Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2012, Article ID 303960, 30 pages
doi:10.1155/2012/303960

Review Article

Linearization: Geometric, Complex, and Conditional

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Received 21 September 2012; Accepted 25 November 2012

Academic Editor: Fazal M. Mahomed

Lie symmetry analysis provides a systematic method of obtaining exact solutions of nonlinear (systems of) differential equations, whether partial or ordinary. Of special interest is the procedure that Lie developed to transform scalar nonlinear second-order ordinary differential equations to linear form. Not much work was done in this direction to start with, but recently there have been various developments. Here, first the original work of Lie (and the early developments on it), and then more recent developments based on geometry and complex analysis, apart from Lie's own method of algebra (namely, Lie group theory), are reviewed. It is relevant to mention that much of the work is *not* linearization but uses the base of linearization.

1. Introduction

Symmetry has not only been one of the criteria of aesthetics and beauty but has repeatedly proved extremely useful. It lies at the base of the geometry of the Greeks and is at the base of modern developments in high energy physics and in gravity. It was used by Evariste Galois in 1830 [1] for proving that quartic equations are solvable by means of radicals but that it is impossible to canonically solve higher order polynomial equations by means of radicals. This led to the concept of *groups*. The groups used are now called *Galois groups*. Lie wanted to extend the approach of Galois to deal with differential equations (DEs). Of course, this is a vastly more ambitious programme. Apart from the order of the DEs there are ordinary DEs (ODEs) and partial DEs (PDEs); scalar DEs and vector DEs; initial and boundary conditions to be satisfied. Worse follows; while polynomial equations generically have at most as many solutions as their order, DEs have infinitely many. For ODEs the infinity is tamed because there are arbitrary parameters (constants) that appear, and *they* are as many as the order of the ODEs. However, they remain untamed for PDEs. To extend the use of symmetry to differential equations, Lie (1880/83/91) had to extend from finite groups to continuously infinite groups that could be (at least twice) differentiated [2–5]. These are now called *Lie groups*.

One method Lie adopted was a generalization of the methods for some specific first-order ODEs, changing them to linear form by using an invertible transformation of the dependent and independent variables. He showed that *all* order ODEs can be transformed to linear form by such transformations. He then obtained general criteria for such transformations to exist for *second-order* ODEs. Such transformations are called *point transformations,* and the transformed equation is said to be *linearized*. Equations so transformed are said to be *linearizable*. Lie proved that the necessary and sufficient condition for a scalar nonlinear ODE to be linearizable is that it must have 8 Lie point symmetries. He exploited the fact that all scalar linear second-order ODEs are equivalent under point transformations; that is every linearizable scalar ODE is reducible to the free particle equation. He showed that the ODE had to be cubically semilinear and was able to state criteria that the coefficients must satisfy for the equation to be linearizable. These consisted of a set of four consistency conditions for the four coefficients of the first-derivative terms, involving two auxiliary functions. In 1894 Tresse [6] eliminated the auxiliary functions and reduced them to two conditions. It turns out that all second-order ODEs are transformable to linear form by *contact transformations* (that involve firstderivatives of the dependent variables as well). We will not be concerned with them here.

Lie took this approach no further but considerably later (1937/40) Chern [7, 8] extended the analysis to a class of scalar third-order ODEs by using contact transformations. It was not till much later (1996/97) that the same results were obtained using Lie's classical method by Grebot [9, 10]. In 1990 Mahomed and Leach [11] proved that for mth-order ODEs, $m \geq 3$, there was no unique class of linearizable ODEs. Instead there were *three* equivalence classes with $m+1$, $m+2$, or $m+4$ infinitesimal symmetry generators. Notice that whereas for second-order there are 8 symmetry generators as the maximal number, for third-order there are only 7, but even with 5 or 4 generators the ODE may be linearizable. However, there was no procedure provided to linearize them. The general third-order linearizable ODEs were dealt with only recently (2002, 2005) by Neut and Petitot [12] and independently by Ibragimov and Meleshko [13]. Though the procedure followed was the same as that used by Lie, the calculations become much more complicated, and algebraic computation was needed. The calculation for order 4 is even more complicated and was achieved by Ibragimov et al. [14] only in 2008. Though very useful, they did not provide means of actually solving the equations.

All the work mentioned so far has been for scalar ODEs. Nothing has been said about *systems* of ODEs. In 1988 systems of two second-order ODEs, linearizable to constant coefficient systems, were proved to have three equivalence classes [15] with 7, 8, or 15 dimensional Lie algebras. This result was extended to the general case in 2000 by Wafo Soh and Mahomed [16], giving 5, 6, 7, 8, or 15 dimensional Lie algebras, and was further extended by them in 2001 to $m \geq 2$ second-order ODEs [17], giving $m+3, m+4, \ldots, 2(m+2)$ or $(m+1)(m+2)/2$ dimensional Lie algebras. Thus there are a total of $m+3$ classes. (Notice that the formula would not hold for $m = 1$.) However, once again, there was no procedure provided to obtain the linearizing transformations, the linearized equations, or the solutions in general. While very useful as existence theorems, the methods gave little practical development beyond the general symmetry analysis methods developed by Lie himself.

This is as far as the earlier developments using the classical procedures went. The next step came from the attempt to connect the symmetries of ODEs with the symmetries of geometry. The connection was found by considering geodesic equations, independently by Aminova and Aminov [18, 19] in 2000/06 and by Ferozeet al. [20] in 2006. This connection proved very fruitful. The geodesic equations *are* systems of second-order ODEs, so it dealt with systems. It had been noted in passing by the former authors that the system is linearizable if the space is flat. Independently it was taken much further in 2007 by Mahomed and Qadir [21], who considered a second-order system of ODEs of geodesic type and found that the consistency conditions for the system to be linearizable are to treat the coefficients of the system of ODEs as if they were Christoffel symbols and require that the curvature tensor constructed from them be zero. By projecting the equations down one dimension, using the invariance of the geodesics under translations of the geodetic parameter, one obtains a system of cubically semilinear ODEs that are linearizable if they are obtainable by projection from a system of geodesic-type equations in a flat space [22]. The projection procedure had been discussed by Aminova and Aminov, but its usefulness for linearization had not been noticed by them. There is some further work on linearization that follows from the algebraic linearization and group classification of Lie and from the geometric developments mentioned [23–25], that I will not be going into here. However, these works would be well worth to be followed up.

