Research Article

Dynamic Properties of a Differential-Algebraic Biological Economic System

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We analyze a differential-algebraic biological economic system with time delay. The model has two different Holling functional responses. By considering time delay as bifurcation parameter, we find that there exists stability switches when delay varies, and the Hopf bifurcation occurs when delay passes through a sequence of critical values. Furthermore, we also consider the stability and direction of the Hopf bifurcation by applying the normal form theory and the center manifold theorem. Finally, using Matlab software, we do some numerical simulations to illustrate the effectiveness of our results.

1. Introduction

Recently, dynamics of differential-algebraic equations have become a subject of intense research activities because differential-algebraic equation models can describe many kinds of practical systems such as power system, biological economic system, and singular prey-predator economic model [1, 2]. The dynamic relationship between predators and their preys with harvesting has long been and will continue to be one of the dominant themes in both ecology and mathematical ecology due to its universal existence and importance. Harvesting has a strong impact on the dynamic evolution of a population. Depending on the nature of applied harvesting strategy, the long-run stationary density of the population may be significantly smaller than the long-run stationary density of a population in the absence of harvesting. Time delays of one type or another have been incorporated into mathematical models of population dynamics due to maturation time, capturing time, or other reasons. In general, delay differential equations exhibit much more complicated dynamics than ordinary differential equations since a time delay could cause a stable equilibrium to become unstable and cause the populations to fluctuate. Recently, a number of researches have been devoted to the dynamic properties for delayed predator-prey systems with harvesting. For further details, refer to [3–7].

Kar and Pahari [8] have investigated the dynamical behavior of an exploited system consisting of a prey and a predator that are harvested:

$$\dot{x} = x(t)g(x(t-\tau)) - yp(x(t)) - q_1 E_1 x(t),$$
$$\dot{y} = y(t)(-d + \alpha p(x(t))) - q_2 E_2 y(t),$$

$$(1.1)$$

where $x(t)$ and $y(t)$ respect prey and predator population densities at time t and $g(x)$ is the growth rate of prey when $y(t)$ is zero. $p(x)$ is the response function, d is the death rate of $y(t)$ and α is the conversion factor. $q_1 E_1 x$ and $q_2 E_2 y$ represent harvesting of $x(t)$ and $y(t)$.

In system (1.1) $x(t)$ and $y(t)$ have the same Holling function. But in real life they may have different adaptation for the environment. So we formulate the aforementioned problem with different Holling functions as follows:

$$\dot{x} = \alpha\left(1 - \frac{x(t-\tau)}{k}\right)x(t) - \frac{\beta x(t)y(t)}{1 + bx(t) + cy(t)} - E(t)x(t),$$
$$\dot{y} = -dy(t) + \frac{a\beta x(t)y(t)}{1 + bx(t)}.$$

$$(1.2)$$

In 1954, Gordon analyzed the effect of harvest effort on ecosystem from an economic perspective and proposed the following economic theory [9]:

Net Economic Revenue (NER) = Total Revenue (TR) − Total Cost (TC).

Based on the previous economic theory, we let $E(t)$ represent the harvest effort for $x(t)$ at time t. TR $= pE(t)x(t)$ and TC $= c_0 E(t)$. p represents harvesting reward per unit harvesting effort for unit weight of prey. c_0 represents the cost per unit harvest effort for prey. Then a differential-algebraic model which consists of two differential equations and an algebraic equation can be established as follows:

$$\dot{x} = \alpha\left(1 - \frac{x(t-\tau)}{k}\right)x(t) - \frac{\beta x(t)y(t)}{1 + bx(t) + cy(t)} - E(t)x(t),$$
$$\dot{y} = -dy(t) + \frac{a\beta x(t)y(t)}{1 + bx(t)},$$
$$E(t)(px(t) - c_0) - m = 0.$$

$$(1.3)$$

Define

$$g(x, E) = E(t)(px(t) - c_0) - m.$$

$$(1.4)$$

From the standpoint of biology, we are only interested in the dynamics of model (1.3) in the region

$$R_+^3 = \{(x, y, E) \mid x \geq 0, y \geq 0, E \geq 0\}.$$

$$(1.5)$$

2. Stability Analysis

Firstly, we consider the location and number of the equilibria of model (1.3) in R_+^3. The interior equilibrium point is a root of the following equation:

$$\alpha\left(1 - \frac{x_0}{k}\right)x_0 - \frac{\beta x_0 y_0}{1 + bx_0 + cy_0} - E_0 x_0 = 0,$$

$$-dy_0 + \frac{a\beta x_0 y_0}{1 + bx_0} = 0, \tag{2.1}$$

$$E_0(px_0 - c_0) - m = 0.$$

It is obvious that (2.1) has an only real solution $Y_0 = (x_0, y_0, E_0)$ where

$$x_0 = \frac{d}{a\beta - bd}, \qquad y_0 = \frac{(\alpha(1 - x_0/k) - m/(px_0 - c_0))(1 + bx_0)}{\beta - c(\alpha(1 - x_0/k) - m/(px_0 - c_0))}, \qquad E_0 = \frac{m}{px_0 - c_0}, \tag{2.2}$$

and $a\beta > bd$, $0 < m < \alpha(px_0 - c_0)(1 - x_0/k)$. In order to analyze the local stability of the positive equilibrium point for the system (1.3) we use the linear transformation $Y^T = QN^T$

$$N = \left(u(t), v(t), \overline{E}(t)\right), \qquad Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\dfrac{pE_0}{px_0 - c_0} & 0 & 1 \end{pmatrix}. \tag{2.3}$$

