left(A_j^T h_j\right) E_j^T \left\langle A_i^T \mid C_i^T \right\rangle \right) = 0. \tag{5.9}$$

It follows that

$$\begin{gathered} x_0^T \left\langle A_1^T \mid C_1^T \right\rangle = 0, \\ x_0^T \prod_{j=1}^{i-1} \exp\left(A_j^T h_j\right) E_j^T \left\langle A_i^T \mid C_i^T \right\rangle = 0, \quad i = 2, \dots, k. \end{gathered} \tag{5.10}$$

By Lemma 2.3, we get

$$
\begin{gathered}
C_1 \exp[A_1(t-t_0)]x_0 = 0, \quad t \in (t_0, t_1], \\
C_i \exp[A_i(t-t_{i-1})] \prod_{j=i-1}^{1} E_j \exp(A_j h_j) x_0 = 0, \quad t \in (t_{i-1}, t_i],\ i = 2, \dots, k-1, \\
C_k \exp[A_k(t-t_{k-1})] \prod_{j=k-1}^{1} E_j \exp(A_j h_j) x_0 = 0, \quad t \in (t_{k-1}, t_f].
\end{gathered} \tag{5.11}
$$

This means that $y(t) \equiv 0$, $t \in [t_0, t_f]$. Thus, the system (5.1) is not observable. □

Remark 5.4. Theorem 5.3 is a geometric type condition. By simple transformation, we can get an algebraic type condition. In fact, for $i = 1, 2, \dots,$ denote

$$
O_i = \left[C_i^T, A_i^T C_i^T, \dots, \left(A_i^T\right)^{n-1} C_i^T \right], \tag{5.12}
$$

for $i = 2, \dots, k$, denote

$$
G_i = \prod_{j=1}^{i-1} \exp\left(A_j^T h_j\right) E_j^T O_i, \tag{5.13}
$$

and, finally, denote

$$
O_{[t_0, t_f]} = [O_1, G_2, \dots, G_k]. \tag{5.14}
$$

Then, it is easy to verify that

$$
\operatorname{Im}\left(O_{[t_0,t_f]}\right) = \left\langle A_1^T \mid C_1^T \right\rangle + \sum_{i=2}^{k} \prod_{j=1}^{i-1} \exp\left(A_j^T h_j\right) E_j^T \left\langle A_i^T \mid C_i^T \right\rangle. \tag{5.15}
$$

Thus, we get the following algebraic type criterion.

Corollary 5.5. *The system* (5.1) *is observable on* $[t_0, t_f]$*, where* $t_f \in (t_{k-1}, t_k]$*, if and only if*

$$
\operatorname{rank}\left(O_{[t_0,t_f]}\right) = n. \tag{5.16}
$$

Next, we establish a criterion for determinability.

Theorem 5.6. *The system* (5.1) *is determinable on* $[t_0, t_f]$*, where* $t_f \in (t_{k-1}, t_k]$*, if and only if*

$$
\operatorname{Im}\left(\prod_{j=1}^{k-1} \exp\left(A_j^T h_j\right) E_j^T\right) \subseteq \sum_{i=2}^{k} \prod_{j=1}^{i-1} \exp\left(A_j^T h_j\right) E_j^T \left\langle A_i^T \mid C_i^T \right\rangle + \left\langle A_1^T \mid C_1^T \right\rangle. \tag{5.17}
$$

Proof. First, by Lemma 2.4, we know that (5.17) is equivalent to

$$\prod_{j=k-1}^{1} E_j \exp(A_j h_j) \left(\sum_{i=2}^{k} \prod_{j=1}^{i-1} \exp\left(A_j^T h_j\right) E_j^T \left\langle A_i^T \mid C_i^T \right\rangle + \left\langle A_1^T \mid C_1^T \right\rangle \right)_{\perp} = 0. \tag{5.18}$$

Similar to the proof of Theorem 5.3, we prove the complementary proposition of Theorem 5.6, that is, the system (5.1) is not determinable on $[t_0, t_f]$, where $t_f \in (t_{k-1}, t_k]$, if and only if

$$\prod_{j=k-1}^{1} E_j \exp(A_j h_j) \left(\sum_{i=2}^{k} \prod_{j=1}^{i-1} \exp\left(A_j^T h_j\right) E_j^T \left\langle A_i^T \mid C_i^T \right\rangle + \left\langle A_1^T \mid C_1^T \right\rangle \right)_{\perp} \neq 0. \tag{5.19}$$