Very recently it was noted [26, 27] that whereas Lie had used complex DEs for complex functions, he had not explicitly used their analyticity in real terms. The fact that the equations are in the complex domain is very relevant for geometric purposes, as the topology of the manifold is changed by going from the real to the complex. In fact, as pointed out by Penrose [28], the topology becomes simpler. Thus for example, if a point is removed from the real line the manifold is broken into two pieces. Hence, if we want to construct a Lie group under multiplication from it we are unable to do so (as there is no inverse for the element 0 available). However, for the complex "line", taking out a single point leaves a multiply connected space. We can then make a Lie group under multiplication from it by leaving out the complex number 0. This makes the complex domain for Lie groups very important for differential geometry, but the corresponding aspect for DEs had not been exploited. It was realized that the dependent variables must be analytic for a DE. As such, when considering the scalar complex DEs broken into a system of real DEs, we have to include the Cauchy-Riemann equations (CREs) in the system. This will change the symmetry structure of the system substantially, thus leading to many unforeseen consequences.

There have been many new developments using geometry and complex analysis, with linearization proper and with methods developed that do not give linearization but use it to get solutions. The main thrust of this paper is to discuss these recent developments. We limit ourselves to point transformations and do not discuss the developments involving contact or higher order transformations or discuss the developments for PDEs, except in so far as the new methods give some results for them. The plan of the paper is as follows: in the next section we provide some preliminaries giving the notation and terminology used for symmetry analysis of systems of ODEs. In the subsequent section the original method of Lie and the algebraic methods used are discussed. In Section 4 the recent geometric methods and in Section 5 the complex methods are presented. In the next section after that some other developments regarding conditional linearization are given. These lead to a proposal for the systems of ODEs along the lines of Lie's original intention. However, it is found that the proposal needs other ingredients. This is discussed in Section 7. Finally, in Section 8 a summary and discussion of some ongoing work are provided.

2. Preliminaries

To get more concrete, let us define the terms mentioned previously for general motivation. If an algebraic expression $F(x,y)$ is form invariant under a transformation of both the dependent (y) and independent (x) variables given by

$$s = f(x,y), \qquad t = g(x,y), \tag{2.1}$$

that is, it converts to $F(s,t)$; we say that it is symmetric under the transformation. If we were to regard the variables as coordinates in a 2-dimensional space this would amount to the expression being invariant under coordinate transformations. As such, it would have geometrical significance as representing something on the manifold. (It is in this sense that it is not clear what the geometric significance of contact symmetries would be, and hence the power of geometry that will shortly appear would not be available for it.)

We are particularly interested in symmetries of DEs and not of only algebraic expressions. For that purpose we need invertible transformations that can be reduced to the identity. Then the symmetry generators would form a group that is connected to the identity as we could always invert the transformations to get the inverse element. The group would be a Lie group if it applies to DEs. Thus we define

$$\overline{x} = x + \epsilon\xi(x,y), \qquad \overline{y} = y + \epsilon\eta(x,y), \tag{2.2}$$

where ϵ is an infinitesimal quantity that can be taken to zero continuously. This leads to the infinitesimal generator of symmetry

$$\mathbf{X} = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}. \tag{2.3}$$

This generator will only give infinitesimal symmetries of algebraic equations but not of DEs. The point is that now we need to treat the derivatives of the dependent variable as independent variables. As such, for the purpose of the symmetry of the algebraic expression involved in the DE, we need to *prolong* or *extend* it to include derivatives with respect to the relevant derivatives. Thus for mth order scalar ODEs we need the prolonged generator

$$\mathbf{X}^{[n]} = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} + \eta^{[1]}(x,y)\frac{\partial}{\partial y'} + \cdots + \eta^{[n]}(x,y)\frac{\partial}{\partial y^{(n)}}. \tag{2.4}$$

Now it is required that for $\mathbf{X}$ to be a symmetry generator for a DE of order n, $E(x,y;y',\ldots,y^{(n)}) = 0$,

$$\mathbf{X}^{[n]}\mathbf{E}|_{\mathbf{E}=0} = 0. \tag{2.5}$$

The same procedure can be used in the case of several dependent variables for one independent variable. Writing the vector of dependent variables as $\mathbf{y} = y^i,\ (i = 1, \ldots, m)$, we would now have a system of ODEs of the same dimension, $\mathbf{E} = E_i = 0$. Then we have

$$\mathbf{X}^{[n]} = \xi\left(x, y^i\right)\frac{\partial}{\partial x} + \eta^j\left(x, y^i\right)\frac{\partial}{\partial y^j} + \eta^{[1]j}\left(x, y^{i\prime}\right)\frac{\partial}{\partial y^{j\prime}} + \cdots + \eta^{[n]j}\left(x, y^i\right)\frac{\partial}{\partial y^{j(n)}}. \tag{2.6}$$

Now we will require that

$$\mathbf{X}^{[n]}\mathbf{E}|_{\mathbf{E}=0} = 0. \tag{2.7}$$

Two DEs are said to be *equivalent* if one can be mapped into the other by point transformations. All first-order ODEs are equivalent under point transformations and hence all can be linearized. This is not true for second-order ODEs. However, all linear scalar second-order ODEs *are* equivalent [29]. Further, there are different classes of *linear* ODEs of order $m \geq 3$. As such, there are different classes of higher order linearizable nonlinear ODEs. For systems of ODEs we have to regard the space as $(m + 1)$-dimensional (m independent variables and one dependent variable) instead of 2-dimensional. Clearly the earlier results for ODEs of higher order (than 3) would not generally apply to systems. In fact little is known about them. In particular, it is not clear how the numbers will change with dimensions and order, though one can make guesses by naively putting the two modifications together. The problem is that the number of symmetry generators keeps pulling new surprises. (Remember the change in going from second to third order.) A rigorous proof for the general formulae would be very useful.

As mentioned before, a connection was found between the symmetries of systems of second-order ODEs and isometries of a space by considering systems of geodesic equations. As such, it is worthwhile to briefly review the differential geometry involved in isometries and systems of geodesic equations.

For differential geometry one normally considers a manifold of dimension n with a Riemannian metric, $\mathbf{g}$ defined on it. However, it is not necessary to limit ourselves to a positive definite metric, and we can consider indefinite metrics as well. For a manifold it is necessary that locally there is a homeomorphism from the space to $\mathbb{R}^n$. Using the homeomorphism we can assign n coordinates on it, x^i. In general one can transform coordinates at will and often needs to change coordinates in going from one element of the open cover of the manifold to another. In any chosen coordinates the metric tensor can then be written as a symmetric matrix $g_{ij}(\mathbf{x})$. For our purposes we need differentiable manifolds and hence need that there be *diffeomorphisms* instead of only homeomorphisms.