Then we have

$$u(t) = x(t), \qquad v(t) = y(t), \qquad \overline{E}(t) = \frac{pE_0}{px_0 - c_0}x(t) + E(t), \qquad D_y g(x)Q = (0, 0, px_0 - c_0), \tag{2.4}$$

for which system (1.3) yields

$$\dot{u} = \alpha\left(1 - \frac{u(t - \tau)}{k}\right)u(t) - \frac{\beta u(t)v(t)}{1 + bu(t) + cv(t)} - \overline{E}(t)u(t) + \frac{pE_0 u^2(t)}{px_0 - c_0},$$

$$\dot{v} = -dv(t) + \frac{a\beta u(t)v(t)}{1 + bu(t)}, \tag{2.5}$$

$$\left(\overline{E}(t) - \frac{pE_0 u(t)}{px_0 - c_0}\right)(pu(t) - c_0) - m = 0.$$

Now we derive the formula for determining the properties of the positive equilibrium point of the system (2.5). Firstly, we consider the local parametric [10] ψ, which is defined as follows:

$$\psi(z(t)) = N_0^T + U_0 z(t) + V_0 h(z(t)), \tag{2.6}$$

where $U_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$, $V_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $z(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$, and $h = \begin{pmatrix} 0 \\ 0 \\ h_3 \end{pmatrix}$, $h_3 = m/(p(u_0 + y_1(t)) - c_0) + pE_0(u_0 + y_1(t))/(px_0 - c_0) - E_0$. Introducing perturbations $u(t) = u_0 + y_1$ and $v(t) = v_0 + y_2$ and defining $\psi(z(t)) = (u(t), v(t), \overline{E}(t))^T$, $\overline{E}(t) = E_0 + h_3(y_{(1)}, y_{(2)})$, and $g(\psi(z(t))) = 0$, then we can obtain the parametric system of system (2.5) as follows:

$$
\dot{y}_1 = \alpha\left(1 - \frac{y_1(t-\tau) + u_0}{k}\right)(y_1(t) + u_0) - \frac{\beta(y_1(t) + u_0)(y_2(t) + v_0)}{1 + b(y_1(t) + u_0) + c(y_2(t) + v_0)}
$$

$$
- \left(\overline{E}_0 + h_3(y_1(t), y_2(t))\right)(y_1(t) + u_0) + \frac{pE_0(y_1(t) + u_0)}{px_0 - c_0}, \tag{2.7}
$$

$$
\dot{y}_2 = -d(y_2 + v_0) + \frac{a\beta(y_1(t) + u_0)(y_2(t) + v_0)}{1 + b(y_1(t) + u_0)}.
$$

Neglecting 2nd- and higher-order products of y_1 and y_2, we get

$$
\dot{y}_1 = \left(\alpha\left(1 - \frac{u_0}{k}\right) - \frac{\beta v_0(1 + cv_0)}{(1 + bv_0 + cv_0)^2} - E_0 + \frac{pE_0 u_0}{px_0 - c_0}\right)y_1(t)
$$

$$
- \frac{\alpha u_0}{k}y_1(t - \tau) - \frac{\beta u_0(1 + bu_0)}{(1 + bu_0 + cv_0)^2}y_2(t), \tag{2.8}
$$

$$
\dot{y}_2 = \frac{a\beta v_0}{(1 + bu_0)^2}y_2(t).
$$

The characteristic equation of system (2.8) at $(0,0)$ takes the form

$$
\lambda^2 + \left(A - \frac{\alpha u_0}{k} + \frac{\alpha u_0}{k}e^{(-\lambda\tau)}\right)\lambda + B = 0. \tag{2.9}
$$

First of all, we know that the roots of (2.9) with $\tau = 0$ have always negative real parts when $A < 0, B > 0$, where

$$
A = -\alpha\left(1 - \frac{2u_0}{k}\right) - \frac{\beta v_0(1 + cv_0)}{(1 + bu_0 + cv_0)^2} - E_0 + \frac{pE_0 u_0}{pu_0 - c}, \qquad B = \frac{a\beta v_0}{(1 + bu_0 + cv_0)^2(1 + bu_0)^2}. \tag{2.10}
$$