Necessity: if the system (5.1) is not determinable on $[t_0, t_f]$, where $t_f \in (t_{k-1}, t_k]$, then there exists a terminal $x_f \in \mathbb{R}^n$, nonzero, such that $y(t) = 0$, $t \in [t_0, t_f]$. Then, there exists a nonzero $x_0 \in \mathbb{R}^n$ as the initial state such that the system is driven from $x(t_0) = x_0$ to $x(t_f) = x_f$, that is, $x_f = \exp[A_k(t_f - t_{k-1})] \prod_{j=k-1}^{1} E_j \exp(A_j h_j) x_0$. This means that

$$\begin{gathered} C_1 \exp[A_1(t - t_0)] x_0 = 0, \quad t \in (t_0, t_1], \\ C_i \exp[A_i(t - t_{i-1})] \prod_{j=i-1}^{1} E_j \exp(A_j h_j) x_0 = 0, \quad t \in (t_{i-1}, t_i],\ i = 2, \ldots, k-1, \\ C_k \exp[A_k(t - t_{k-1})] \prod_{j=k-1}^{1} E_j \exp(A_j h_j) x_0 = 0, \quad t \in (t_{k-1}, t_f]. \end{gathered} \tag{5.20}$$

By Lemma 2.3, we get

$$\begin{gathered} x_0^T \left\langle A_1^T \mid C_1^T \right\rangle = 0, \\ x_0^T \prod_{j=1}^{i-1} \exp\left(A_j^T h_j\right) E_j^T \left\langle A_i^T \mid C_i^T \right\rangle = 0, \quad i = 2, \ldots, k. \end{gathered} \tag{5.21}$$

It follows that

$$x_0^T \left(\left\langle A_1^T \mid C_1^T \right\rangle + \sum_{i=2}^{k} \prod_{j=1}^{i-1} \exp\left(A_j^T h_j\right) E_j^T \left\langle A_i^T \mid C_i^T \right\rangle \right) = 0. \tag{5.22}$$

This implies that

$$x_0 \in \left(\left\langle A_1^T \mid C_1^T \right\rangle + \sum_{i=2}^{k} \prod_{j=1}^{i-1} \exp\left(A_j^T h_j\right) E_j^T \left\langle A_i^T \mid C_i^T \right\rangle \right)_{\perp}. \tag{5.23}$$

Since $\exp[A_k(t_{k-1}-t_f)]x_f = \prod_{j=k-1}^{1} E_j \exp(A_j h_j)x_0$, we know that

$$\begin{aligned}
&\exp\left[A_k\left(t_{k-1}-t_f\right)\right]x_f \\
&\quad \in \prod_{j=k-1}^{1} E_j \exp\left(A_j h_j\right)\left(\left\langle A_1^T \mid C_1^T\right\rangle + \sum_{i=2}^{k}\prod_{j=1}^{i-1}\exp\left(A_j^T h_j\right)E_j^T\left\langle A_i^T \mid C_i^T\right\rangle\right)_{\perp}.
\end{aligned} \tag{5.24}$$

It implies that

$$\prod_{j=k-1}^{1} E_j \exp\left(A_j h_j\right)\left(\left\langle A_1^T \mid C_1^T\right\rangle + \sum_{i=2}^{k}\prod_{j=1}^{i-1}\exp\left(A_j^T h_j\right)E_j^T\left\langle A_i^T \mid C_i^T\right\rangle\right)_{\perp} \neq 0. \tag{5.25}$$

Hence, (5.19) holds.

Sufficiency: on the contrary, if (5.19) holds, then we know that

$$\prod_{j=k-1}^{1} E_j \exp\left(A_j h_j\right)\left(\left\langle A_1^T \mid C_1^T\right\rangle + \sum_{i=2}^{k}\prod_{j=1}^{i-1}\exp\left(A_j^T h_j\right)E_j^T\left\langle A_i^T \mid C_i^T\right\rangle\right)_{\perp} \neq 0. \tag{5.26}$$

