Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2012, Article ID 302830, 17 pages
doi:10.1155/2012/302830

Research Article

On Approximate Coincidence Point Properties and Their Applications to Fixed Point Theory

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Received 29 October 2011; Revised 21 January 2012; Accepted 23 January 2012

Academic Editor: Giuseppe Marino

We first establish some existence results concerning approximate coincidence point properties and approximate fixed point properties for various types of nonlinear contractive maps in the setting of cone metric spaces and general metric spaces. From these results, we present some new coincidence point and fixed point theorems which generalize Berinde-Berinde's fixed point theorem, Mizoguchi-Takahashi's fixed point theorem, and some well-known results in the literature.

1. Introduction

Let (X, d) be a metric space. For each $x \in X$ and $A \subseteq X$, let $d(x, A) = \inf_{y\in A} d(x, y)$. Denote by $\mathcal{N}(X)$ the class of all nonempty subsets of X, $\mathcal{C}(X)$ the collection of all nonempty closed subsets of X, and $\mathcal{CB}(X)$ the family of all nonempty closed and bounded subsets of X. A function $\mathcal{H} : \mathcal{CB}(X) \times \mathcal{CB}(X) \to [0, \infty)$ defined by

$$\mathcal{H}(A, B) = \max\left\{ \sup_{x\in B} d(x, A), \sup_{x\in A} d(x, B) \right\} \tag{1.1}$$

is said to be the *Hausdorff metric* on $\mathcal{CB}(X)$ induced by the metric d on X. Let $T : X \to \mathcal{N}(X)$ be a multivalued map. A point x in X is a fixed point of T if $x \in Tx$. The set of fixed points of T is denoted by $\mathcal{F}(T)$. Let $g : X \to X$ be a single-valued self-map and $T : X \to \mathcal{N}(X)$ be a multivalued map. A point x in X is said to be a *coincidence point* (see, for instance, [1, 2]) of g and T if $gx \in Tx$. The set of coincidence point of g and T is denoted by $\mathcal{COP}(g, T)$. Throughout this paper, we denote by $\mathbb{N}$ and $\mathbb{R}$ the sets of positive integers and real numbers, respectively.

Let f be a real-valued function defined on $\mathbb{R}$. For $c \in \mathbb{R}$, we recall that

$$\begin{aligned} \limsup_{x \to c} f(x) &= \inf_{\varepsilon>0} \sup_{0<|x-c|<\varepsilon} f(x), \\ \limsup_{x \to c^+} f(x) &= \inf_{\varepsilon>0} \sup_{0<x-c<\varepsilon} f(x). \end{aligned} \tag{1.2}$$

Definition 1.1 (see [2–9]). A function $\varphi : [0,\infty) \to [0,1)$ is said to be an $\mathcal{MT}$*-function* (or $\mathcal{R}$*-function*) if

$$\limsup_{s \to t^+} \varphi(s) < 1, \quad \forall t \in [0,\infty). \tag{1.3}$$

It is obvious that if $\varphi : [0,\infty) \to [0,1)$ is a nondecreasing function or a nonincreasing function, then φ is an $\mathcal{MT}$-function. So the set of $\mathcal{MT}$-functions is a rich class.

Very recently, Du [7] first proved some characterizations of $\mathcal{MT}$-functions.

Theorem D (see [7]). *Let $\varphi : [0,\infty) \to [0,1)$ be a function. Then, the following statements are equivalent.*

(a) *φ is an $\mathcal{MT}$-function.*

(b) *For each $t \in [0,\infty)$, there exist $r_t^{(1)} \in [0,1)$ and $\varepsilon_t^{(1)} > 0$ such that $\varphi(s) \le r_t^{(1)}$ for all $s \in (t, t+\varepsilon_t^{(1)})$.*

(c) *For each $t \in [0,\infty)$, there exist $r_t^{(2)} \in [0,1)$ and $\varepsilon_t^{(2)} > 0$ such that $\varphi(s) \le r_t^{(2)}$ for all $s \in [t, t+\varepsilon_t^{(2)}]$.*

(d) *For each $t \in [0,\infty)$, there exist $r_t^{(3)} \in [0,1)$ and $\varepsilon_t^{(3)} > 0$ such that $\varphi(s) \le r_t^{(3)}$ for all $s \in (t, t+\varepsilon_t^{(3)}]$.*

(e) *For each $t \in [0,\infty)$, there exist $r_t^{(4)} \in [0,1)$ and $\varepsilon_t^{(4)} > 0$ such that $\varphi(s) \le r_t^{(4)}$ for all $s \in [t, t+\varepsilon_t^{(4)})$.*

(f) *For any nonincreasing sequence $\{x_n\}_{n\in\mathbb{N}}$ in $[0,\infty)$, we have $0 \le \sup_{n\in\mathbb{N}} \varphi(x_n) < 1$.*

(g) *φ is a function of contractive factor [5]; that is, for any strictly decreasing sequence $\{x_n\}_{n\in\mathbb{N}}$ in $[0,\infty)$, we have $0 \le \sup_{n\in\mathbb{N}} \varphi(x_n) < 1$.*

It is worth to mention that there exist functions which are not $\mathcal{MT}$-functions. For example, let $\varphi : [0,\infty) \to [0,1)$ be defined by

$$\varphi(t) := \begin{cases} \dfrac{\sin t}{t}, & \text{if } t \in \left(0, \dfrac{\pi}{2}\right], \\ 0, & \text{otherwise.} \end{cases} \tag{1.4}$$

Since $\limsup_{s \to 0^+} \varphi(s) = 1$, φ is not an $\mathcal{MT}$-function.