A vector field on the manifold is a mapping from any point on it to another point on it. We generally take the vectors to be infinitesimal so that they can be taken to lie on the tangent space at the initial point. This way we get a linear vector space for the vectors. We then have the vector given in some coordinate system by its components. However, the vector is an invariant quantity, and the components change with a change of coordinates. As such, a vector field $\mathbf{V}(\mathbf{x})$ is given by a linear combination of its components (which are functions of the coordinates) multiplied by the basis vectors $\mathbf{V} = V^i(\mathbf{x})\mathbf{e}_i$, where we have used the Einstein summation convention that repeated indices are summed over the entire range of values. When a vector field is differentiated we have to not only differentiate the components (to obtain the partial derivative $V^i_{,j} = \partial V^i / \partial x^j$), but we also have to differentiate the basis vectors $\mathbf{e}_i$. The derivatives must be linear combinations of the basis vectors. Thus we can write

$$\frac{\partial \mathbf{e}_i}{\partial x^j} = \Gamma^k_{ij} \mathbf{e}_k. \tag{2.8}$$

For our purposes we can take the Γ^k_{ij} to be the Christoffel symbols that are given by

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left(g_{il,j} + g_{jl,i} - g_{ij,l} \right), \tag{2.9}$$

where g^{kl} is the inverse matrix for g_{il}, that is, $g^{kl} g_{il} = \delta^k_i$, which is the Kronecker delta, being 1 if $k = i$ and 0 otherwise. Notice that neither the partial derivative nor the Christoffel symbols are tensor quantities as they are not invariant. The combination of both gives a tensor representing the geometrical derivative of the vector field, called the *covariant derivative* and is denoted by $V^i_{;j}$. Thus

$$V^i_{;j} = V^i_{,j} + \Gamma^i_{jk} V^k. \tag{2.10}$$

The curvature of a manifold is given by carrying a vector field along one direction and then another and subtracting by carrying the vector field in the reversed order of directions. In components,

$$V^i_{;jk} - V^i_{;kj} = R^i_{lkj} V^l. \tag{2.11}$$

The tensor $\mathbf{R}$ is called the *Riemann curvature tensor*. It can be written in terms of the Christoffel symbols as

$$R^i_{lkj} = \Gamma^i_{lj,k} - \Gamma^i_{lk,j} + \Gamma^i_{pj} \Gamma^p_{lk} - \Gamma^i_{pk} \Gamma^p_{lj}. \tag{2.12}$$

Since it is a fourth-rank tensor one can take different traces. Two are zero and four are either equal or the negative of each other. That is called the *Ricci tensor* $R_{lj} = R^i_{lij}$. *Its* trace is called the *Ricci scalar*. In 2 dimensions it is twice, and in n-dimensions it is $n!$ times, the Gaussian curvature.

In a flat space the shortest path between two points is a straight line. This is not true for a curved manifold. In that case we can vary the action integral for the arc length and obtain the equation for the shortest path. It turns out that it is the straightest available path in that the derivative of the tangent vector along the path is zero; that is, it does not change direction. The equation for this path comes out to be

$$\ddot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k = 0, \tag{2.13}$$

which is called the *geodesic equation*.

The geometric methods developed for the study and linearization of ODEs use these quantities extensively. Further background geometry is provided where needed.

3. Algebraic Linearization

3.1. Lie's Original Approach

Lie proved that the general homogeneous, linear, scalar, second-order ODE:

$$y''(x) + P(x)y'(x) + Q(x)y(x) = 0, \tag{3.1}$$

can always be reduced to the free particle equation (in which $P(x), Q(x) = 0$) by transforming the independent and dependent variables appropriately. He did this by postulating that there was some transformation that would do so and showing that the requirement could always be consistently met. As such, *all* linear second-order ODEs are equivalent. Next, he used a geometric argument to show that all linear ODEs have 8 infinitesimal generators of symmetry. He first showed that the free-particle equation has 8 and then argued by *reductio ad absurdum* that a 9th linearly independent generator could not be added.

His next point was that the number of infinitesimal symmetry generators remains invariant under any infinitesimal point symmetry transformation. This is because it just amounts to a coordinate transformation in the space of the variables. Since the generators are essentially vector fields, they are invariant, though their coordinate representations may look different. Hence the *number* remains the same. Consequently, *any* nonlinear second-order ODE that can be converted to linear form by a point transformation must have 8 symmetry generators. Now, consider a general nonlinear ODE, and assume that it can be converted to $y'' = 0$ by some point transformation. Put in the presumed transformations and check when the equation can be solved. It turns out that it has to be of the form

$$y''(x) + c(x,y)y'^3(x) + g(x,y)y'^2(x) + h(x,y)y'(x) + d(x,y) = 0. \tag{3.2}$$

Further, it must satisfy some consistency criteria for the same functions and their derivatives to be identifiable with the four coefficients, c, g, h, d, mentioned previously. These consistency criteria involve the two unknown transformation functions, which appear in the constraints as auxiliary functions. One would have liked to be able to determine them. However, finding the functions amounts for solving the original nonlinear ODE. As such, we have to leave the auxiliary functions there. The only way to remove them is by further differentiating the derivatives in different orders and eliminating the unknown functions. This is what Tresse did, to obtain

$$\begin{aligned} &3(ch)_x + 3dc_y - 2gg_x - hg_y - 3c_{xx} - 2g_{xy} - h_{yy} = 0, \\ &3(dg)_y + 3cd_x - 2hh_y - gh_x - 3d_{yy} - 2h_{xy} - g_{xx} = 0. \end{aligned} \tag{3.3}$$

Note the symmetry between the two equations by interchanging (c, g, x) and (d, h, y). Note also that these equations, however, complicated they may look, are only *constraint* equations to be checked and not DEs to be solved.

Symmetry generators were not only useful for the purposes of linearization but could be directly used to reduce the order of the ODE by 1. Thus, if we have enough symmetry generators we can solve any ODE. Even if there are not enough, we can reduce the order. If, for example, a second-order ODE has a single infinitesimal symmetry generator that maps this generator to a translation generator, by a point transformation it can be reduced to a first-order ODE. Now, by the fundamental theorem of calculus, it can be solved if it is semilinear. As such, we would have proved that it can be solved and could then use numerical methods to solve it. Lie developed the methods further. In the process he used the power of group theory to derive very general results for the solution of ODEs by means of symmetries. In fact, he needed to classify the ODEs by their groups. If the ODE had a solvable Lie group of the correct order it could be solved. Group classification became one of the staples of symmetry analysis. This played a role for the purpose of linearization later.