Next, we will investigate the distribution of the roots of (2.9). If $i\omega(\omega > 0)$ is a root of (2.9), then we have

$$
-\omega^2 + \left(A - \frac{\alpha u_0}{k} + \frac{\alpha u_0}{k}e^{i\omega\tau}\right)i\omega + B = 0. \tag{2.11}
$$

Separating the real and imaginary parts, we have

$$\frac{\alpha \omega u_0}{k} \sin(\omega \tau) = \omega^2 - B,$$

$$\frac{\alpha \omega u_0}{k} \cos(\omega \tau) = \omega \left(\frac{\alpha u_0}{k} - A \right),$$

(2.12)

which lead to

$$\omega^4 + \left[\left(\frac{\alpha u_0}{k} - A \right)^2 - 2B - \left(\frac{\alpha u_0}{k} \right)^2 \right] \omega^2 + B^2 = 0.$$

(2.13)

When $(\alpha u_0/k - A)^2 < 2B + (\alpha u_0/k)^2$ and $[(\alpha u_0/k - A)^2 - 2B - (\alpha u_0/k)^2]^2 > 4B^2$ hold, we can get that (2.13) has two positive roots ω^+ and ω^-.

According to the analysis previous, we have the following lemma 2.1:

Lemma 2.1. (1) *If $A < 0, B > 0$ and $(\alpha u_0/k - A)^2 > 2B + (\alpha u_0/k)^2$, then all the roots of (2.9) have negative real parts for all $\tau \geq 0$.*

(2) *If $(\alpha u_0/k - A)^2 < 2B + (\alpha u_0/k)^2$ and $[(\alpha u_0/k - A)^2 - 2B - (\alpha u_0/k)^2]^2 > 4B^2$, then (2.13) has two positive roots ω^+, ω^-. Furthermore, one has*

$$\tau^{\pm} = \frac{1}{\omega^{\pm}} \arccos \left(\frac{k}{\alpha u_0} \left(\frac{\alpha u_0}{k} - A \right) \right) + \frac{2n\pi}{\omega^{\pm}}, \quad n = 0, 1, 2, \ldots.$$

(2.14)

Differentiating the two sides of (2.9) and noticing that λ is a function of τ, we have

$$\left(\frac{d\lambda}{d\tau} \right)^{-1} = \frac{2\lambda - (\alpha u_0/k)\lambda \tau e^{-\lambda \tau} + (A - \alpha u_0/k + (\alpha u_0/k)e^{-\lambda \tau})}{(\alpha u_0/k)\lambda^2 e^{-\lambda \tau}},$$

(2.15)

and then we can obtain

$$\text{sign} \left\{ \text{Re} \left(\frac{d\lambda}{d\tau} \right) \right\}_{\tau = \tau^+, \omega = \omega^+} > 0, \qquad \text{sign} \left\{ \text{Re} \left(\frac{d\lambda}{d\tau} \right) \right\}_{\tau = \tau^-, \omega = \omega^-} < 0.$$

(2.16)

Therefore, we can obtain the following results about the stability of the positive equilibrium and the Hopf bifurcation of (2.7).

Theorem 2.2. *Assume that $A < 0$, $B > 0$.*

(1) *If $(\alpha u_0/k - A)^2 < 2B + (\alpha u_0/k)^2$ and $[(\alpha u_0/k - A)^2 - 2B - (\alpha u_0/k)^2]^2 > 4B^2$ are satisfied, then there exists an integer N such that the zero solution of system (2.7) is asymptotically stable when $\tau \in [0, \tau_0^+) \cup (\tau_0^-, \tau_1^+) \cup (\tau_1^-, \tau_2^+) \cup \cdots \cup (\tau_{N-1}^-, \tau_N^+)$ and unstable when $\tau \in (\tau_0^+, \tau_0^-) \cup (\tau_1^+, \tau_1^-) \cup \cdots \cup (\tau_N^+, +\infty)$.*

(2) *When $\tau = \tau_n^{\pm}, n = 0, 1, 2 \ldots$, there are Hopf bifurcations at the equilibrium point of system (2.7).*

3. Direction and the Stability of Hopf Bifurcation

In the previous section, we have already obtained the conditions which ensure that model (2.7) undergoes the Hopf bifurcation at the critical values τ_n ($n = 0, 1, \ldots$). In this section, we will study the direction and stability of the Hopf bifurcation based on the normal form approach theory and center manifold theory introduced by Hassard et al. [11].