Let $g : X \to X$ be a single-valued self-map. Recall that a multivalued map $T : X \to \mathcal{N}(X)$ is called

(1) a *multivalued k-contraction* [1, 10, 11], if there exists a number $0 < k < 1$ such that

$$\mathcal{H}(Tx, Ty) \leq kd(x, y), \quad \forall x, y \in X, \tag{1.5}$$

(2) a *multivalued (θ, L)-almost contraction* [1, 12], if there exist two constants $\theta \in (0, 1)$ and $L \geq 0$ such that

$$\mathcal{H}(Tx, Ty) \leq \theta d(x, y) + Ld(y, Tx), \quad \forall x, y \in X, \tag{1.6}$$

(3) a *generalized multivalued almost contraction* [1, 12], if there exists an $\mathcal{MT}$-function φ and $L \geq 0$ such that

$$\mathcal{H}(Tx, Ty) \leq \varphi(d(x, y))d(x, y) + Ld(y, Tx), \quad \forall x, y \in X, \tag{1.7}$$

(4) a *multivalued almost g-contraction* [1, 12, 13], if there exist two constants $\theta \in (0, 1)$ and $L \geq 0$ such that

$$\mathcal{H}(Tx, Ty) \leq \theta d(gx, gy) + Ld(gy, Tx), \quad \forall x, y \in X, \tag{1.8}$$

(5) a *generalized multivalued almost g-contraction* [1], if there exists an $\mathcal{MT}$-function φ and $L \geq 0$ such that

$$\mathcal{H}(Tx, Ty) \leq \varphi(d(gx, gy))d(gx, gy) + Ld(gy, Tx), \quad \forall x, y \in X, \tag{1.9}$$

In 1989, Mizoguchi and Takahashi [14] proved the following fixed point theorem which is a generalization of Nadler's fixed point theorem [10] and the celebrated Banach contraction principle (see, e.g., [11]). It is worth to mention that Mizoguchi-Takahashi's fixed point theorem gave a partial answer of Problem 9 in Reich [15] and it's primitive proof is difficult. Recently, Suzuki [16] presented a very simple proof of Mizoguchi-Takahashi's fixed point theorem.

Theorem MT (Mizoguchi and Takahashi). *Let (X, d) be a complete metric space, $T : X \to \mathcal{CB}(X)$ be a multivalued map, and $\varphi : [0, \infty) \to [0, 1)$ be a $\mathcal{MT}$-function. Assume that*

$$\mathcal{H}(Tx, Ty) \leq \varphi(d(x, y))d(x, y), \quad \forall x, y \in X, \tag{1.10}$$

Then, $\mathcal{F}(T) \neq \emptyset$.

Since then a number of generalizations in various different directions of Mizoguchi-Takahashi's fixed point theorem have been investigated by several authors in the past. In 2007, M. Berinde and V. Berinde [12] proved the following interesting fixed point theorem to generalize Mizoguchi-Takahashi's fixed point theorem.

Theorem BB (M. Berinde and V. Berinde). *Let (X, d) be a complete metric space, $T : X \to \mathcal{CB}(X)$ be a multivalued map, $\varphi : [0, \infty) \to [0, 1)$ be a $\mathcal{MT}$-function, and $L \geq 0$. Assume that*

$$\mathcal{H}(Tx, Ty) \leq \varphi(d(x, y))d(x, y) + Ld(y, Tx), \quad \forall x, y \in X, \tag{1.11}$$

that is, T is a generalized multivalued almost contraction. Then, $\mathcal{F}(T) \neq \emptyset$.

In [3], the author established some new fixed point theorems for nonlinear multivalued contractive maps by using τ^0-metrics (see [3, Def. 1.2]), τ^0-metrics (see [3, Def. 1.3]), and $\mathcal{MT}$-functions. Applying those results, the author gave the generalizations of Berinde-Berinde's fixed point theorem, Mizoguchi-Takahashi's fixed point theorem, Nadler's fixed point theorem, and Banach contraction principle, Kannan's fixed point theorems, and Chatterjea's fixed point theorems for nonlinear multivalued contractive maps in complete metric spaces; for more detail, one can refer to [3].

Very recently, Du [17] first introduced the concepts of TVS-cone metric and TVS-cone metric space to improve and extend the concept of cone metric space in the sense of Huang and Zhang [18].

Definition 1.2 (see [17]). Let X be a nonempty set and Y a locally convex Hausdorff t.v.s. with its zero vector θ, K a proper, closed, convex, and pointed cone in Y, and $\precsim_K$ a partial ordering with respect to K defined by

$$x \precsim_K y \Longleftrightarrow y - x \in K. \tag{1.12}$$

A vector-valued function $p : X \times X \to Y$ is said to be a *TVS-cone metric*, if the following conditions hold:

(C1) $\theta \precsim_K p(x, y)$ for all $x, y \in X$ and $p(x, y) = \theta$ if and only if $x = y$,

(C2) $p(x, y) = p(y, x)$ for all $x, y \in X$,

(C3) $p(x, z) \precsim_K p(x, y) + p(y, z)$ for all $x, y, z \in X$.

The pair (X, p) is then called a *TVS-cone metric space*.

In this paper, we first establish some existence results concerning approximate coincidence point property and approximate fixed point property for various types of nonlinear contractive maps in the setting of cone metric spaces and general metric spaces. From these results, we present some new coincidence point and fixed point theorems which generalize Berinde-Berinde's fixed point theorem and Mizoguchi-Takahashi's fixed point theorem. Our results generalize and improve some recent results in [1–6, 10–19] and references therein.

2. Preliminaries

Let E be a topological vector space (t.v.s. for short) with its zero vector θ_E. A nonempty subset K of E is called a *convex cone* if $K + K \subseteq K$ and $\lambda K \subseteq K$ for $\lambda \geq 0$. A convex cone K is said to be *pointed* if $K \cap (-K) = \{\theta_E\}$. For a given proper, pointed, and convex cone K in E, we can define a partial ordering $\precsim_K$ with respect to K by

$$x \precsim_K y \Longleftrightarrow y - x \in K. \tag{2.1}$$

$x \prec_K y$ will stand for $x \precsim_K y$ and $x \neq y$, while $x \ll_K y$ will stand for $y - x \in \operatorname{int} K$, where $\operatorname{int} K$ denotes the interior of K.

In the following, unless otherwise specified, we always assume that Y is a locally convex Hausdorff t.v.s. with its zero vector θ, K is a proper, closed, convex, and pointed cone in Y with $\operatorname{int} K \neq \emptyset$, $\precsim_K$ is a partial ordering with respect to K and $e \in \operatorname{int} K$ is fixed.