Then, there exists a nonzero x_f satisfying

$$\exp\left[A_k\left(t_{k-1}-t_f\right)\right]x_f \in \prod_{j=k-1}^{1} E_j \exp\left(A_j h_j\right)\left(\left\langle A_1^T \mid C_1^T\right\rangle + \sum_{i=2}^{k}\prod_{j=1}^{i-1}\exp\left(A_j^T h_j\right)E_j^T\left\langle A_i^T \mid C_i^T\right\rangle\right)_{\perp} \tag{5.27}$$

such that there exists a nonzero x_0 satisfying

$$\begin{gathered}
\exp\left[A_k\left(t_{k-1}-t_f\right)\right]x_f = x_0, \\
x_0^T\left(\left\langle A_1^T \mid C_1^T\right\rangle + \sum_{i=2}^{k}\prod_{j=1}^{i-1}\exp\left(A_j^T h_j\right)E_j^T\left\langle A_i^T \mid C_i^T\right\rangle\right) = 0.
\end{gathered} \tag{5.28}$$

It follows that

$$\begin{gathered}
x_0^T\left\langle A_1^T \mid C_1^T\right\rangle = 0, \\
x_0^T\prod_{j=1}^{i-1}\exp\left(A_j^T h_j\right)E_j^T\left\langle A_i^T \mid C_i^T\right\rangle = 0, \quad i = 2,\ldots,k.
\end{gathered} \tag{5.29}$$

By Lemma 2.3, we get

$$\begin{aligned}
&C_1 \exp[A_1(t-t_0)]x_0 = 0, \quad t \in (t_0, t_1],\\
&C_i \exp[A_i(t-t_{i-1})]\prod_{j=i-1}^{1} E_j \exp(A_j h_j)x_0 = 0, \quad t \in (t_{i-1}, t_i],\ i = 2,\dots,k-1,\\
&C_k \exp[A_k(t-t_{k-1})]\prod_{j=k-1}^{1} E_j \exp(A_j h_j)x_0 = 0, \quad t \in (t_{k-1}, t_f].
\end{aligned} \tag{5.30}$$

This means that $y(t) \equiv 0$, $t \in [t_0, t_f]$. Thus, we find a nonterminal nonzero state x_f such that the output $y(t)$ remains zero. Hence, the system (5.1) is not determinable. □

Similar to the controllability and reachability case, under some simple condition, we can show that for the system (5.1), observability is equivalent to determinability.

Corollary 5.7. *For the system* (5.1)*, if E_i is nonsingular, $i = 1, 2, \dots, k-1$, then the following statements are equivalent:*

(a) *the system is observable,*

(b) *the system is determinable,*

(c) $\sum_{i=k}^{2}\prod_{j=1}^{i-1}\exp(A_j^T h_j)E_j^T\langle A_i^T \mid C_i^T\rangle + \langle A_1^T \mid C_1^T\rangle = \mathbb{R}^n$.

Proof. If E_i is nonsingular, $i = 1, 2, \dots, k-1$, then we know that $\prod_{j=k-1}^{1} E_j \exp(A_j h_j)$ is nonsingular. Hence, we get (5.3) and (5.17) are equivalent. □

Remark 5.8. For system (2.1), assume that $A_i = A$, $B_i = B$, $i = 1, \dots, k$. Then, it is easy to see that Theorem 4.3 concludes the results of Theorem 4.2 in [15].

Remark 5.9. For system (2.1), assume that $E_i = I$, $F_i = 0$, $i = 1, \dots, k$. Then, it is easy to see that Theorem 2 in [20] is a special case of Corollary 5.7.

6. Examples

In this section, we give two numerical examples to illustrate how to utilize our criteria.

Example 6.1. Consider a 3-dimensional linear piecewise constant impulsive system with

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad C_1 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix},$$

$$D_1 = 0, \qquad E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad F_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad C_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix},$$
$$D_2 = 0, \qquad E_2 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad F_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},$$
$$A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad C_3 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix},$$
$$D_3 = 0, \qquad E_3 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad F_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \tag{6.1}$$

where $t_0 = 0$, $t_1 = 1$, $t_2 = 2$, and $t_3 = 3$.

Now, we try to use our criteria to investigate the reachability, controllability, observability, and determinability on $[0, t_f]$, where $t_f \in (2, 3]$, of the system in Example 6.1.

First, we consider the reachability. By a simple calculation, we have

$$\begin{aligned} &E_2 \exp(A_2)(E_1\langle A_1 \mid B_1\rangle + \mathcal{I}m(F_1)) + E_2\langle A_2 \mid B_2\rangle + \mathcal{I}m(F_2) + \langle A_3 \mid B_3\rangle \\ &\quad = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}. \end{aligned} \tag{6.2}$$

By Theorem 3.3, the system should not be reachable. In fact, for any piecewise continuous input $u(t)$, $t \in [0, t_f]$, and any nonzero initial state $x_0 = [x_1^0 \ \ x_2^0 \ \ x_3^0]^T$, we have

$$x(t_f) = \begin{bmatrix} * \\ * \\ 0 \end{bmatrix}. \tag{6.3}$$

This fact shows that the system is indeed not reachable.