First of all, by the transformation $y_1 = u - u_0$, $y_2 = v - v_0$, $t = t/\tau$, $\tau = \tau_n + \mu$, the parametric system (2.7) is equivalent to the following Functional Differential Equation (FDE) system in $C = C([-1, 0], R^2)$:

$$\dot{z}(t) = L_\mu(z_t) + f(\mu, z_t), \tag{3.1}$$

where $z(t) = (y_1(t), \ y_2(t))^T$, $z_t(\theta) = z(t + \theta)$ for $\theta \in [-1, 0]$, and $L_\mu : C \to R : f : R \times C \to R$.

$$L_\mu(\phi) = (\tau_n + \mu)\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & 0 \end{pmatrix}\phi^T(0) + (\tau_n + \mu)\begin{pmatrix} b_{11} & 0 \\ 0 & 0 \end{pmatrix}\phi^T(-1),$$

$$a_{11} = \left(\alpha - \frac{\alpha u_0}{k}\right) - \frac{\beta v_0(1 + cv_0)}{(1 + bu_0 + cv_0)^2} - E_0 + \frac{pE_0 u_0}{pu_0 - c_0}, \tag{3.2}$$

$$a_{12} = \frac{-\beta u_0(1 + bu_0)}{(1 + bu_0 + cv_0)^2}, \qquad a_{21} = \frac{\alpha\beta v_0}{(1 + bv_0)^2}, \qquad b_{11} = -\frac{\alpha u_0}{k}.$$

Let $\phi = (\phi_1, \phi_2)^T$. Then

$$f(\mu, \phi) = (\tau_n + \mu)(f_{11}, f_{22})^T, \tag{3.3}$$

with

$$f_{11} = \left(\frac{pE_0}{px_0 - c_0} - \frac{p^2 E_0 u_0}{(px_0 - c_0)^2} + \frac{\beta b v_0}{(1 + bu_0 + cv_0)^3}\right)\phi_1(0)^2 - \frac{\alpha}{k}\phi_1(0)\phi_1(-1)$$

$$- \frac{\beta}{(1 + bu_0 + cv_0)^2}\phi_1(0)\phi_2(0), \tag{3.4}$$

$$f_{22} = -\frac{\beta a b v_0}{(1 + bu_0 + cv_0)^3}\phi_1(0)^2 + \frac{a\beta}{(1 + bu_0 + cv_0)^2}\phi_1(0)\phi_2(0).$$

By the Riesz representation theorem, there exists a matrix whose components are bounded variation functions $\eta(\theta, \mu)$ such that

$$L_\mu = \int_{-1}^{0} d\eta(\theta, \mu)\phi(\theta), \quad \phi \in C, \ \theta \in [-1, 0]. \tag{3.5}$$

In fact, we can choose

$$\eta(\theta, \mu) = (\tau_n + \mu)\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & 0 \end{pmatrix}\delta(\theta) + (\tau_n + \mu)\begin{pmatrix} b_{11} & 0 \\ 0 & 0 \end{pmatrix}\delta(\theta + 1), \tag{3.6}$$

where

$$\delta(\theta) = \begin{cases} 0, & \theta \neq 0, \\ 1, & \theta = 0, \end{cases} \tag{3.7}$$

for $\phi \in C^1([-1,0], R^2)$. Define

$$A(\mu)\phi(\theta) = \begin{cases} \dfrac{d\phi(\theta)}{d\theta}, & -1 \leq \theta < 0, \\ \displaystyle\int_{-1}^{0} d\eta(\theta, \mu)\phi(\theta), & \theta = 0, \end{cases} \tag{3.8}$$

$$R(\mu)\phi = \begin{cases} 0, & \theta \in [-1, 0), \\ f(\mu, \phi), & \theta = 0. \end{cases}$$

Then the system (3.1) is equivalent to

$$\dot{z}_t = A(\mu)z_t + R(\mu)z_t, \tag{3.9}$$

for $\psi \in C([0,1], (R^2)^*)$, the adjoint operator A^* of A is

$$A^*\psi(s) = \begin{cases} -\dfrac{d\psi(s)}{ds}, & 0 < s \leq 1, \\ \displaystyle\int_{-1}^{0} d\eta^T(s, 0)\psi(-s), & s = 0, \end{cases} \tag{3.10}$$

and a bilinear inner product is given by

$$\langle \psi(s), \phi(\theta) \rangle = \overline{\psi}(0)\phi(0) - \int_{\theta=1}^{0} \int_{\xi=0}^{\theta} \overline{\psi}(\xi - \theta) d\eta(\theta)\phi(\xi)d\xi. \tag{3.11}$$

By the discussion in Section 2, we know that $\pm i\tau_n\omega^{\pm}$ are eigenvalues of $A(0)$. Thus, they are also eigenvalues of A^*. It is easy to calculate that

$$q(\theta) = (1, r)^T e^{i\omega \pm \tau_n \theta}, \qquad q^*(s) = G(r^*, 1)e^{i\omega \pm \tau_n s}, \tag{3.12}$$

where

$$r = \frac{i\omega - a_{11} - b_{11}e^{-i\omega\tau_n}}{a_{12}}, \qquad \overline{G} = \frac{1}{r + \overline{r^*} - \tau_n \overline{r^*}b_{11}e^{-i\omega\tau_n}}, \tag{3.13}$$

$$\langle q^*(s), q(\theta) \rangle = 1, \qquad \langle q^*(s), \overline{q}(\theta) \rangle = 0.$$