Recall that the nonlinear scalarization function $\xi_e : Y \to \mathbb{R}$ is defined by

$$\xi_e(y) = \inf\{r \in \mathbb{R} : y \in re - K\}, \quad \forall y \in Y. \tag{2.2}$$

Theorem 2.1 (see [6, 17, 20, 21]). *For each $r \in \mathbb{R}$ and $y \in Y$, the following statements are satisfied:*

(i) $\xi_e(y) \leq r \Leftrightarrow y \in re - K$,

(ii) $\xi_e(y) > r \Leftrightarrow y \notin re - K$,

(iii) $\xi_e(y) \geq r \Leftrightarrow y \notin re - \operatorname{int} K$,

(iv) $\xi_e(y) < r \Leftrightarrow y \in re - \operatorname{int} K$,

(v) $\xi_e(\cdot)$ *is positively homogeneous and continuous on Y,*

(vi) *if $y_1 \in y_2 + K$ (i.e., $y_2 \precsim_K y_1$), then $\xi_e(y_2) \leq \xi_e(y_1)$,*

(vii) $\xi_e(y_1 + y_2) \leq \xi_e(y_1) + \xi_e(y_2)$ *for all $y_1, y_2 \in Y$.*

Clearly, $\xi_e(\theta) = 0$. Notice that the reverse statement of (vi) in Theorem 2.1 (i.e., $\xi_e(y_2) \leq \xi_e(y_1) \Rightarrow y_1 \in y_2 + K$) does not hold in general. We illustrate the truth with the following simple example.

Example A. Let $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2 : x, y \geq 0\}$, and $e = (1, 1)$. Then K is a proper, closed, convex, and pointed cone in Y with $\operatorname{int} K = \{(x, y) \in \mathbb{R}^2 : x, y > 0\} \neq \emptyset$ and $e \in \operatorname{int} K$. For $r = 1$, it is easy to see that $y_1 = (10, -30) \notin re - \operatorname{int} K$, and $y_2 = (0, 0) \in re - \operatorname{int} K$. By applying (iii) and (iv) of Theorem 2.1, we have $\xi_e(y_2) < 1 \leq \xi_e(y_1)$ while $y_1 \notin y_2 + K$.

Definition 2.2 (see [17]). Let (X, p) be a *TVS*-cone metric space, $x \in X$, and $\{x_n\}_{n\in\mathbb{N}}$ a sequence in X.

(i) $\{x_n\}$ is said to *TVS-cone converge* to x if, for every $c \in Y$ with $\theta \ll_K c$, there exists a natural number $\mathbb{N}_0$ such that $p(x_n, x) \ll_K c$ for all $n \geq \mathbb{N}_0$. We denote this by *cone*-$\lim_{n\to\infty} x_n = x$ or $x_n \overset{\text{cone}}{\to} x$ as $n \to \infty$ and call x the *TVS-cone limit* of $\{x_n\}$.

(ii) $\{x_n\}$ is said to be a *TVS-cone Cauchy sequence* if, for every $c \in Y$ with $\theta \ll_K c$, there is a natural number $\mathbb{N}_0$ such that $p(x_n, x_m) \ll_K c$ for all $n, m \geq \mathbb{N}_0$.

(iii) (X, p) is said to be *TVS-cone complete* if every *TVS*-cone Cauchy sequence in X is *TVS-cone* convergent in X.

In [17], the author proved the following important results.

Theorem 2.3 (see [17]). *Let (X, p) be a TVS-cone metric spaces. Then, $d_p : X \times X \to [0, \infty)$ defined by $d_p := \xi_e \circ p$ is a metric.*

Example B. Let $X = [0, 1]$, $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2 : x, y \geq 0\}$, $e = (1, 1)$, and $\theta = (0, 0)$. Define $p : X \times X \to Y$ by

$$p(x, y) = (|x - y|, 8|x - y|). \tag{2.3}$$

Then, (X, p) is a TVS-cone complete metric space and

$$d_p(x, y) = \xi_e(p(x, y)) = \inf\{r \in \mathbb{R} : p(x, y) \in re - K\} = 8|x - y|. \tag{2.4}$$

So d_p is a metric on X and (X, d_p) is a complete metric space.

Theorem 2.4 (see [17]). *Let (X, p) be a TVS-cone metric space, $x \in X$, and $\{x_n\}_{n \in \mathbb{N}}$ a sequence in X. Then, the following statements hold.*

(a) *If $\{x_n\}$ TVS-cone converges to x (i.e., $x_n \xrightarrow{\text{cone}} x$ as $n \to \infty$), then $d_p(x_n, x) \to 0$ as $n \to \infty$ (i.e., $x_n \xrightarrow{d_p} x$ as $n \to \infty$).*

(b) *If $\{x_n\}$ is a TVS-cone Cauchy sequence in (X, p), then $\{x_n\}$ is a Cauchy sequence (in usual sense) in (X, d_p).*

Definition 2.5 (see [1, 22]). Let (X, d) be a metric space. A multivalued map $T : X \to \mathcal{N}(X)$ is said to have an approximate fixed point property provided

$$\inf_{x \in X} d(x, Tx) = 0, \tag{2.5}$$

or, equivalently, for any $\epsilon > 0$, there exists $z \in X$ such that

$$d(z, Tz) \leq \epsilon, \tag{2.6}$$

or, equivalently, for any $\epsilon > 0$, there exists $x_\epsilon \in X$ such that

$$T(x_\epsilon) \cap B(x_\epsilon, \epsilon) \neq \emptyset, \tag{2.7}$$

where $B(x, r)$ denotes a closed ball of radius r centered at x.

It is known that every generalized multivalued almost contraction in a metric space (X, d) has the approximate fixed point property (see [1, Lemma 2.2]).

Remark 2.6. It is obvious that $\mathcal{F}(T) \neq \emptyset$ implies that T has the approximate fixed point property.

Definition 2.7 (see [1]). Let (X, d) be a metric space, $g : X \to X$ a single-valued self-map, and $T : X \to \mathcal{N}(X)$ a multivalued map. The maps g and T are said to have an approximate coincidence point property provided

$$\inf_{x \in X} d(gx, Tx) = 0, \tag{2.8}$$

or, equivalently, for any $\epsilon > 0$, there exists $z \in X$ such that

$$d(gz, Tz) \leq \epsilon. \tag{2.9}$$

It is known that every generalized multivalued almost g-contraction in a metric space (X, d) has the approximate coincidence point property provided each Tx is g-invariant (i.e., $g(Tx) \subseteq Tx$) for each $x \in X$ (see [1, Theorem 2.7]).