Presumably, Lie would have noted that the number of symmetry generators for third-order linear ODEs was not the same. For the ODE $y'''(x) = 0$ the number is 7. For $y'''(x) + \lambda y(x) = 0$, where λ is a complex number, the number of generators is 5, and for $y'''(x) + f(x)y(x) = 0$, where $f(x)$ is a nonconstant function, the number is 4. As such, Lie's own argument did not work, and he did not give any results on the linearization of third-order ODEs.

It was left to Chern to use the Lie method with contact symmetries for the first two cases. However, he did not solve the general problem. He was limited to the special case when the coefficient of the dependent variable is a constant. As mentioned earlier, Grebot managed to use the classical Lie approach for the same classes of ODEs. Neut and Petitot used Lie's method to deal with the general case. Later, but independently, Ibragimov and Meleshko used the same methods with the help of algebraic computing for a more thorough treatment of the general case. Though there are three classes as regards the symmetries of the equations, there are two types of linearizable equations obtainable. These are

$$y''' + (A_1 y' + A_0) y'' + B_3 y'^3 + B_2 y'^2 + B_1 y' + B_0 = 0, \tag{3.4}$$

subject to the linearizability criteria

$$A_{0y} - A_{1x} = 0, \qquad \left(3B_1 - A_0^2 - 3A_{0x}\right)_y = 0,$$

$$3A_{1x} + A_0 A_1 - 3B_2 = 0, \qquad 3A_{1y} + A_1^2 - 9B_3 = 0,$$

$$\left(9B_1 - 6A_{0x} - 2A_0^2\right) A_{1x} + 9(B_{1x} - A_1 B_0)_y + 3B_{1y} A_0 - 27B_{0yy} = 0,$$

$$y''' + \frac{1}{y' + r}\left[-3y''^2 + \left(C_2 y'^2 + C_1 y' + C_0\right) y'' + D_5 y'^5 + D_4 y'^4 + D_3 y'^3 + D_2 y'^2 + D_1 y' + D_0\right] = 0, \tag{3.5}$$

subject to much more complicated linearizability criteria, where A_i, B_j, C_k, D_l, r are some given functions of x and y. In fact, r has to be the ratio of the partial derivatives of the new independent variable relative to y and x. If this new variable does not depend on x the function would be undefined. In that case the simpler transformation applies.

Neither Neut and Petitot nor Ibragimov and Meleshko connected the symmetries of the ODE to the linearizing procedure, so that the classification of linearizable third-order ODEs was left incomplete.

The extension to the fourth-order follows exactly the same procedure as does the third-order. Here, again, there are the two types of linearizable equations, but this time there are four classes. The point is that for the third order we can get rid of the first second-derivative terms by appropriate choice of the linearizing transformations. The extra class here comes because the second and third derivatives can be gotten rid of, but the first derivative cannot. Laguerre [30, 31] says that the same formula applies for relating all fourth- *and higher* order linear ODEs. As such, they argue that there is no need to extend beyond the fourth-order. Again, they did not connect their work with the number or algebra of the symmetry generators. It turns out that the 8 symmetries are for the simpler type, and all the other three classes are contained in the more complicated case. We will not go into further details with it here.

3.2. Group Classification of Higher Order ODEs

It is worthwhile, at this stage, to go back to the original problem from which group theory arose: the solution of algebraic equations. When solving a quadratic equation we can think of y as the quadratic function of x: we translate along the x-axis to the point about which the function is symmetric and then translate along the y-axis to the point at which there is exactly one root. Transforming back to the original variables gives the solution. When solving for cubic equations there is a problem. There are three classes of function: one of which is monotonically increasing or decreasing; the second of which has a point of inflection, and the third-has two turnovers. We cannot solve by reducing the equation to quadratic form directly and first transform variables so as to eliminate the second-order term. We now follow the previous strategy and go to the point about which the function has symmetry under x reflections and then use the requirement that the graph of the curve is invariant under rotation through π radians. This can *always* be done for the cubic, as may be seen by looking at its graph in any of the three cases. One can find the point by looking for the maxima and minima of the function and finding the central point by translating along the x- and y-axes.

However, this strategy does not work for the quartic as symmetry is no longer guaranteed here. This may be seen in the function

$$y = f(x) = 3x^4 - 8x^3 - 6x^2 + 24x + 3, \tag{3.6}$$

which has local minima $f(-1) = -16$, $f(2) = 11$ (and a maximum $f(1) = 16$). It is clear from its graph that this function will remain asymmetric under translations. As such, one needs to be cleverer with this equation. Essentially, one transforms to reduce it to a square of a quadratic which can then be solved by the usual quadratic formula. The solutions for the cubic and the quartic were obtained by Omar Khayyam (better known as a poet through Fitzgerald's translations of his Persian quartets with rhyme scheme $aaba$) in the real domain. It was Cardano who introduced the imaginary for the solution of the cubic and later Tartaglia solved the general quartic.

The natural next step was to solve the quintic. However, all attempts failed. To try to find the general solution Abel and Galois independently considered the function as a product of 5 (complex) factors, which are the roots of the quintic equation, $(x - r_1)(x - r_2)(x - r_3)(x - r_4)(x - r_5)$. It is clear that the function "does not care" which root is called which, as multiplication is commutative. It seems that we could then reduce this to a quartic by taking away one of the factors and solve the quartic. The hurdle is the requirement of symmetry under interchange of the roots. It was Galois who showed under what conditions on the coefficients the required symmetry would hold to allow this procedure to work. Both of them demonstrated that the general quintic cannot be solved by means of radicals. Galois was also able to show that this hurdle would apply to all higher order polynomial equations. The lesson for higher order scalar ODEs should be clear. Many classes are *generically* to be expected. Only for the second order there is no problem of classes. As regards systems, it may be noted that a system of two first-order ODEs "is equivalent" (in some sense) to a scalar second-order ODE. As such, the generic problem of many classes can again be expected to recur. Indeed it does!