Next, we consider the controllability. By a simple calculation, we have

$$\mathcal{I}m(E_2 \exp(A_2) E_1 \exp(A_1)) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}. \tag{6.4}$$

It is easy to see that

$$\begin{aligned} &\mathcal{I}m(E_2 \exp(A_2) E_1 \exp(A_1)) \\ &\quad \subseteq E_2 \exp(A_2)(E_1\langle A_1 \mid B_1\rangle + \mathcal{I}m(F_1)) + E_2\langle A_2 \mid B_2\rangle + \mathcal{I}m(F_2) + \langle A_3 \mid B_3\rangle. \end{aligned} \tag{6.5}$$

By Theorem 4.3, the system should be controllable. In fact, we can take the piecewise constant input

$$u(t) = \begin{cases} c_1, & t \in (0,1], \\ 0, & t \in (1,2], \\ c_3, & t \in (2,3]. \end{cases} \tag{6.6}$$

Then, for any nonzero initial state $x_0 = [x_1^0 \;\; x_2^0 \;\; x_3^0]^T$, we have

$$x(t_f) = \begin{bmatrix} x_1^0 + 0.5c_1 \\ x_2^0 + 1.5c_1 + \left(2 - 2t_f + 0.5t_f^2\right)c_3 \\ 0 \end{bmatrix}. \tag{6.7}$$

Obviously, if $c_1 = -2x_1^0$, $c_3 = (-x_2^0 - 1.5c_1)/(2 - 2t_f + 0.5t_f^2)$, then $x(t_f) = 0$. This fact shows that the system is indeed controllable.

Next, we consider the observability. By a simple calculation, we have

$$\begin{aligned} &\left\langle A_1^T \mid C_1^T \right\rangle + \exp\left(A_1^T\right)E_1^T\left\langle A_2^T \mid C_2^T \right\rangle + \exp\left(A_1^T\right)E_1^T \exp\left(A_2^T\right)E_2^T\left\langle A_3^T \mid C_3^T \right\rangle \\ &\quad = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}. \end{aligned} \tag{6.8}$$

By Theorem 5.3, the system should not be observable. In fact, for any piecewise continuous input $u(t)$, $t \in [0, t_f]$, and nonzero initial state $x_0 = [0 \;\; 0 \;\; 1]^T$, we have

$$y(t) \equiv 0, \quad t \in [0, t_f). \tag{6.9}$$

This fact shows that the system is indeed not observable.

Finally, we consider the determinability. By a simple calculation, we have

$$\begin{aligned} &E_2 \exp(A_2) E_1 \exp(A_1) \\ &\quad \times \left(\left\langle A_1^T \mid C_1^T \right\rangle + \exp\left(A_1^T\right)E_1^T\left\langle A_2^T \mid C_2^T \right\rangle + \exp\left(A_1^T\right)E_1^T \exp\left(A_2^T\right)E_2^T\left\langle A_3^T \mid C_3^T \right\rangle\right)_{\perp} \\ &\quad = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} = 0. \end{aligned} \tag{6.10}$$

It follows that

$$E_2 \exp(A_2) E_1 \exp(A_1) \times \left(\left\langle A_1^T \mid C_1^T \right\rangle + \exp\left(A_1^T\right) E_1^T \left\langle A_2^T \mid C_2^T \right\rangle + \exp\left(A_1^T\right) E_1^T \exp\left(A_2^T\right) E_2^T \left\langle A_3^T \mid C_3^T \right\rangle \right)_{\perp} = 0. \tag{6.11}$$

By Theorem 5.6, the system should be determinable. In fact, for any nonzero terminal state $x_f = [x_1^f \;\; x_2^f \;\; x_3^f]^T$, there must exist a nonzero initial state $x_0 = [x_1^0 \;\; x_2^0 \;\; x_3^0]^T$ such that

$$\exp\left[A_3(2-t_f)\right] x_f = E_2 \exp(A_2) E_1 \exp(A_1) x_0. \tag{6.12}$$

It follows that

$$(2-t_f) x_f = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x_0. \tag{6.13}$$

This means that $x_3^f = x_3^0 = 0$ and $|x_1^0| + |x_2^0| \neq 0$. It is easy to verify that, for any initial state x_0 satisfying $|x_1^0| + |x_2^0| \neq 0$, we have $y(t) \not\equiv 0$, $t \in (0, t_f)$. This fact shows that the system is indeed determinable.