Let z_t be the solution of (3.9) when $\mu = 0$. Define

$$\dot{z}(t) = \langle q^*, z_t \rangle, \qquad w(t, \theta) = z_t - 2\operatorname{Re}\{z_t q(\theta)\}. \tag{3.14}$$

On the center manifold C_0, we have

$$w(t,\theta) = w(z(t),\overline{z}(t),\theta) = w_{20}^{(\theta)}\frac{z^2}{2} + w_{11}^{(\theta)}z\overline{z} + w_{02}^{(\theta)}\frac{\overline{z}^2}{2} + \cdots. \tag{3.15}$$

Note that w is real. When $\mu = 0$ we have

$$\dot{z} = i\omega\tau_n z + \left\langle q^{*(\theta)}, f\left(0, \omega + 2\operatorname{Re} z(t)q(\theta)\right)\right\rangle. \tag{3.16}$$

Then

$$\dot{z} = i\omega\tau_n z + g(z,\overline{z}), \tag{3.17}$$

where

$$g(z,\overline{z}) = g_{20}(\theta)\frac{z^2}{2} + g_{11}(\theta)z\overline{z} + g_{02}(\theta)\frac{\overline{z}^2}{2} + \cdots. \tag{3.18}$$

From (3.9) and (3.16), we have

$$\dot{W} = \dot{z}_t - \dot{z}q - \dot{\overline{z}}\,\overline{q} = \begin{cases} Aw - 2\operatorname{Re}\overline{q}^*(0)f(z,\overline{z})q(\theta), & -1 \le \theta \le 0, \\ Aw - 2\operatorname{Re}\overline{q}^*(0)f(z,\overline{z})q(\theta) + f, & \theta = 0. \end{cases} \tag{3.19}$$

Rewrite

$$\dot{W} = AW + H(z,\overline{z},\theta), \tag{3.20}$$

where

$$H(z,\overline{z},\theta) = H_{20}^{(\theta)}\frac{z^2}{2} + H_{11}^{(\theta)}z\overline{z} + H_{02}^{(\theta)}\frac{\overline{z}^2}{2} + \cdots. \tag{3.21}$$

We can obtain

$$(A - 2i\omega\tau_n)w_{20} = -H_{20}^{(\theta)}, \qquad Aw_{11}^{(\theta)} = -H_{11}^{(\theta)}. \tag{3.22}$$

Now we have g_{20}, g_{11}, g_{02}, and g_{21}:

$$g_{20} = 2\overline{G}\tau_n\left[F\overline{r}^* - \frac{\beta r\overline{r}^* - ar\beta}{(1 + bu_0 + cv_0)^2} - \frac{\alpha\overline{r}^*}{k}e^{-i\omega\tau_n\theta} - \frac{\beta abv_0}{(1 + bu_0 + cv_0)^3}\right],$$

$$g_{11} = \overline{G}\tau_n\left[2F\overline{r}^* + \left(2aB - 2B\overline{r}^*\right)\operatorname{Re}(r) - \frac{2\beta abv_0}{(1 + bu_0 + cv_0)^3} - \frac{2\alpha\overline{r}^*}{k}\operatorname{Re}\left(e^{i\omega\tau_n\theta}\right)\right],$$

$$g_{02} = 2\overline{G}\tau_n\left[F\overline{r}^* - \frac{\beta r\overline{r}^* - a\beta\overline{r}}{(1 + bu_0 + cv_0)^2} - \frac{\alpha\overline{r}^*}{k}e^{i\omega\tau_n\theta} - \frac{\beta abv_0}{(1 + bu_0 + cv_0)^3}\right],$$

$$g_{21} = 2\overline{G}\tau_n\left[\left(2F\overline{r^*} - \frac{\alpha\overline{r^*}}{k}e^{-i\omega\tau_n\theta} - \frac{2\beta abv_0}{(1 + bu_0 + cv_0)^3} + \frac{a\beta r - \beta r\overline{r^*}}{(1 + bu_0 + cv_0)^2}\right)W_{20}^{(1)}(0)\right.$$

$$+ \frac{\left(a\beta - \beta\overline{r^*}\right)}{(1 + bu_0 + cv_0)^2}W_{11}^{(2)}(0)$$

$$+ \left(F\overline{r^*} - \frac{\beta abv_0}{(1 + bu_0 + cv_0)^3} - \frac{\alpha\overline{r}}{2k}e^{i\omega\tau_n\theta} + \frac{a\beta\overline{r} - \beta\overline{r}\overline{r^*}}{2(1 + bu_0 + cv_0)^2}\right)W_{20}^{(1)}(0)$$