3. Main Results

For any locally convex Hausdorff t.v.s. Y with its zero vector θ, let τ denote the topology of Y and let $\mathcal{U}_\tau$ be the base at θ consisting of all absolutely convex neighborhood of θ. Let

$$\mathcal{L} = \{\ell : \ell \text{ is a Minkowski functional of } U \text{ for } U \in \mathcal{U}_\tau\}. \tag{3.1}$$

Then, $\mathcal{L}$ is a family of seminorms on Y. For each $\ell \in \mathcal{L}$, put

$$V(\ell) = \{y \in Y : \ell(y) < 1\} \tag{3.2}$$

and denote

$$\begin{gathered} \mathcal{U}_{\mathcal{L}} = \{U : U = r_1 V(\ell_1) \cap r_2 V(\ell_2) \cap \cdots \cap r_n V(\ell_n), \\ r_k > 0,\ \ell_k \in \mathcal{L},\ 1 \leq k \leq n,\ n \in \mathbb{N}\}. \end{gathered} \tag{3.3}$$

Then, $\mathcal{U}_{\mathcal{L}}$ is a base at θ and the topology $\Gamma_{\mathcal{L}}$ generated by $\mathcal{U}_{\mathcal{L}}$ is the weakest topology for Y such that all seminorms in $\mathcal{L}$ are continuous and $\tau = \Gamma_{\mathcal{L}}$. Moreover, given any neighborhood $\mathcal{O}_\theta$ of θ, there exists $U \in \mathcal{U}_{\mathcal{L}}$ such that $\theta \in U \subset \mathcal{O}_\theta$; for more detail, we refer the reader to ([23, Theorem 12.4 in II.12, Page 113]).

The following lemmas are very crucial to our main results.

Lemma 3.1. *Let O_θ be a neighborhood of θ, $\omega \in K$, and $\{\alpha_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} \alpha_n = 0$. Suppose that $U \in \mathcal{U}_{\mathcal{L}}$ satisfies $\theta \in U \subset O_\theta$, then there exists $n_0 \in \mathbb{N}$, such that $\pm\alpha_n \omega \in U$ for all $n \geq n_0$.*

Proof. Clearly, if $\omega = \theta$, then $\pm\alpha_n \omega = \theta \in U$ for all $n \in \mathbb{N}$, and we are done. Suppose $\omega \in K \setminus \{\theta\}$. Since $U \in \mathcal{U}_{\mathcal{L}}$,

$$U = r_1 V(\ell_1) \cap r_2 V(\ell_2) \cap \cdots \cap r_s V(\ell_s), \tag{3.4}$$

for some $r_i > 0$, $\ell_i \in \mathcal{L}$ and $1 \leq i \leq s$. Let

$$\begin{aligned} \delta &= \min\{r_i : 1 \leq i \leq s\} > 0, \\ \rho &= \max\{\ell_i(\omega) : 1 \leq i \leq s\}. \end{aligned} \tag{3.5}$$

We need consider two possible cases.

Case 1. If $\rho = 0$, since each ℓ_i is a seminorm, we have $\ell_i(\omega) = 0$ and

$$\ell_i(\pm\alpha_n\omega) = \alpha_n\ell_i(\omega) = 0 < r_i \tag{3.6}$$

for all $1 \leq i \leq s$ and all $n \in \mathbb{N}$.

Case 2. If $\rho > 0$, since $\lim_{n\to\infty}\alpha_n = 0$, there exists $n_0 \in \mathbb{N}$ such that $\alpha_n < \delta/\rho$ for all $n \geq n_0$. So, for each $i \in \{1, 2, \ldots, s\}$ and any $n \geq n_0$, we obtain

$$\begin{aligned} \ell_i(\pm\alpha_n\omega) &= \alpha_n\ell_i(\omega) \\ &< \frac{\delta}{\rho}\ell_i(\omega) \\ &\leq \delta \\ &\leq r_i. \end{aligned} \tag{3.7}$$

Hence, by Cases 1 and 2, we get that, for any $n \geq n_0$, $\pm\alpha_n\omega \in r_iV(\ell_i)$ for all $1 \leq i \leq s$. Therefore, $\pm\alpha_n\omega \in U$ for all $n \geq n_0$. □

Lemma 3.2 (see [24, Lemma 2.5]). *Let E be a t.v.s., K a convex cone with $\operatorname{int} K \neq \emptyset$ in E, and $a, b, c \in E$. Then, the following statements hold.*

(a) $\operatorname{int} K + K \subseteq \operatorname{int} K$.

(b) *If $a \precsim_K b$ and $b \ll_K c$, then $a \ll_K c$.*

In this section, we first establish an existence theorem related to approximate coincidence point property for maps in *TVS*-cone complete metric space which is one of the main results of this paper. It will have many applications to study metric fixed point theory.

It is worth observing that the following existence theorem does not require the *TVS*-cone completeness assumption on *TVS*-cone metric space (X, p).

Theorem 3.3. *Let (X, p) be a TVS-cone metric space, $T : X \to \mathcal{N}(X)$ a multivalued map, $h : X \to X$ a self-map, and $d_p := \xi_e \circ p$. Suppose that*

(D1) *there exists an $\mathcal{MT}$-function $\mu : [0,\infty) \to [0,1)$ such that, for each $x \in X$, if $y \in Tx$ with $y \neq x$ then there exists $z \in Ty$ such that*

$$p(hy, hz) \precsim_K \mu(d_p(hx, hy))p(hx, hy), \tag{3.8}$$

(D2) *Tx is h-invariant (i.e., $h(Tx) \subseteq Tx$) for each $x \in X$.*

Then, the following statements hold.

(a) *h and T have the d_p-approximate coincidence point property on X (i.e.,* $\inf_{x\in X} d_p(hx, Tx) = 0$*).*

(b) *There exists a sequence $\{z_n\}$ in (X, p) such that $\{z_n\}$ is a TVS-cone Cauchy sequence and* $\lim_{n\to\infty} d_p(z_n, z_{n+1}) = \inf_{x\in X} d_p(hx, Tx) = 0$.

Proof. By Theorem 2.3, we know that d_p is a metric on X. Let $x_1 \in X$ and $x_2 \in Tx_1$. If $x_1 = x_2$, then, by (D2), we have $hx_1 \in Tx_1$. Since

$$\inf_{x\in X} d_p(hx, Tx) \le d_p(hx_1, Tx_1) = 0, \tag{3.9}$$

we have $\inf_{x\in X} d_p(hx, Tx) = 0$. Let $z_n = hx_1$ for all $n \in \mathbb{N}$. Then, $\{z_n\} \subset X$ is a *TVS*-cone Cauchy sequence and $\lim_{n\to\infty} d_p(z_n, z_{n+1}) = 0 = \inf_{x\in X} d_p(hx, Tx)$. Hence, all conclusions are proved in the case of $x_1 = x_2$. If $x_2 \ne x_1$ or $p(x_1, x_2) \ne \theta$, then, by (D1), there exists $x_3 \in Tx_2$ such that

$$p(hx_2, hx_3) \precsim_K \mu(d_p(hx_1, hx_2))p(hx_1, hx_2). \tag{3.10}$$

If $x_2 = x_3$, then the conclusions also hold by following a similar argument as above. If $x_3 \ne x_2$, then there exists $x_4 \in Tx_3$ such that

$$p(hx_3, hx_4) \precsim_K \mu(d_p(hx_2, hx_3))p(hx_2, hx_3). \tag{3.11}$$

By induction, we can obtain a sequence $\{x_n\}$ in X satisfying the following: for each $n \in \mathbb{N}$,

$$\begin{gathered} x_{n+1} \in Tx_n, \\ p(hx_{n+1}, hx_{n+2}) \precsim_K \mu(d_p(hx_n, hx_{n+1}))p(hx_n, hx_{n+1}). \end{gathered} \tag{3.12}$$