Note the use of groups to classify ODEs according to the number of their symmetries. For algebraic equations the permutation group sufficed. Here we need *Lie* groups. This means that we now consider the number of symmetry generators instead of the order of the permutation group. Generally we do not know the topology of the associated manifold. As such we have only a *local* realization of the group. It is the *Lie algebra* associated to the group that is relevant. Recall that there is a unique Lie algebra associated with any Lie group, but there can be many Lie groups (with different topologies) for a given Lie algebra. Mahomed and Leach [11] classified mth order ODEs and found that there remain 3 (as for second order), with $m+1, m+2, m+4$ generators. Though technically much more complicated, the principle remains the same; invariance under a point transformation depends on the nature of the ODE, independent of the "coordinates" used. Thus we obtain a canonical form into which all equivalent ODEs can be transformed (as we factorized the algebraic function). Now we can check the symmetries of the different canonical forms of ODEs according to their symmetry algebras. Notice that different Lie algebras can have the same number of generators, and hence the *number* of classes does not need be the distinctive feature. The Lie algebra of the mth order ODE is [11] $\mathbb{R}^{m-1} \oplus_s \mathrm{gl}(2, \mathbb{R})$, (where $\oplus_s$ is the semidirect sum, meaning that the two subalgebras do not commute).

3.3. Meleshko's "Linearization" of Third-Order ODEs

For solving an autonomous third-order ODE, Meleshko [32] provided an alternative method that he called "linearization". This seemed odd, because shortly *before* it Ibragimov and he had characterized all possible third-order linearizable ODEs, and this new method dealt with equations that did not satisfy those criteria. What Meleshko does is to convert this third-order ODE to a second-order ODE *and then linearize that* (if it is linearizable). Of course, *it cannot be linearizable if it does not satisfy the IM linearizability criteria* as IM have pointed out. This use of linearization, without being linearization itself, is what this review paper is about and will be relevant for a proposal for classifying ODEs by their solvability by transformations according to the number of initial conditions they satisfy. I give the essence of Meleshko's method here.

Consider the general third-order semilinear ODE

$$y''' = f(y, y', y'', y'''). \tag{3.7}$$

Since the independent variable does not occur in it, we are free to treat y as the independent variable and its derivative as the dependent variable, $u(y) = y'$. In this case we clearly have a second-order ODE in u remaining

$$u^2u'' + uu'^2 = f(y, u, u', u''), \tag{3.8}$$

which can then be tested for Lie linearizability. If it can, we go ahead and solve it for $u(y)$. Our problem will then be to determine x in terms of y. We do this by writing $dx/dy = 1/u$. Now a simple quadrature will give x in terms of y.

Notice that there are two arbitrary constants guaranteed by the linearizability of the second-order ODE, and one more will come from the quadrature. Despite having its full quota of arbitrary constants in the general solution, the ODE does not need to be Lie linearizable, that is convertible to linear form by point transformations. An example given by Meleshko is mentioned here. Consider the Kortevegde Vries (KdV) equation for $y(x,t)$

$$y_t + yy_x + Ky_{xxx} = 0 \quad (K > 0). \tag{3.9}$$

For a traveling wave it reduces to the ODE

$$Ku'''' + (u + c_1)u' = 0 \tag{3.10}$$

can be reduced by the previous method to a linearizable second-order ODE. However, its integration yields the second order

$$KH'' + \frac{H^2}{2} + c_1H = c_2, \tag{3.11}$$

which does not satisfy Lie's test. Hence the third-order ODE is not Lie linearizable.

3.4. Systems of ODEs

For the system of two second-order ODEs we follow the usual Lie procedure of first considering the linear equations and then considering those that could, in principle, be transformed to linear form by point transformations. It turns out [16, 17, 29] that the general linear system of two ODEs

$$\ddot{\mathbf{v}} = \mathbf{A}\dot{\mathbf{v}} + \mathbf{B}\mathbf{v} + \mathbf{c}, \tag{3.12}$$

where **A**, **B**, are given $n \times n$ matrix functions $(n \geq 2)$, c is a given n-dimensional vector function, and the dot represents differentiation relative to the new variable, can be invertibly transformed to one of two types of system of ODEs: either to

$$\ddot{\mathbf{w}} = \mathbf{K}\dot{\mathbf{w}} \tag{3.13}$$

or to

$$\ddot{\mathbf{x}} = \mathbf{L}\mathbf{x}, \tag{3.14}$$

where $\mathbf{K}, \mathbf{L}$, are arbitrary $n \times n$ matrix functions, but the two cannot be transformed into each other. It is clear that the two will have a very different character in general.

The number of classes is another matter. We need to then consider the symmetries of the equation. It can generally be proved, see for example [33], that the symmetry algebra for the n-dimensional system of free-particle equations, $y'' = 0$, is $\mathrm{sl}(n+2, \mathbb{R})$, and hence it has $n^2 + 4n + 3$ generators. For $n = 2$ it is 15. The group classification has to be resorted to again, and we need to distinguish between the various algebras that have the same dimension but are different. For example, the algebra of rotations in 3 dimensions is so(3), of symmetries of the plane is $\mathrm{so}(2) \oplus_s \mathbb{R}^2$, and the group of translations in 3 dimensions is $\mathbb{R}^3$. The first is a simple algebra, the second is semisimple, and the third is abelian. There are 5 classes of different dimensions in this case.

Gonzalez Gascon and Gonzalez-Lopez[34] gave the maximal symmetry for linearizable $n \times n$ systems. Gorringe and Leach [15] considered general systems and showed that linearizable 2-dimensional second-order systems with constant coefficients lie in one of three classes with 7, 8, and 15 generators. (The canonical form of the last one has no coefficients as it has the maximal Lie algebra.) Wafo Soh and Mahomed [16] allowed variable coefficients and found two more classes (with 5- and 6-dimensional Lie algebras). They then generalized to n-dimensional systems [16] by using group classification and found that the number of classes increases by one with each increased dimension. The number of generators in the minimal case is $n + 3$ and for the highest submaximal case is $2n + 4$. The maximal, of course, has $n^2 + 3n + 4$ generators, as mentioned earlier.

4. Geometric Linearization

The use of geometry for Lie symmetry analysis dates back to Lie's own work. Also, Lie's work lies at the base of modern differential geometry. However, the two areas diverged after Lie. The methods of modern differential geometry can be very effectively used for systems of second-order ODEs. This use is based on the connection noted between symmetries of differential equations and systems of geodesic equations projected down one dimension, by Aminova and Aminov and by Feroze, Mahomed, and Qadir.