$$\left. - \frac{\alpha\overline{r^*}}{k}W_{11}^{(1)}(-1) + \frac{a\beta - \beta\overline{r^*}}{2(1 + bu_0 + cv_0)^2}W_{20}^{(2)}(0) - \frac{\alpha\overline{r^*}}{2k}W_{20}^{(1)}(-1)\right],$$

$$(3.23)$$

where

$$F = \frac{pE_0}{px_0 - c_0} - \frac{p^2 E_0 u_0}{(px_0 - c_0)^2} + \frac{\beta bv_0}{(1 + bu_0 + cv_0)^3},$$

$$W_{20}(\theta) = \frac{ig_{20}}{\tau_n\omega}q(0)e^{i\omega\tau_n\theta} + \frac{i\overline{g}_{02}}{3\omega\tau_n}\overline{q}(0)e^{-i\omega\tau_n\theta} + M_1 e^{2i\omega\tau_n\theta},$$

$$W_{11}(\theta) = -\frac{ig_{11}}{\tau_n\omega}q(0)e^{i\omega\tau_n\theta} + \frac{i\overline{g}_{11}}{\omega\tau_n}\overline{q}(0)e^{-i\omega\tau_n} + M_2,$$

$$M_1^{(1)} = -\frac{4A_1^{(1)}i\omega + 2a_{12}A_1^{(2)}}{a_{21}a_{12} + 2b_{11}i\omega e^{-2i\omega\tau_n} + 4\omega^2 + 2a_{11}i\omega},$$

$$M_1^{(2)} = -\frac{2A_1^{(1)}a_{21} + \left(4i\omega - 2a_{21} - 2b_{11}e^{-2i\omega\tau_n}\right)A_1^{(2)}}{a_{21}a_{12} + 2b_{11}i\omega e^{-2i\omega\tau_n} + 4\omega^2 + 2a_{11}i\omega},$$

$$M_2^{(1)} = -\frac{2A_2^{(2)}}{a_{21}}, \qquad M_2^{(2)} = \frac{-2a_{21}A_2^{(1)} + 2(a_{11} + b_{11})A_2^{(2)}}{a_{12}a_{21}}.$$

$$A_1^{(1)} = F - \frac{\beta r}{(1 + bu_0 + cv_0)^2} - \frac{\alpha}{k}e^{-i\omega\tau_n\theta}, \qquad A_1^{(2)} = \frac{a\beta r}{(1 + bu_0 + cv_0)^2} - \frac{\beta abv_0}{(1 + bu_0 + cv_0)^3},$$

$$A_2^{(1)} = F - \frac{\beta\operatorname{Re}(r)}{(1 + bu_0 + cv_0)^2} - \frac{\alpha}{k}\operatorname{Re}\left(e^{i\omega\tau_n\theta}\right), \qquad A_2^{(2)} = \frac{a\beta\operatorname{Re}(\gamma)}{(1 + bu_0 + cv_0)^2} - \frac{\beta abv_0}{(1 + bu_0 + cv_0)^3}.$$

$$(3.24)$$

At last, we can calculate

$$C_1(0) = \frac{i}{2\omega\tau_n}\left(g_{11}g_{20} - 2\left|g_{11}^2\right| - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2},$$

$$\mu_2 = -\frac{\operatorname{Re}\{c_1(0)\}}{\operatorname{Re}\{\lambda'(\tau_n)\}}, \qquad \beta_2 = 2\operatorname{Re}\{c_1(0)\},$$

$$(3.25)$$

$$T_2 = -\frac{\operatorname{Im}\{c_1(0)\} + \mu_2\operatorname{Im}\{\lambda'(\tau_n)\}}{\omega\tau_n}.$$

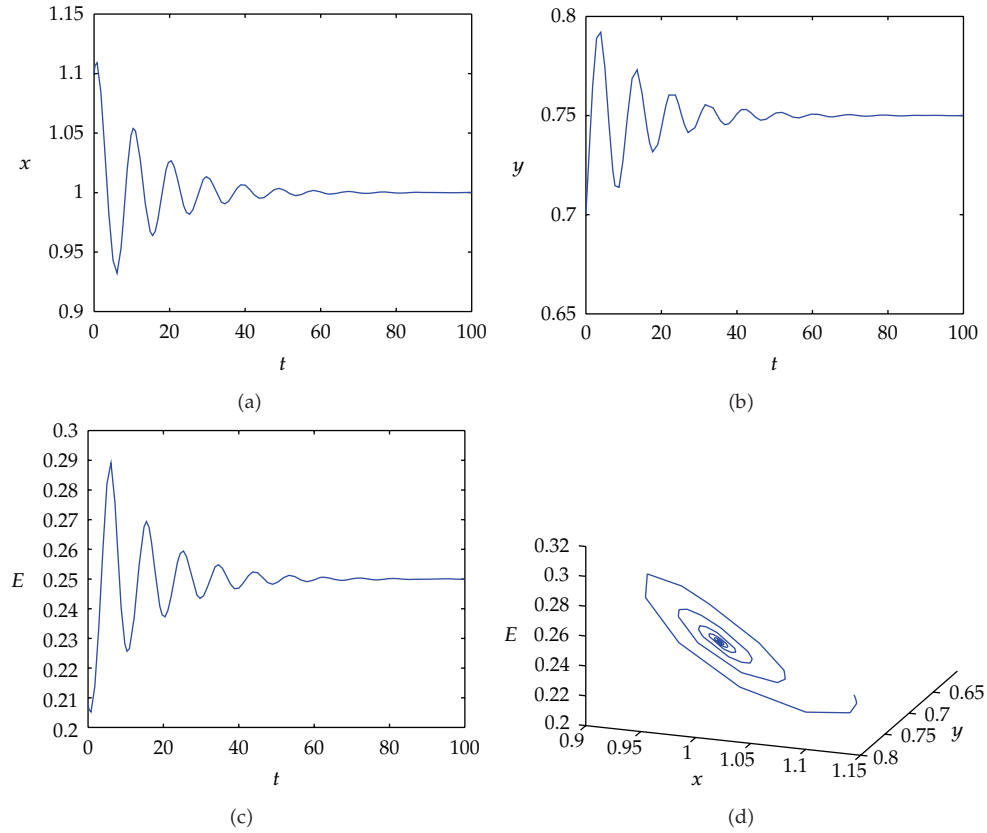Now we have a conclusion.

