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Research Article

Ruin Probability in Compound Poisson Process with Investment

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We consider that the surplus of an insurer follows compound Poisson process and the insurer would invest its surplus in risky assets, whose prices satisfy the Black-Scholes model. In the risk process, we decompose the ruin probability into the sum of two ruin probabilities which are caused by the claim and the oscillation, respectively. We derive the integro-differential equations for these ruin probabilities these ruin probabilities. When the claim sizes are exponentially distributed, third-order differential equations of the ruin probabilities are derived from the integro-differential equations and a lower bound is obtained.

1. Introduction

In classical risk models, the surplus process is defined as $U(t) = u + ct - \sum_{i=1}^{N(t)} X_i$, where $U(0) = u \geq 0$ is the initial surplus, $c > 0$ is the premium rate, $\{N(t),\ t \in \mathbb{R}^+\}$ is a homogeneous Poisson process with rate $\lambda > 0$, and $\{X_i,\ i \in \mathbb{N}^+\}$ is a sequence of independent and identically distributed (i.i.d.) nonnegative random variables with distribution F, denoting claim sizes.

In this paper, we assume that the surplus can be invested in risky assets. The price of the nth risky asset follows

$$dR_n(t) = \mu_n R_n(t)dt + \sigma_n R_n(t)dB_n(t), \quad R(0) > 0, \tag{1.1}$$

where $R(0)$ denotes the initial value of the risky asset, μ_n and σ_n are fixed constants, and $\{B_n(t),\ t > 0\}$ are standard Brownian motions. Moreover, $B_n(t)$ and $B_m(t)$ are correlated with $dB_m(t)dB_n(t) = \rho_{mn}dt,\ n, m = 1, 2, \ldots$. Furthermore, we assume that the insurance company invests a fixed proportion ω_n in nth risky asset. It is natural that $\sum_{n=1}^{N}\omega_n = 1$, but here we think that $\omega_n < 0$ is reasonable, which means that the investor can borrow money from risky market. Thus, the surplus of the insurance company can be expressed as

$$S(t) = u + ct - \sum_{i=1}^{N(t)} X_i + \sum_{n=1}^{N}\int_0^t \frac{\omega_n S(s)}{R_n(s)} dR_n(s),$$
$$S(0) = u. \tag{1.2}$$

Letting T denote the time of ruin, we have $T = \inf\{t \mid S(t) < 0\}$ and $T = \infty$ for all $t \geq 0$. The probability of ruin from initial surplus u is defined as

$$\Psi(u) = P(T < \infty \mid S(0) = u). \tag{1.3}$$

In this paper, we decompose the ruin probability into the sum of two ruin probabilities which are caused by the claim and the oscillation, respectively (see [1–3]). We denote by $T_s = \inf\{t \mid S(t) < 0,\ S(h) > 0,\ 0 < h < t\}$ and $T_s = \infty$ for all $t \geq 0$; namely, T_s is the ruin time at which ruin is caused by a claim. We also denote by $T_d = \inf\{t \mid S(t) = 0,\ S(h) > 0,\ 0 < h < t\}$ and $T_d = \infty$ for all $t \geq 0$; namely, T_d is the ruin time at which ruin is caused by oscillation. Clearly, $T = \min\{T_s, T_d\}$. Moreover, we denote ruin probabilities in the two situations, respectively, by $\Psi_s(u) = P(T_s < \infty \mid S(0) = u)$ and $\Psi_d(u) = P(T_d < \infty \mid S(0) = u)$. Clearly, the ruin probability $\Psi(u)$ can be decomposed as

$$\Psi(u) = \Psi_s(u) + \Psi_d(u). \tag{1.4}$$

In addition, it follows from the oscillating nature of the sample path of $\{S(t)\}$ that

$$\Psi_d(0) = \Psi(0) = 1, \qquad \Psi_s(0) = 0. \tag{1.5}$$

The ruin in the compound Poisson process by geometric Brownian motion has been studied extensively in the literature (see [4–7] and references therein). In [8–11], ruin in the compound Poisson risk process under interest force has been studied. In this paper, we investigate the ruin probabilities in the risk process (1.2). We first derive integro-differential equation for $\Psi(u)$, $\Psi_s(u)$, $\Psi_d(u)$. Secondly, we consider the special case when the claim sizes are exponentially distributed. Finally, we obtain a lower bound of $\Psi(u)$.

2. Integro-Differential Equation for Ruin Probability

In this section, we will derive integro-differential equations for $\Psi(u)$, $\Psi_s(u)$, $\Psi_d(u)$. These equations will be used in Section 3.