Applying (v) and (vi) of Theorem 2.1, the inequality (3.12) implies

$$\begin{aligned} d_p(hx_{n+1}, hx_{n+2}) &= \xi_e(p(hx_{n+1}, hx_{n+2})) \\ &\le \xi_e(\mu(d_p(hx_n, hx_{n+1}))p(hx_n, hx_{n+1})) \\ &\le \mu(d_p(hx_n, hx_{n+1}))\xi_e(p(hx_n, hx_{n+1})) \\ &= \mu(d_p(hx_n, hx_{n+1}))d_p(hx_n, hx_{n+1}). \end{aligned} \tag{3.13}$$

Since $\mu(t) < 1$ for all $t \in [0, \infty)$, the sequence $\{d_p(hx_n, hx_{n+1})\}_{n\in\mathbb{N}}$ is strictly decreasing in $[0, \infty)$. Since μ is an $\mathcal{MT}$-function, by (g) of Theorem D,

$$0 \le \sup_{n\in\mathbb{N}} \mu(d_p(hx_n, hx_{n+1})) < 1. \tag{3.14}$$

Let $\lambda := \sup_{n\in\mathbb{N}} \mu(d_p(hx_n, hx_{n+1}))$. So $\lambda \in [0, 1)$. From (3.12), we get

$$p(hx_{n+1}, hx_{n+2}) \precsim_K \mu(d_p(hx_n, hx_{n+1}))p(hx_n, hx_{n+1}) \precsim_K \lambda p(hx_n, hx_{n+1}). \tag{3.15}$$

It follows from (3.15) that

$$p(hx_{n+1}, hx_{n+2}) \precsim_K \lambda p(hx_n, hx_{n+1}) \precsim_K \cdots \precsim_K \lambda^n p(hx_1, hx_2) \quad \text{for each } n \in \mathbb{N}. \tag{3.16}$$

Let $z_n := hx_n$, $n \in \mathbb{N}$. Applying (v) and (vi) of Theorem 2.1, the inequality (3.16) implies

$$d_p(z_{n+1}, z_{n+2}) \leq \lambda^n d_p(z_1, z_2) \quad \text{for each } n \in \mathbb{N}. \tag{3.17}$$

Since $\lambda \in [0,1)$, by (3.17), we have $\lim_{n\to\infty} d_p(z_n, z_{n+1}) = 0$. By (3.12) and (D2), we obtain $z_{n+1} = hx_{n+1} \in Tx_n$, $n \in \mathbb{N}$. It implies that

$$\inf_{x\in X} d_p(hx, Tx) \leq d_p(hx_n, Tx_n) \leq d_p(z_n, z_{n+1}), \quad \forall n \in \mathbb{N}. \tag{3.18}$$

Since $d_p(z_n, z_{n+1}) \to 0$ as $n \to \infty$, it follows from (3.18) that $\inf_{x\in X} d_p(hx, Tx) = 0$ and the conclusion (a) is proved.

To see (b), it suffices to prove that $\{z_n\}$ is a *TVS*-cone Cauchy sequence in (X, p). For $m, n \in \mathbb{N}$ with $m > n$, it follows from (3.16) that

$$p(z_n, z_m) \precsim_K \sum_{j=n}^{m-1} p(z_j, z_{j+1}) \precsim_K \frac{\lambda^{n-1}}{1-\lambda} p(z_1, z_2). \tag{3.19}$$

Given $c \in Y$ with $\theta \ll_K c$ (i.e, $c \in \operatorname{int} K = \operatorname{int}(\operatorname{int} K)$), there exists a neighborhood N_θ of θ such that $c + N_\theta \subseteq \operatorname{int} K$. Therefore, there exists $U_c \in \mathcal{U}_{\mathcal{L}}$ with $U_c \subseteq N_\theta$ such that

$$c + U_c \subseteq c + N_\theta \subseteq \operatorname{int} K. \tag{3.20}$$

Let $\omega = p(z_1, z_2)$ and $\alpha_n = \lambda^{n-1}/(1-\lambda)$, $n \in \mathbb{N}$. Since $\lambda \in [0,1)$, we have $\{\alpha_n\} \subset [0, \infty)$ and $\lim_{n\to\infty} \alpha_n = 0$. Applying Lemma 3.1, there exists $n_0 \in \mathbb{N}$, such that $-\alpha_n \omega \in U_c$ for all $n \geq n_0$. So, by (3.20), we obtain

$$c - \alpha_n \omega \in c + U_c \subseteq \operatorname{int} K \tag{3.21}$$

or

$$\alpha_n \omega \ll_K c \tag{3.22}$$

for all $n \geq n_0$. For $m, n \in \mathbb{N}$ with $m > n \geq n_0$, since $p(z_n, z_m) \precsim_K \alpha_n \omega$ from (3.19) and $\alpha_n \omega \ll_K c$, it follows from Lemma 3.2 that

$$p(z_m, z_n) \ll_K c. \tag{3.23}$$

Hence, we prove that $\{z_n\}$ is a *TVS-cone* Cauchy sequence in (X, p). The proof is completed. $\square$

Theorem 3.4. *Let (X, p) be a TVS-cone metric space, $T : X \to \mathcal{N}(X)$ a multivalued map, and $d_p := \xi_e \circ p$. Suppose that*

(D3) *there exists an $\mathcal{MT}$-function $\mu : [0, \infty) \to [0, 1)$ such that, for each $x \in X$, if $y \in Tx$ with $y \neq x$, then there exists $z \in Ty$ such that*

$$p(y, z) \precsim_K \mu(d_p(x, y))p(x, y). \tag{3.24}$$

Then, there exists a nonempty proper subset V of X, such that $\inf_{x\in X} d_p(x, Tx) = \inf_{x\in V} d_p(x, Tx) = 0$ (i.e., T has the d_p-approximate fixed point property on V and X.)