Example 6.2. Consider a 3-dimensional linear piecewise constant impulsive system with

$$\begin{gathered} A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, \\ D_1 = 0, \quad E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad F_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \\ A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \\ D_2 = 0, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \\ A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C_3 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \\ D_3 = 0, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad F_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \end{gathered} \tag{6.14}$$

where $t_0 = 0$, $t_1 = 1$, $t_2 = 2$, and $t_3 = 3$.

Now, we try to use our criteria to investigate the reachability and controllability on $[0, t_f]$, where $t_f \in (2, 3]$, of the system in Example 6.2.

First, we consider reachability. By a simple calculation, we have

$$E_2 \exp(A_2)(E_1\langle A_1 \mid B_1\rangle + \mathcal{I}m(F_1)) + E_2\langle A_2 \mid B_2\rangle + \mathcal{I}m(F_2) + \langle A_3 \mid B_3\rangle$$
$$= \operatorname{span}\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\} = \mathbb{R}^3. \tag{6.15}$$

By Theorem 3.3, the system should be reachable. In fact, we take the piecewise constant input

$$u(t) = \begin{cases} c_1, & t \in (0, 1], \\ c_2, & t \in (1, 2], \\ c_3, & t \in (2, 3]. \end{cases} \tag{6.16}$$

Then, letting $x(0) = 0$, for any nonzero terminal state $x(3) = [x_1^f \quad x_2^f \quad x_3^f]^T$, we have

$$\begin{bmatrix} x_1^f \\ x_2^f \\ x_3^f \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}. \tag{6.17}$$

Obviously, we can select suitable c_1, c_2, and c_3 such that x_f is any state in $\mathbb{R}^3$. This fact shows that the system is indeed reachable.

Next, by Theorem 4.3, the system should be reachable. In fact, we take the piecewise constant input

$$u(t) = \begin{cases} c_1, & t \in (0, 1], \\ c_2, & t \in (1, 2], \\ c_3, & t \in (2, 3]. \end{cases} \tag{6.18}$$

Then, for any nonzero initial state $x(0) = [x_1^0 \quad x_2^0 \quad x_3^0]^T$, letting $x(3) = 0$, we have

$$0 = \begin{bmatrix} x_1^0 \\ x_2^0 \\ x_3^0 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}. \tag{6.19}$$

Obviously, we can select suitable c_1, c_2, and c_3 such that x_0 is any state in $\mathbb{R}^3$. This fact shows that the system is indeed controllable.

Finally, according to the conclusion in Corollary 4.4, since the matrices E_1, E_2, and E_3 in Example 6.1 are singular, we know that the reachability might not be equivalent to the controllability in this example. However, the reachability should be equivalent to the controllability in Example 6.2 since the matrices E_1, E_2, and E_3 in this example are non-singular. From the above analysis, all these statements are correct indeed.

7. Conclusion

This paper has studied the controllability and observability of linear piecewise constant impulsive systems. Necessary and sufficient criteria for reachability and controllability have been established, respectively. Moreover, it has been proved that the reachability is equivalent to the controllability under some mild conditions. Then, necessary and sufficient criteria for the observability and determinability of such systems have been established, respectively. It has been also proved that the observability is equivalent to the determinability under some mild conditions. Our criteria are of the geometric type, and they can be transformed into algebraic type conveniently. Finally, a numerical example has been given to illustrate the utility of our criteria.

Appendices

A. Proof of Lemma 2.2

By Theorem 7.8.1 in [17], we have

$$\begin{aligned}&\left\{x \mid x = \int_{t_0}^{t_f} \exp\left[A\left(t_f - s\right)\right] Bu(s)ds,\ \forall \text{ piecewise continuous } u\right\}\\&\quad = \langle A \mid B\rangle.\end{aligned} \tag{A.1}$$

Thus, it is easy to see that

$$\begin{aligned}&\left\{x \mid x = E\int_{t_0}^{t_f} \exp\left[A\left(t_f - s\right)\right] Bu(s)ds + Fu\left(t_f\right),\ \forall \text{ piecewise continuous } u\right\}\\&\quad \subseteq E\langle A \mid B\rangle + \mathcal{I}m(F).\end{aligned} \tag{A.2}$$