First consider the system of geodesic equations (2.13). Note that the Christoffel symbols are symmetric in the lower two indices. As such, there are $n^2(n+1)/2$ coefficients for the quadratic terms in the first derivatives. Even for 2 dimensions, that is, 6 independent coefficients and for 3 dimensions it is already 18. A general system of semilinear second-order ODEs, quadratic in the first-derivative, can be written as

$$x^{i\prime\prime} + \gamma^i_{jk} x^{j\prime} x^{k\prime} + \beta^i_j x^{j\prime} + \alpha^i = 0, \tag{4.1}$$

where α, β, γ are functions of the independent and dependent variables. We will call this quadratically semilinear system *of geodesic type* if $\alpha = \beta = 0$. It is not necessary that every system of geodesic type comes from a system of geodesics. The point is that the number of metric coefficients for the system is $n(n+1)/2$. Though there *are* n first derivatives involved in the Christoffel symbols, leading to the total number being what it is, it should be clear that there is no guarantee that these coefficients can be consistently obtained from a metric tensor by (2.9).

One could ask for the consistency conditions that they could come from geometry. It turns out that explicit criteria are not so easy to state. If the criteria are fulfilled then one should be able to write down the metric corresponding to the coefficients of the system of ODEs. Thus, given the system of ODEs one should be able to construct the space on which they can be regarded as systems of geodesics. In other words, given the geodesics, one should be able to construct the space on which they lie. A mathematica code for this purpose has been written [35].

4.1. Linearization of Second-Order Systems of Geodesic Type

The interesting fact is [21] that a sufficient (though not necessary) condition for consistency is that *the curvature tensor constructed from the coefficients, treated as Christoffel symbols, is ze) andro*! Further, that *in this case, the system of geodesic equations, regarded as a system of second-order ODEs, is linearizable*! Thus we already know the solution in appropriate variables; it is the straight line in Cartesian coordinates, $\widehat{x}^a(s) = A^a s + B^a$. We also know the metric tensor in these coordinates. It is a Kronecker delta if we are guaranteed that the symmetry group is compact (so that the metric tensor is positive definite), and otherwise it is a diagonal matrix $\widehat{g}_{ab} = \operatorname{diag}(1, \pm 1, \ldots, \pm 1)$. We have also constructed it in the given variables. Now all we have to do is to write down the coordinate transformations from the computed metric tensor to the $\widehat{g}_{ab}$. The transformations are simply given by

$$\delta_i^a = \frac{\partial \widehat{x}^a}{\partial x^i}, \tag{4.2}$$

or their inverse, depending on the context. These are the linearizing transformations. Thus we will get

$$g_{ij} = \delta_i^a \delta_j^b \widehat{g}_{ab}. \tag{4.3}$$

A procedure is provided [36] to determine the δ_i^a given the g_{ij}. This enables us to write down the solution directly.

Example 4.1. The 2-dimensional system,

$$\begin{gathered} x'' - x'^2 - y'^2 = 0, \\ y'' - 2x'y' = 0, \end{gathered} \tag{4.4}$$

corresponds to the metric tensor

$$\begin{gathered} g_{11} = g_{22} = 2c_1^2 e^{-2y-2x-2y} + 2c_2^2 e^{2y-2x}, \\ g_{12} = g_{21} = 2c_1^2 e^{-2y-2x-2y} - 2c_2^2 e^{2y-2x}, \end{gathered} \tag{4.5}$$

with the corresponding linearizing transformation

$$u = c_1 e^{-y-x} + c_2 e^{y-x}, \qquad v = c_1 e^{-y-x} - c_2 e^{y-x}, \tag{4.6}$$

leading to the solution

$$v = c_3 + c_4 u \quad \text{or} \quad u = 0. \tag{4.7}$$

Example 4.2. The system,

$$\begin{aligned} x'' + \sin x \cos x y'^2 &= 0, \\ y'' - 2x'y' \cot x &= 0, \end{aligned} \tag{4.8}$$

is not geometrizable.

Example 4.3. The 3-dimensional system,

$$\begin{aligned} x'' &= \frac{x'^2}{x} + y'^2 \frac{xy + x}{y^2}, \\ y'' &= -2y'^2, \\ z'' &= -z'^2 - 2y'z', \end{aligned} \tag{4.9}$$

has a coordinate singularity at $x = y = 0$, and the linearizing transformation is

$$u = \ln xy, \qquad v = e^y, \qquad w = e^{y+z}, \tag{4.10}$$

yielding the straight line equation as a solution.

4.2. Linearization of Cubically Semilinear Second-Order Systems

Since the geodesic equations are invariant under translation and rescaling of the geodetic parameter, we can use the translational symmetry to project down one dimension, replacing the geodetic parameter by one of the dependent variables (say x^n) [18, 19]. One might have thought that the other symmetry generator could be used to project down one more dimension, but that is not so. The reason is that the two generators are degenerate, as they only depend on one and the same variable. When I was thinking about the problem I had not considered this method. Instead, I wanted to embed a general n-dimensional system into an $(n + 1)$-dimensional manifold and ask that the embedded equations be of geodesic type. When I later found that Aminova and Aminov had already got the same results by projection, I did not publish the other idea. However, there is no bar that I can see on a double embedding, which may lead to interesting systems of equations.

Since there is one symmetry, we can use one of the dependent variables, say x^n as the independent variable and now treat all the other dependent variables, x^a, as functions of the new independent variable. The projection procedure now puts

$$x^{a\prime} = \frac{dx^a}{dx^n} x^{n\prime} \quad (a = 1, \dots, n-1), \tag{4.11}$$

and hence

$$x^{a\prime\prime} = \frac{d^2 x^a}{d(x^n)^2} \left(x^{n\prime}\right)^2 + \frac{dx^a}{dx^n} x^{n\prime\prime}. \tag{4.12}$$

The resulting projected geodesic equations can be written as

$$x^{a\prime\prime} + A_{bc} x^{a\prime} x^{b\prime} x^{c\prime} + B^a_{bc} x^{b\prime} x^{c\prime} + C^a_b x^{b\prime} + D^a = 0, \tag{4.13}$$

where **A**, **B**, **C**, and **D** are functions of the independent and dependent variables.

$$A_{bc} = \Gamma^n_{bc}, \qquad B^a_{bc} = \Gamma^a_{bc} - 2\delta^a_{(c}\Gamma^n_{b)n}, \qquad C^a_b = 2\Gamma^a_{nb} - \delta^a_b \Gamma^n_{nn}, \qquad D^a = \Gamma^a_{nn}, \qquad (a, b, c = 1, \dots, n-1). \tag{4.14}$$

The linearization conditions are that the curvature tensor constructed from the Christoffel symbols is zero.

Remark 4.4. There are more Christoffel symbols than coefficients in the system. This means that there is an arbitrariness of choice of Christoffel symbols for a given system. However, one can provide a canonical procedure for making the choice. It is not clear that sometimes another choice may not be more convenient.