Proof. Let $h \equiv id$ be the identity map on X. Then, the conditions (D1) and (D2) as in Theorem 3.3 hold. Following the same argument as in the proof of Theorem 3.3, we obtain two sequences $\{x_n\}$ and $\{z_n\}$ in X such that

(i) $x_{n+1} \in Tx_n$, $z_n = hx_n$ and $z_{n+1} \in Tx_n$ for each $n \in \mathbb{N}$,

(ii) $\lim_{n\to\infty} d_p(z_n, z_{n+1}) = 0$.

Since $h \equiv id$, $\{x_n\}$ and $\{z_n\}$ are identical, we have $\lim_{n\to\infty} d_p(x_n, x_{n+1}) = 0$. Put $V = \{x_n\}_{n\in\mathbb{N}} \subset X$. Since

$$\inf_{x\in X} d_p(x, Tx) \leq \inf_{x\in V} d_p(x, Tx) \leq d_p(x_n, Tx_n) \leq d_p(x_n, x_{n+1}), \quad \forall n \in \mathbb{N}, \tag{3.25}$$

we obtain $\inf_{x\in X} d_p(x, Tx) = \inf_{x\in V} d_p(x, Tx) = 0$. Hence, T has the d_p-approximate fixed point property on V and X. □

New, we establish the following approximate coincidence point property for maps in general metric spaces by applying Theorem 3.3.

Theorem 3.5. *Let (X, d) be a metric space, $T : X \to \mathcal{N}(X)$ a multivalued map, and $h : X \to X$ a self-map. Suppose that*

(A1) *there exists an $\mathcal{MT}$-function $\mu : [0, \infty) \to [0, 1)$ such that, for each $x \in X$, if $y \in Tx$ with $y \neq x$, then there exists $z \in Ty$ such that*

$$d(hy, hz) \leq \mu(d(hx, hy))d(hx, hy), \tag{3.26}$$

(A2) *Tx is h-invariant (i.e., $h(Tx) \subseteq Tx$) for each $x \in X$.*

Then, h and T have the approximate coincidence point property on X.

Proof . In Theorem 3.3, let $Y = \mathbb{R}$, $K = [0, \infty) \subset \mathbb{R}$, and $e = 1$. Therefore, the conclusion is immediate from Theorem 3.3. □

The following approximate fixed point property for maps is immediate from Theorem 3.4.

Theorem 3.6. *Let (X, d) be a metric space and $T : X \to \mathcal{N}(X)$ a multivalued map. Suppose that*

(A3) *there exists an $\mathcal{MT}$-function $\mu : [0, \infty) \to [0, 1)$ such that, for each $x \in X$, if $y \in Tx$ with $y \neq x$, then there exists $z \in Ty$ such that*

$$d(y, z) \leq \mu(d(x, y))d(x, y). \tag{3.27}$$

Then, T has the approximate fixed point property on X.

Theorem 3.7. *In Theorem 3.6, if the condition (A3) is replaced by $(A3)_H$, where*

$(A3)_H$ *there exists an $\mathcal{MT}$-function $\mu : [0, \infty) \to [0, 1)$ such that, for each $x \in X$,*

$$d(y, Ty) \leq \mu(d(x, y))d(x, y), \quad \forall y \in Tx, \tag{3.28}$$

then, T has the approximate fixed point property on X.

Proof. Define $\tau : [0, \infty) \to [0, 1)$ by $\tau(t) = (1 + \mu(t))/2$. Then, by [3, Lemma 2.1], τ is also an $\mathcal{MT}$-function. For each $x \in X$, let $y \in Tx$ with $y \neq x$. Then, $d(x, y) > 0$. By $(A3)_H$, we have

$$d(y, Ty) < \tau(d(x, y))d(x, y). \tag{3.29}$$

It follows that there exists $z \in Ty$ such that

$$d(y, z) < \tau(d(x, y))d(x, y), \tag{3.30}$$

which shows that (A3) holds. Therefore, the conclusion follows from Theorem 3.6. □

Theorem 3.8. *Theorems 3.6 and 3.7 are equivalent.*

Proof. We have shown that Theorem 3.6 implies Theorem 3.7. So it suffices to prove that Theorem 3.7 implies Theorem 3.6. If (A3) holds, then it is easy to verify that $(A3)_H$ also holds. Hence, Theorem 3.7 implies Theorem 3.6 and we get the desired result. □

Remark 3.9 (see [1, Lemma 2.2]). Is a special cases of Theorems 3.6 and 3.7.

Let (X, p) be a *TVS*-cone metric space. By Theorem 2.3, we know that $d_p := \xi_e \circ p$ is a metric on X. So we can obtain the topology Γ_{d_p} on X induced by d_p and hence define d_p-open subsets, d_p-closed subsets, and d_p-compact subsets of X.

Here, we denote $\mathcal{C}_{d_p}(X)$ by the collection of all nonempty d_p-closed subsets of X.

Theorem 3.10. *Let (X, p) be a TVS-cone complete metric space, $T : X \to \mathcal{C}_{d_p}(X)$ a multivalued map, $h : X \to X$ a self-map, and $d_p := \xi_e \circ p$. Suppose that the conditions (D1) and (D2) as in Theorem 3.3 hold and further assume one of the following conditions hold:*

(L1) *(X, d_p) is d_p-compact and the function $f : X \to [0, \infty)$ defined by $f(x) = d_p(hx, Tx)$ is l.s.c.,*

(L2) *$\mathcal{G} = \{(hx, y) \in X \times X : x \in X, y \in Tx\}$ is a d_p-closed subset of $X \times X$.*

Then, the following statements hold.

(a) *There exists a nonempty subset* $\mathcal{M}$ *of* X*, such that* $(\mathcal{M}, d_p)$ *is a complete metric space.*

(b) $\mathcal{COP}(h, T) \neq \emptyset$.

Proof. Following the same argument as in the proof of Theorem 3.3, one can obtain that there exist two sequences $\{x_n\}$ and $\{z_n\}$ in X such that for any $n \in \mathbb{N}$, $z_n = hx_n$, $z_{n+1} \in Tx_n$, $\{z_n\}$ is a TVS-cone Cauchy sequence in (X, p) and $\lim_{n\to\infty} d_p(z_n, z_{n+1}) = \inf_{x\in X} d_p(hx, Tx) = 0$. By the TVS-cone completeness of (X, p), there exists $v \in X$, such that $\{z_n\}$ TVS-cone converges to v. On the other hand, applying Theorem 2.4, we obtain that $\{z_n\}$ is a Cauchy sequence in (X, d_p) and $d_p(z_n, v) \to 0$ or $z_n \xrightarrow{d_p} v$ as $n \to \infty$. Let $\mathcal{M} = \{z_n\}_{n\in\mathbb{N}} \cup \{v\}$. Then $(\mathcal{M}, d_p)$ is a complete metric space and the conclusion (a) holds.