(a)

(b)

(c)

(d)

Figure 1: When $\tau = 0.1$, the positive equilibrium point Y_0 is asymptotically stable.

Conclusion 1. (1) The sign of μ_2 determines the direction of Hopf bifurcation. When $\mu_2 > 0$, the Hopf bifurcation is supercritical; when $\mu_2 < 0$, the Hopf bifurcation is subcritical.

(2) β_2 determines the stability of bifurcated periodic solutions. When $\beta_2 < 0$, the periodic solutions are stable; when $\beta_2 > 0$, the periodic solutions is unstable.

(3) T_2 determines the period of bifurcated periodic solutions. When $T_2 > 0$, the period increases; when $T_2 < 0$, the period decreases.

4. Numerical Simulations

In this section, we use some numerical simulations to illustrate the analytical results we obtained in previous sections.

Let $\alpha = \beta = 2$, $k = p = 2$, $a = b = c_0 = d = 1$, $c = 0$, and $m = 1/4$. The system (1.3) is

$$\dot{x} = 2\left(1 - \frac{x(t-\tau)}{2}\right)x(t) - \frac{2x(t)y(t)}{1+x(t)} - E(t)x(t),$$

$$\dot{y} = -y(t) + \frac{2x(t)y(t)}{1+x(t)}, \tag{4.1}$$

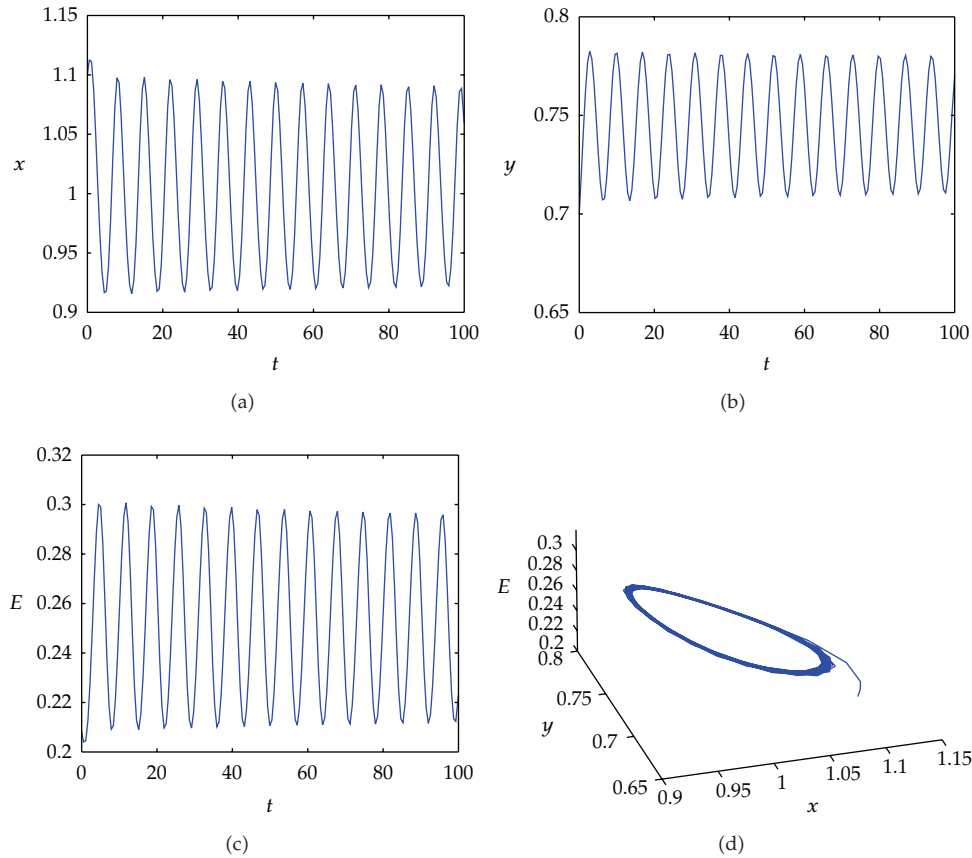$$0 = E(t)(2x(t) - 1) - \frac{1}{4}.$$

(a)