Theorem 2.1. *Assume that $\Psi(u)$ is twice continuously differentiable. Then, for any $u > 0$, $\Psi(u)$ satisfies the following integro-differential equation*

$$\lambda\Psi(u) = \left(c + \sum_{n=1}^{N}\omega_n\mu_n u\right)\Psi'(u) + \frac{1}{2}\left(\sum_{n=1}^{N}\omega_n^2\sigma_n^2\sum_{m=1}^{N}\rho_{mn}\right)u^2\Psi''(u) + \lambda\int_0^u \Psi(u-x)dF(x) + \lambda\overline{F}(u) \tag{2.1}$$

and the boundary condition is as follows:

$$\begin{aligned}\Psi(+\infty) &= 0,\\ \Psi(0) &= 1.\end{aligned} \tag{2.2}$$

Proof. Denote

$$U_t = u + ct + \sum_{n=1}^{N}\int_0^t \frac{\omega_n S(s)}{R_n(s)}dR_n(s), \qquad W_t = \sum_{i=1}^{N(t)} X_i. \tag{2.3}$$

Consider the surplus process $S(t)$ in an infinitesimal time interval $(0,t]$. Note that $N(t)$ is a homogeneous Poisson process, and there are three possible cases in $(0,t]$ as follows.

Case one. The probability is $1-\lambda t$ if here is no claim in $(0,t]$.

Case two. The probability is λt if there is only one claim in $(0,t]$.

Case three. The probability is $o(t)$ if there is more than one claim in $(0,t]$.

Therefore,

$$\begin{aligned}\Psi(u) &= (1-\lambda t)E[\Psi(U_t)] + \lambda t E\left[\int_0^\infty \Psi(U_t - x)dF(x)\right] + o(t)\\ &= (1-\lambda t)E[\Psi(U_t)] + \lambda t E\left[\int_0^{U_t} \Psi(U_t - x)dF(x)\right] + \lambda t E\left[\overline{F}(U_t)\right] + o(t).\end{aligned} \tag{2.4}$$

Let $\Delta = ct + \sum_{n=1}^{N}\int_0^t(\omega_n S(s)/R_n(s))dR_n(s)$. It follows from Taylor's formula that

$$E[\Psi(U_t)] = E[\Psi(u+\Delta)] = \Psi(u) + \Psi'(u)E(\Delta) + \frac{1}{2}\Psi''(u)E\left(\Delta^2\right) + E\left(o\left(\Delta^2\right)\right). \tag{2.5}$$

By Ito's formula, we have

$$\begin{aligned}\lim_{t\to 0}\frac{E(\Delta)}{t} &= \lim_{t\to 0}\frac{ct + E\left[\sum_{n=1}^{N}\int_0^t(\omega_n S(s)/R_n(s))dR_n(s)\right]}{t} = c + \sum_{n=1}^{N}\omega_n\mu_n u,\\ \lim_{t\to 0}\frac{E(\Delta^2)}{t} &= \lim_{t\to 0}\frac{E\left[\sum_{n=1}^{N}\int_0^t(\omega_n S(s)/R_n(s))dR_n(s)\right]^2 + o(t)}{t} = \sum_{n=1}^{N}\omega_n^2\sigma_n^2\mu_n u\sum_{m=1}^{N}\rho_{mn}u^2.\end{aligned} \tag{2.6}$$

Similarly,

$$\lim_{t\to 0}\frac{E\left[o(\Delta^2)\right]}{t}=0. \tag{2.7}$$

Hence,

$$\lim_{t\to 0}\frac{E[\Psi(U_t)]-\Psi(u)}{t}=\left(c+\sum_{n=1}^{N}\omega_n\mu_n u\right)\Psi'(u)+\frac{1}{2}\left(\sum_{n=1}^{N}\omega_n^2\sigma_n^2\sum_{m=1}^{N}\rho_{mn}\right)u^2\Psi''(u). \tag{2.8}$$

According to (2.4) and (2.8), we have

$$\begin{aligned}\lambda\Psi(u)&=\left(c+\sum_{n=1}^{N}\omega_n\mu_n u\right)\Psi'(u)+\frac{1}{2}\left(\sum_{n=1}^{N}\omega_n^2\sigma_n^2\sum_{m=1}^{N}\rho_{mn}\right)u^2\Psi''(u)\\&\quad+\lambda\int_0^u\Psi(u-x)dF(x)+\lambda\overline{F}(u).\end{aligned} \tag{2.9}$$

From the definition of $\Psi(u)$, we easily obtain the boundary condition. □

Theorem 2.2. *Assume that $\Psi_s(u)$ is twice continuously differentiable. Then, for any $u>0$, $\Psi_s(u)$ satisfies the following integro-differential equation:*

$$\begin{aligned}\lambda\Psi_s(u)&=\left(c+\sum_{n=1}^{N}\omega_n\mu_n u\right)\Psi_s'(u)+\frac{1}{2}\left(\sum_{n=1}^{N}\omega_n^2\sigma_n^2\sum_{m=1}^{N}\rho_{mn}\right)u^2\Psi_s''(u)\\&\quad+\lambda\int_0^u\Psi_s(u-x)dF(x)+\lambda\overline{F}(u),\end{aligned} \tag{2.10}$$

and the boundary condition is as follows:

$$\begin{aligned}\Psi_s(+\infty)&=0,\\ \Psi_s(0)&=0.\end{aligned} \tag{2.11}$$

Proof. Since the proof of this theorem is similar to that of Theorem 2.1, we omit it. □