Remark 4.5. By taking geodesics in $n = 2$ and projecting down *we obtain a scalar cubically semilinear ODE, and the linearization conditions are precisely the Lie conditions*. The auxiliary functions here appear because of the degeneracy in the choice of Christoffel symbols. Obviously, the Tressé conditions follow.

Remark 4.6. The most general cubically semilinear system is

$$x^{a\prime\prime} + \Delta^a_{bcd} x^{b\prime} x^{c\prime} x^{d\prime} + \Lambda^a_{bc} x^{b\prime} x^{c\prime} + \Upsilon^a_b x^{b\prime} + A^a = 0, \tag{4.15}$$

which has more possible combinations of the cubic term. Those extra systems are not geometrically linearizable.

5. Complex Linearization

Whereas Lie had used complex DEs of complex variables in his analysis, he did not use the analyticity properties embodied in the CREs. At first sight it may be expected that this would not give anything new. However, there was one curious fact that attracted my attention. The algebra of the real linearizable scalar second-order ODEs is $sl(3,\mathbb{R})$, which has 8 real linearly independent generators. For the complex case it would be $sl(3,\mathbb{C})$, which has 8 complex linearly independent generators and hence 16 real independent operators. As explained later, these operators do not form a Lie algebra and are therefore not symmetry generators. The corresponding canonical ODE is the 2-dimensional real second-order system of ODEs with symmetry algebra $sl(4,\mathbb{R})$, which has 15 generators and not 16. Where did the extra generator go? One might think that one could drop one of the 16 generators to get the required subalgebra. This expectation is based on the intuition of dropping one generator of $gl(4,\mathbb{R})$ to get $sl(4,\mathbb{R})$. However, the real system cannot get the symmetry generators in this naive way, as the generators of the real system appear in pairs. What happens is that one leaves out one of the complex generators gets a set of only 14 generators, and then requiring the closure of the algebra, obtains the 15-generator algebra. This strange behaviour seemed to me to deserve more attention. On investigation [26, 27] it was found that splitting complex generators into real and imaginary parts yields interesting insights. For example, the complex scaling symmetry contains the real scaling and the *real rotation* in 2 dimensions!

Of course, a complex dependent variable will split into two real dependent variables and the complex independent variable into two real independent variables. Thus the scalar ODE would split into a system of two PDEs [26, 37]. The CREs would apply not only between the independent and the dependent variables but also between the independent variables and the derivatives of the dependent variables, to the relevant order. Here we are only concerned with the splitting of the complex scalar ODEs into systems of two real *ODEs*. I briefly explain the basics of the splitting procedure.

5.1. Complex Symmetry Analysis

To obtain ODEs we restrict the independent variable to the real line [26, 27]. Now a problem arises with the CREs. They normally apply to the derivatives with respect to the real and imaginary parts of the independent variables. To see how the CREs come in, consider the complex scalar ODE for a real variable written in semilinear form; say the second-order ODE $w'' = f(x,w,w')$. Now writing $w = y + \iota z$, and $f = f^r + \iota f^i$ we get the second-order system

$$y'' = f^r(x,y,z;y',z'), \qquad z'' = f^i(x,y,z;y',z'). \tag{5.1}$$

The point is that we can now ask for the function f to be analytic. As such, its derivatives with respect to the dependent variables and their derivatives must satisfy the CREs.

The symmetry operator for the complex scalar equation will also split into a real and imaginary part as

$$\begin{aligned}\mathbf{X} &:= \xi(x,w)\frac{\partial}{\partial x} + \eta(x,w)\frac{\partial}{\partial w}\\ &= \xi^r(x,y,z)\frac{\partial}{\partial x} + \frac{1}{2}\left[\eta^r(x,y,z)\frac{\partial}{\partial y} + \eta^i(x,y,z)\frac{\partial}{\partial z} + \iota\left\{\eta^i(x,y,z)\frac{\partial}{\partial y} - \eta^r(x,y,z)\frac{\partial}{\partial z}\right\}\right]\\ &:= \mathbf{X}_r + \iota\mathbf{X}_i.\end{aligned} \tag{5.2}$$

Thus, to every symmetry generator of the complex scalar ODE, there correspond two symmetry operators of the real 2-dimensional system of ODEs. *This* is where the problem of the lost extra generator comes from. The 8 for the complex equation $w'' = 0$, which has the symmetry algebra $\mathrm{sl}(3,\mathbb{C})$, will split into 16 operators. However, for the corresponding system, $y'' = 0, z'' = 0$, the maximal algebra is $\mathrm{sl}(4,\mathbb{R})$, which has 15 generators. It should now be obvious why one cannot just remove one of the 16, as it will take "its partner" out with it.

Not every 2-dimensional system of ODEs can be written as a complex scalar ODE of the same order. For example, for the linearizable second-order system there will generally be 18 coefficients of the terms involving first derivatives and 2 for the terms that only depend on the independent and dependent variables (which must satisfy the generalized Lie conditions for the system). However, for the corresponding scalar ODE there are only 4 complex coefficients in all, which become 8 real coefficients instead of 20. As such, the complex scalar ODE will only yield a class of all linearizable 2-dimensional linearizable systems. This class will be of the general form

$$\begin{aligned} y'' &= A_1\left(y'^3 - 3y'z'^2\right) - A_2\left(3y'^2z' - z'^3\right) + B_1\left(y'^2 - z'^2\right) - 2B_2y'z' + C_1y' - C_2\zeta' + D_1, \\ z'' &= A_1\left(3y'^2z' - z'^3\right) + A_2\left(y'^3 - 3y'z'^2\right) + 2B_1y'z' + B_2\left(y'^2 - z'^2\right) + C_2y' + C_1\zeta' + D_2, \end{aligned} \tag{5.3}$$

which must satisfy the generalized Lie constraints. As these are too long to convey much wisdom, they are left out here. (They are given in [37].) The important point to note is that there are now 8 coefficients as required and that the conditions can be written in the Tresse form.

The scalar second-order ODE may come from a variational principle. Classification may be done not only for the Lie symmetries but also for the symmetries of the action integral are called *Noether symmetries*. (Notice that these are distinct from the symmetries of the Lagrangian.) A connection between geometry and Noether symmetries has been explored, for example, in [38]. Noether symmetries are of special interest as they give double reduction of the Euler-Lagrange equation. It would be of interest to study the linearizability of these ODEs. The question then arises as to what happens to the Lagrangian when we split into real and imaginary parts. In general, we should then get a *complex Lagrangian*. However, that seems to become meaningless, as the variational procedure requires an ordered set for the action to be minimized, but the complex field is only *partially* ordered. Also, it might be thought that the *physical* quantity is a real Lagrangian. However, it turns out that complex Lagrangians, and correspondingly complex Hamiltonians, *have* been used in the literature, for example by Bender and others, in which they also explain some anomalies in atomic phenomena [39–44]. The variational principle has been used in complex symmetry analysis [45] but is not reviewed here, due to space considerations.