Moreover, we have

$$\begin{aligned}&\left\{x \mid x = \int_{t_0}^{t_E} \exp[A(t_E - s)] Bu(s)ds,\ \forall \text{ piecewise continuous } u\right\}\\&\quad = \langle A \mid B\rangle,\end{aligned} \tag{A.3}$$

where $t_E = (t_0 + t_f)/2$. Then, for any $x \in E\langle A \mid B\rangle + \mathcal{I}m(F)$, there exist a piecewise continuous function $u(t)$, $t \in [t_0, t_E]$, and $y \in \mathbb{R}^n$ such that

$$x = E\int_{t_0}^{t_E} \exp[A(t_E - s)] Bu(s)ds + Fy. \tag{A.4}$$

Then, we can take

$$v(t) = \begin{cases} u(t), & t \in [t_0, t_E], \\ 0, & t \in (t_E, t_f), \\ y, & t = t_f, \end{cases} \tag{A.5}$$

such that

$$x = E\int_{t_0}^{t_f} \exp\left[A\left(t_f - s\right)\right] Bv(s)ds + Fv\left(t_f\right). \tag{A.6}$$

This implies that

$$x \in \left\{ x \mid x = E\int_{t_0}^{t_f} \exp\left[A\left(t_f - s\right)\right] Bu(s)ds + Fu\left(t_f\right),\ \forall \text{ piecewise continuous } u \right\}. \tag{A.7}$$

It follows that

$$\begin{aligned} &\left\{ x \mid x = E\int_{t_0}^{t_f} \exp\left[A\left(t_f - s\right)\right] Bu(s)ds + Fu\left(t_f\right),\ \forall \text{ piecewise continuous } u \right\} \\ &\quad \supseteq E\langle A \mid B\rangle + \mathcal{I}m(F). \end{aligned} \tag{A.8}$$

By (A.2) and (A.8), we know that (2.8) holds.

B. Proof of Lemma 2.3

((a) $\Rightarrow$ (b)) If $C\exp[A(t - t_0)]x = 0$, $t \in [t_0, t_f]$, we get $Cx = 0$. Then, for $i = 1, \ldots, n-1$, calculating the ith derivative of $C\exp[A(t - t_0)]x$ with respect to t at $t = t_0$, we get

$$CA^i x = 0. \tag{B.1}$$

Thus, we know that

$$x^T\left[C^T, C^T A^T, \ldots, C^T\left(A^T\right)^{n-1}\right] = 0. \tag{B.2}$$

Hence, $x^T\langle A^T \mid C^T\rangle = 0$.

((a) $\Leftarrow$ (b)) If $x^T\langle A^T \mid C^T\rangle = 0$, it follows that $x^T[C^T, C^T A^T, \ldots, C^T(A^T)^{n-1}] = 0$. That is

$$CA^i x = 0, \quad i = 0, 1, \ldots, n-1. \tag{B.3}$$

Then, it is easy to prove that $C\exp[A(t - t_0)]x = 0$, $t \in [t_0, t_f]$.

C. Proof of Lemma 2.4

Given a matrix $A \in \mathbb{R}^{n\times n}$ and a linear subspace $\mathcal{W} \subseteq \mathbb{R}^n$, the following two statements are equivalent:

(a) $\mathcal{I}m(A) \subseteq \mathcal{W}$,

(b) $A^T W_\perp = 0$.

((a) ⇒ (b)) Assume that $\mathcal{I}m(A) \subseteq \mathcal{W}$ ((b) ⇒ (a)). It is equivalent to $\mathcal{I}m(A) \bigcap \mathcal{W}_\perp = 0$. It follows that, for any $x \in \mathcal{W}_\perp$, $x^T A = 0$. That is, $A^T x = 0$. This implies that $A^T W_\perp = 0$.

((b) ⇒ (a)) Assume that $A^T W_\perp = 0$. It follows that, for any $x \in \mathcal{W}_\perp$, $A^T x = 0$. That is, $x^T A = 0$. This implies that $\mathcal{I}m(A) \subseteq \mathcal{W}$.

Acknowledgments

The authors are grateful to Professor Xinghuo Yu, the Associate Editor, and the reviewers for their helpful and valuable comments and suggestions for improving this paper. This work is supported by National Natural Science Foundation (NNSF) of China (60774089, 10972003, and 60736022). This work is also supported by N2008-07 (08010702014) from Beijing Institute of Petrochemical Technology.

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