Theorem 2.3. *Assume that $\Psi_d(u)$ is twice continuously differentiable. Then, for any $u>0$, $\Psi_d(u)$ satisfies the following integro-differential equation:*

$$\begin{aligned}\lambda\Psi_d(u)&=\left(c+\sum_{n=1}^{N}\omega_n\mu_n u\right)\Psi_d'(u)+\frac{1}{2}\left(\sum_{n=1}^{N}\omega_n^2\sigma_n^2\sum_{m=1}^{N}\rho_{mn}\right)u^2\Psi_d''(u)\\&\quad+\lambda\int_0^u\Psi_d(u-x)dF(x),\end{aligned} \tag{2.12}$$

and the boundary condition is as follows:

$$\Psi_d(+\infty) = 0,$$
$$\Psi_d(0) = 1. \tag{2.13}$$

Proof. It follows from (1.4) that $\Psi'(u) = \Psi'_s(u) + \Psi'_d(u)$ and $\Psi''(u) = \Psi'_s(u) + \Psi''_d(u)$. Adding (2.1) to (2.10), we obtain (2.12). From the definition of $\Psi_d(u)$, we easily obtain the boundary condition. □

3. Ruin with Exponential Claim Sizes

In this section, we will derive the differential equations for $\Psi(u)$, $\Psi_s(u)$, and $\Psi_d(u)$ with exponential claim sizes, respectively. When $\lambda = 0$, the surplus process (1.2) reduces to a pure diffusion risk model, in which the surplus follows Brownian motions with drift, and we can get lower bounds of the ruin probabilities. For simplicity, we denote

$$h_1(u) = c + au,$$
$$h_2(u) = \frac{1}{2}bu^2, \tag{3.1}$$

where

$$a = \sum_{n=1}^{N} \omega_n \mu_n,$$
$$b = \sum_{n=1}^{N} \omega_n^2 \sigma_n^2 \sum_{m=1}^{N} \rho_{mn}. \tag{3.2}$$

Corollary 3.1. *Under the assumptions of Theorems 2.1–2.3, if $F(x) = 1 - e^{-\beta x}$, $x > 0$, $\beta > 0$, then, for any $u > 0$, $\Psi(u)$, $\Psi_s(u)$, and $\Psi_d(u)$ satisfy the following integro-differential equation:*

$$h_2(u)\xi'''(u) + \left[h_1(u) + h_2'(u) + \beta h_2(u)\right]\xi''(u) + \left[\beta h_1(u) + h_1'(u) - \lambda\right]\xi'(u) = 0, \tag{3.3}$$

where $\xi(u)$ is any of $\Psi(u)$, $\Psi_s(u)$, and $\Psi_d(u)$. The boundary conditions of $\Psi(u)$, $\Psi_s(u)$ and $\Psi_d(u)$ are as follows:

$$c\Psi'(0^+) = 0,$$
$$c\Psi'_s(0^+) = -\lambda, \tag{3.4}$$
$$c\Psi'_d(0^+) = \lambda.$$

Proof. When F is a exponential distribution, (2.1) can be expressed as Theorems 2.1–2.3, and we obtain the following equation:

$$\lambda\Psi(u) - h_1(u)\Psi'(u) - h_2(u)\Psi''(u) = \lambda e^{-\beta u}\left(\beta\int_0^u \Psi(x)e^{\beta x}dx + 1\right). \tag{3.5}$$

Taking derivative about u on both sides of (3.5), we obtain

$$\begin{aligned}\lambda\beta\Psi(u) &- \left(\lambda - h_1'(u)\right)\Psi'(u) + \left[h_1(u) + h_2'(u)\right]\Psi''(u) + h_2(u)\Psi'''(u)\\ &= \lambda\beta e^{-\beta u}\left(\beta\int_0^u \Psi(x)e^{\beta x}dx + 1\right).\end{aligned} \tag{3.6}$$

According to the above expression and (2.1), (3.3) holds. The boundary condition of $\Psi(u)$ in (3.4) can be derived if we let $u \to 0^+$. Similarly, we can derive the third-order differential equations for $\Psi_s(u)$ and $\Psi_d(u)$. □

If $\lambda = 0$, then the surplus process (1.2) reduces to a pure diffusion risk model, in which the surplus follows Brownian motions with drift. Denote the ruin probability in this pure diffusion risk process by $\Psi_0(u)$. Clearly,

$$\Psi(u) \geq \Psi_0(u), \quad u \geq 0. \tag{3.7}$$

Equation (2.1) implies that $\Psi_0(u)$ satisfies the following differential equation:

$$(c + au)\Psi'(u) + \frac{1}{2}bu^2\Psi''(u) = 0. \tag{3.8}$$

The solution of (3.8) is given by $\Psi_0(u) = 1 - g(u)/g(+\infty)$, where

$$g(u) = \int_0^u \left(bx^2\right)^{-(a/b)} e^{(2c/bx)}dx. \tag{3.9}$$

Thus, we obtain a lower bound for $\Psi(u)$.

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