Now, we verify the conclusion (b). Suppose that (L1) holds. Then the infimum $\inf_{x\in X} f(x)$ is attained. Since $\inf_{x\in X} f(x) = \inf_{x\in X} d_p(hx, Tx) = 0$, there exists $\zeta \in X$ such that

$$d_p(h\zeta, T\zeta) := f(\zeta) = \inf_{x\in X} f(x) = 0. \tag{3.31}$$

Since $T\zeta$ is a d_p-closed subset of X, it implies from (3.31) that $h\zeta \in T\zeta$. Hence, $\zeta \in \mathcal{COP}(h, T)$.

Suppose that (L2) holds. For any $n \in \mathbb{N}$, since $z_{n+1} \in Tx_n$, we know $(z_n, z_{n+1}) \in \mathcal{G}$. Since $\mathcal{G}$ is d_p-closed in $X \times X$ and $z_n \xrightarrow{d_p} v$ as $n \to \infty$, we have $(v, v) \in \mathcal{G}$. Hence, there exists $a \in X$ such that $v = ha$ and $v \in Ta$, which say that $a \in \mathcal{COP}(h, T)$. The proof is completed. □

Theorem 3.11. *Let* (X, p) *be a TVS-cone complete metric space,* $T : X \to \mathcal{C}_{d_p}(X)$ *a multivalued map, and* $d_p := \xi_e \circ p$*. Suppose that the condition (D3) as in Theorem 3.4 holds and further assume one of the following conditions hold:*

(H1) T *is* d_p*-closed; that is,* $GrT := \{(x, y) \in X \times X : y \in Tx\}$*, the graph of* T*, is a* d_p*-closed subset of* $X \times X$*,*

(H2) *the function* $f : X \to [0, \infty)$ *defined by* $f(x) = d_p(x, Tx)$ *is l.s.c.,*

(H3) $\inf\{d_p(x, z) + d_p(x, Tx) : x \in X\} > 0$ *for every* $z \notin \mathcal{F}(T)$.

Then, there exists a nonempty subset W *of* X*, such that*

(a) (W, d_p) *is a complete metric space,*

(b) $\mathcal{F}(T) \cap W \neq \emptyset$.

Proof. Following a similar argument as in the proof of Theorem 3.3, we can obtain a sequence $\{x_n\}$ in X such that, for any $n \in \mathbb{N}$,

(i) $x_{n+1} \in Tx_n$,

(ii) $p(x_{n+1}, x_{n+2}) \precsim_K \gamma p(x_n, x_{n+1})$, where $\gamma := \sup_{n\in\mathbb{N}} \mu(d_p(x_n, x_{n+1})) \in [0, 1)$,

(iii) $p(x_n, x_m) \precsim_K (\gamma^{n-1}/(1-\gamma))\, p(x_1, x_2)$, for $m, n \in \mathbb{N}$ with $m > n$.

(iv) $\{x_n\}$ is a TVS-cone Cauchy sequence in (X, p).

Applying Theorem 2.1, the inequality (iii) implies that, for $m, n \in \mathbb{N}$ with $m > n$, we have

$$\begin{aligned} d_p(x_n, x_m) &= \xi_e(p(x_n, x_m)) \\ &\leq \xi_e\left(\frac{\gamma^{n-1}}{1-\gamma} p(x_1, x_2)\right) \\ &= \frac{\gamma^{n-1}}{1-\gamma} d_p(x_1, x_2). \end{aligned} \tag{3.32}$$

Since $\lim_{n\to\infty}\gamma^{n-1}/(1-\gamma) = 0$, it follows from (3.32) that $\{x_n\}$ is a Cauchy sequence in (X, d_p). By the TVS-cone completeness of (X, p) and (iv), there exists $v \in X$, such that $\{x_n\}$ TVS-cone converges to v. Applying Theorem 2.4, we have $d_p(x_n, v) \to 0$ or $x_n \xrightarrow{d_p} v$ as $n \to \infty$. Let $W = \{x_n\}_{n\in\mathbb{N}} \cup \{v\}$. Then, (W, d_p) is a complete metric space and the conclusion (a) holds.

Finally, in order to complete the proof, it suffices to show that $v \in \mathcal{F}(T)$. Suppose that (H1) holds. Since $x_{n+1} \in Tx_n$, we have $(x_n, x_{n+1}) \in GrT$ for each $n \in \mathbb{N}$. By (H1) and $x_n \xrightarrow{d_p} v$ as $n \to \infty$, we get $v \in \mathcal{F}(T)$.

If (H2) holds, by the lower semicontinuity of f and (i), we obtain

$$\begin{aligned} d_p(v, Tv) &= f(v) \\ &\leq \liminf_{n\to\infty} f(x_n) \\ &\leq \liminf_{n\to\infty} d_p(x_n, Tx_n) \\ &\leq \lim_{n\to\infty} d_p(x_n, x_{n+1}) = 0, \end{aligned} \tag{3.33}$$

which implies $d_p(v, Tv) = 0$. Since Tv is a d_p-closed subset of X, $v \in \mathcal{F}(T)$.

Let (H3) holds. Suppose $v \notin \mathcal{F}(T)$. Since d_p is a metric on X and $x_m \xrightarrow{d_p} v$ as $m \to \infty$, by (3.32), we get

$$d_p(x_n, v) \leq \frac{\gamma^{n-1}}{1-\gamma} d_p(x_1, x_2) \quad \text{for any } n \in \mathbb{N}. \tag{3.34}$$

From (3.32) and (3.34), we have

$$\begin{aligned} 0 &< \inf_{x\in X}\{d_p(x, v) + d_p(x, Tx)\} \\ &\leq \inf_{n\in\mathbb{N}}\{d_p(x_n, v) + d_p(x_n, Tx_n)\} \\ &\leq \inf_{n\in\mathbb{N}}\{d_p(x_n, v) + d_p(x_n, x_{n+1})\} \\ &\leq \lim_{n\to\infty} \frac{2\gamma^{n-1}}{1-\gamma} d_p(x_1, x_2) \\ &= 0, \end{aligned} \tag{3.35}$$

which leads a contradiction. Therefore, $v \in \mathcal{F}(T)$. The proof is completed. □

The following result is immediate from Theorem 3.10.