(b)

(c)

(d)

Figure 2: When $\tau = 0.5618$, periodic solutions occur from Y_0.

The only positive equilibrium point of (4.1) is $Y_0 = (1, 3/4, 1/4)$. By simple computation we have $\omega^- = 0.4164$ and $\omega^+ = 0.9005$. Substituting these parameters into (2.14) gives $\tau_0^- = 1.2136$, $\tau_0^+ = 0.5612$, $\tau_1^+ = 7.5386$, and $\tau_1^- = 16.2129$. Applying Theorem (2.5), we have that the positive equilibrium point of (4.1) is asymptotically stable when $\tau \in [0, 0.5612) \cup (1.2136, 7.5386)$. The results can be illustrated in Figures 1 and 3.

When $\tau > \tau_0^+ = 0.5612$ and is sufficiently near τ_0^+, the bifurcating periodic solution from positive equilibrium point of (4.1) occurs. Furthermore, we consider the properties of the Hopf bifurcation at $\tau = \tau_0^+ = 0.5612$. Based on Theorem (2.5) and conclusion 1, we get $c_1(0) = 0.1673 - 0.6835i$, and $\lambda'(\tau_0) = 0.9563 - 0.1856i$. This implies that

$$\mu_2 = -0.1749, \qquad \beta_2 = 0.3346, \qquad T_2 = 1.2883. \tag{4.2}$$

We conclude that the bifurcating periodic solution is subcritical and unstable. The results are illustrated in Figure 2.
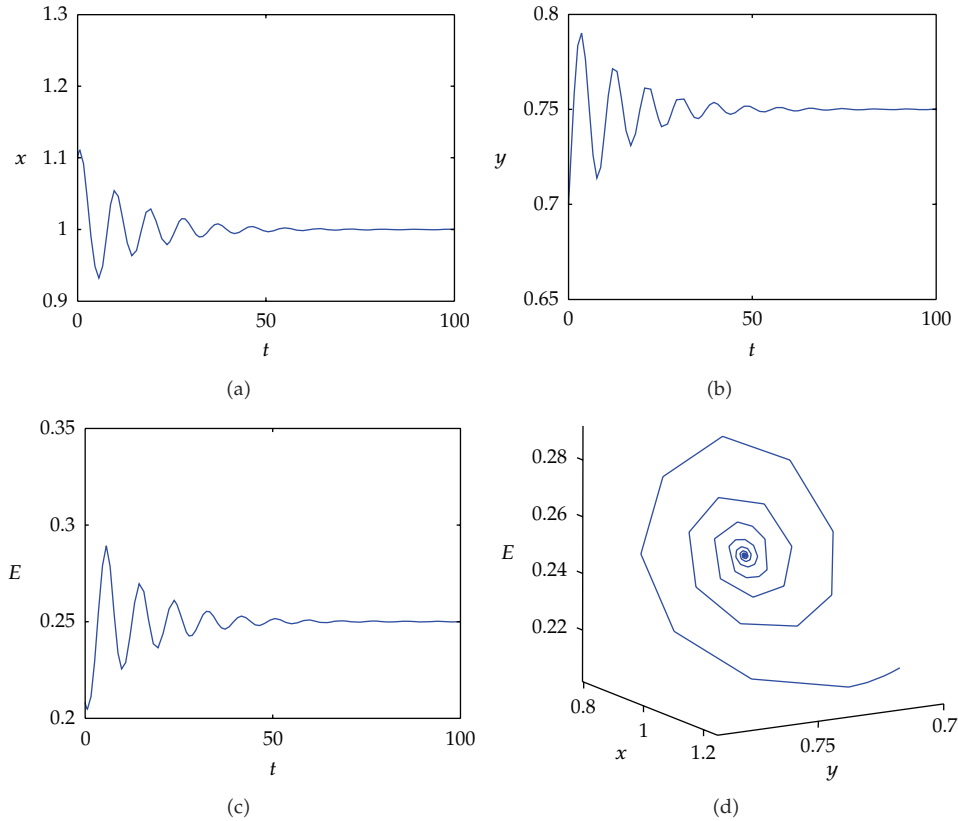
(a)

(b)

(c)

(d)

Figure 3: When $\tau = 1.3$, the positive equilibrium point Y_0 is stable.

Acknowledgment

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