Theorem 3.12. *Let (X,d) be a complete metric space, $T : X \to \mathcal{C}(X)$ a multivalued map, and $h : X \to X$ a self-map. Suppose that the conditions (A1) and (A2) as in Theorem 3.5 hold and further assume one of the following conditions hold:*

(i) *(X,d) is compact and the function $f : X \to [0,\infty)$ defined by $f(x) = d(hx, Tx)$ is l.s.c.;*

(ii) *$\{(hx,y) \in X \times X : x \in X, y \in Tx\}$ is a closed subset of $X \times X$.*

Then, $\mathcal{COP}(h,T) \neq \emptyset$.

Theorem 3.13. *Let (X,d) be a complete metric space and $T : X \to \mathcal{C}(X)$ a multivalued map. Suppose that the condition (A3) or $(A3)_H$ holds and further assume one of the following conditions hold:*

(h1) *T is closed,*

(h2) *the function $f : X \to [0,\infty)$ defined by $f(x) = d(x,Tx)$ is l.s.c.,*

(h3) *$\inf\{d(x,z) + d(x,Tx) : x \in X\} > 0$ for every $z \notin \mathcal{F}(T)$,*

(h4) *for each sequence $\{x_n\}$ in X with $x_{n+1} \in Tx_n$, $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = v$, we have $\lim_{n\to\infty} \mathcal{H}(Tx_n, Tv) = 0$.*

Then, $\mathcal{F}(T) \neq \emptyset$.

Proof. The conclusion is immediate from Theorem 3.11 if $Y = \mathbb{R}$, $K = [0,\infty) \subset \mathbb{R}$, $e = 1$, and one of conditions (h1), (h2), and (h3) holds. Suppose that (h4) holds. Following a similar argument as in the proof of Theorem 3.3, we can construct a Cauchy sequence $\{x_n\}$ in (X,d) such that $x_{n+1} \in Tx_n$, $n \in \mathbb{N}$, and $\{x_n\}$ converge to some point $v \in X$. Since the function $x \mapsto d(x,Tv)$ is continuous, $x_{n+1} \in Tx_n$ and $x_n \xrightarrow{d} v$ as $n \to \infty$, by (h4), we get

$$d(v,Tv) = \lim_{n\to\infty} d(x_{n+1}, Tv) \leq \lim_{n\to\infty} \mathcal{H}(Tx_n, Tv) = 0, \tag{3.36}$$

which implies $v \in \mathcal{F}(T)$. The proof is completed. □

Remark 3.14.

(a) Let A and B be two topological vector spaces and $T : A \to 2^B$ a multivalued map. Recall that T is u.s.c. if and only if for any open set V in B, $T^+(V) := \{x \in A : T(x) \subseteq V\}$ is open in A. It is known that if T is u.s.c. with closed values, then T is closed (see [25]). Hence, Theorem 3.13 is true if $(X, \|\cdot\|)$ is a Banach space and $T : X \to C(X)$ is u.s.c.;

(b) let G be a nonempty subset of a metric space (X,d) and $T : X \to \mathcal{C}(X)$ u.s.c. Then, the function $\phi : G \to [0,\infty)$ defined by $\phi(x) = d(x,Tx)$ is l.s.c.; for detail, see [26, Lemma 2] or [27, Lemma 3.1];

(c) it is known that any single-valued map of Kannan's type or Chatterjea's type satisfies (h3); for more detail, one can see [28, Corollary 3] or [3, Remark 3.1].

Applying Theorem 3.13, we can prove the following generalization of Berinde-Berinde's fixed point theorem [12].

Theorem 3.15. *Let (X,d) be a complete metric space, $T : X \to \mathcal{CB}(X)$ a multivalued map, and $G : X \to [0,\infty)$ a function. Suppose that there exists an $\mathcal{MT}$-function $\mu : [0,\infty) \to [0,1)$ such that*

$$\mathcal{H}(Tx,Ty) \le \mu(d(x,y))d(x,y) + G(y)d(y,Tx), \quad \forall x,y \in X. \tag{3.37}$$

Then, $\mathcal{F}(T) \neq \emptyset$.

Proof. Let $x \in X$. If $y \in Tx$, then $d(y,Tx) = 0$. So (3.37) implies the inequality

$$d(y,Ty) \le \mu(d(x,y))d(x,y), \quad \forall y \in Tx. \tag{3.38}$$

Hence, the condition $(\mathrm{A3})_H$ of Theorem 3.13 holds. Let $\{x_n\}$ in X with $x_{n+1} \in Tx_n$, $n \in \mathbb{N}$, and $\lim_{n\to\infty} x_n = v$. By (3.37), we obtain

$$\lim_{n\to\infty} \mathcal{H}(Tx_n,Tv) \le \lim_{n\to\infty}\left\{\varphi(d(x_n,v))d(x_n,v) + G(v)d(v,x_{n+1})\right\} = 0, \tag{3.39}$$

which says that the condition (h4) of Theorem 3.13 also holds. Therefore, the conclusion follows from Theorem 3.13. □

Remark 3.16.

(a) Theorems 3.11, 3.13, and 3.15 all generalize Berinde-Berinde's fixed point theorem [12].

(b) In Theorem 3.15, if $G(x) = 0$ for all $x \in X$, then we can obtain Mizoguchi-Takahashi's fixed point theorem [14].

(c) In Theorem 3.15, if $G(x) = 0$ for all $x \in X$, and $\mu : [0,\infty) \to [0,1)$ is defined by $\mu(t) = \gamma$ for some $\gamma \in [0,1)$, then we can obtain Nadler's fixed point theorem [10].

(d) [1, Theorem 2.6] is a special case of Theorem 3.15.

(e) Notice that, in [1, Theorem 2.6], the authors showed that a generalized multivalued almost contraction T in a metric space (X,d) has $\mathcal{F}(T) \neq \emptyset$ provided either (X,d) is compact and the function $f(x) = d(x,Tx)$ is l.s.c. or T is closed and compact. But reviewing Theorem 3.15, we know that the conditions in [1, Theorem 2.6] are redundant.

Acknowledgments

The author would like to express his sincere thanks to the anonymous referee for their valuable comments and useful suggestions in improving the paper. This research was supported partially by Grant no. NSC 100-2115-M-017-001 of the National Science Council of the Republic of China.

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