While the complex splitting may be an intriguing curiosity, at first sight it does not seem to provide a method for *solving* systems of ODEs. However, it can be used by converting a complex scalar ODE that can be easily solved to a system of two real ODEs and then reading off the solution of the system. It seems trivial as stated, but one can start with a general 2-dimensional system and check to see if it corresponds to a scalar ODE. One might feel that this inverse procedure "is cheating" and that it lacks generality, but the same objection could be raised on basic analytic integration methods that just invert the differentiation procedure. It is to be borne in mind that there is an *enormous* reduction in the amount of work to be done to solve a system of ODEs by symmetry methods if one can convert it to a scalar ODE.

It is especially useful for second-order systems as they can be linearized and the power of geometry used. Bear in mind that the prescriptions for solving by other symmetry methods are tedious and cumbersome.

Example 5.1. The system of two ODEs,

$$
\begin{aligned}
y'' &= -3(yy' - zz') - \left(y^3 - 3yz^2\right), \\
z'' &= -3(zy' + yz') - \left(3y^2 z - z^3\right),
\end{aligned}
\tag{5.4}
$$

corresponds to the linearizable complex scalar ODE

$$
u'' + 3uu' + u^3 = 0, \tag{5.5}
$$

whose solution is

$$
u = \frac{2(x-\alpha)}{x^2 - 2\alpha x - 2\beta}, \tag{5.6}
$$

yielding the solution of the system

$$
\begin{aligned}
y &= \frac{2(x-\alpha_1)\left(x^2 - 2\alpha_1 x - 2\beta_1\right) + 4\alpha_2(\alpha_2 x + \beta_2)}{\left(x^2 - 2\alpha_1 x - 2\beta_1\right)^2 + (2\alpha_2 x + 2\beta_2)^2}, \\
z &= \frac{4(x-\alpha_1)(\alpha_2 x + \beta_2) - 2\alpha_2\left(x^2 - 2\alpha_1 x - 2\beta_1\right)}{\left(x^2 - 2\alpha_1 x - 2\beta_1\right)^2 + (2\alpha_2 x + 2\beta_2)^2}.
\end{aligned}
\tag{5.7}
$$

This is not where it ends. One can start with a 2-dimensional complex *system* of ODEs and split it into a 4-dimensional real system [46] and now ask for the 2-dimensional system to be linearizable. Using the power of geometry for the 2-dimensional system one can write down the solution for the 4-dimensional system.

5.2. Classes of Complex Linearizable Systems

There are 5 classes of 2-dimensional linearizable systems of ODEs with 5, 6, 7, 8, or 15 infinitesimal symmetry generators [16]. There was further work done on systems regarding further details of their algebraic classification [23, 47] that I will not discuss further. The class of *geometrically* linearizable ODEs must have an $\mathrm{sl}(3,\mathbb{C})$ symmetry algebra with 15 generators. As such, it misses the other 4 classes of lower symmetry. Where did they go? The answer is that they are linearizable but not *geometrically* so. It would have been great to be able to get all the linearizable classes accessible to the power of geometry. At first sight, it appears to be impossible. It turns out that this is not quite true. Though we have not been able to get at *all* the classes, we can make two more classes accessible to geometry by the use of complex symmetry analysis. To explain this it is necessary to briefly state a result used for the equivalence of systems of ODEs which reduces the number of coefficients to $n^2 - 1$, and then to show that for $n = 2$ with complex methods we get 2 of them instead of 3.

As mentioned in the section on Preliminaries, any system of n second-order nonhomogeneous linear ODEs with $2n^2 + n$ arbitrary coefficients of the form given in (3.12) can be mapped invertibly to one of the forms of n linear homogeneous second-order ODEs with n^2 coefficients, (3.13) or (3.14), and some canonical forms that have fewer arbitrary coefficients. Thus a system of two second-order ODEs ($n = 2$) has 10 coefficients and may reduce to one with 4 arbitrary coefficients of the form,

$$\ddot{y} = a(t)y + b(t)z, \qquad \ddot{z} = c(t)y + d(t)z, \tag{5.8}$$

or the form involving the first derivative. The number of arbitrary coefficients was further reduced to 3 by the change of variables [16]

$$\tilde{y} = \frac{y}{\rho(t)}, \qquad \tilde{z} = \frac{z}{\rho(t)}, \qquad x = \int^t \rho^{-2}(s)ds, \tag{5.9}$$

where ρ satisfies

$$\rho'' - \frac{a+d}{2}\rho = 0, \tag{5.10}$$

to the linear system

$$\tilde{y}'' = \tilde{a}(x)\tilde{y} + \tilde{b}(x)\tilde{z}, \qquad \tilde{z}'' = \tilde{c}(x)\tilde{y} - \tilde{a}(x)\tilde{z}, \tag{5.11}$$

where

$$\tilde{a} = \frac{\rho^3(a-d)}{2}, \qquad \tilde{b} = \rho^3 b, \qquad \tilde{c} = \rho^3 c. \tag{5.12}$$

This procedure of reduction of arbitrary coefficients for linearizable systems simplifies the classification problem enormously. Recall that a general 2-dimensional system of ODEs has 10 arbitrary coefficients which is reduced to 3, a 3-dimensional system of ODEs has 21 which get reduced to 8, a 4-dimensional system has 36 which reduce to 15, and so on. System (5.11) is called the *optimal canonical form* for linear systems of two second-order ODEs, as it has the least number of arbitrary coefficients.

Following the classical Lie procedure, one uses the point transformations

$$X = X(x,y,z), \qquad Y = Y(x,y,z), \qquad Z = Z(x,y,z), \tag{5.13}$$

to invertibly map (at most) cubically semilinear linearizable system of two second-order ODEs [48],

$$y'' = f_1(x,y,z,y',z'), \qquad z'' = f_2(x,y,z,y',z'), \tag{5.14}$$

to the 2-dimensional free particle system of ODEs. This yields

$$y'' + \alpha_{11}y'^3 + \alpha_{12}y'^2z' + \alpha_{13}y'z'^2 + \alpha_{14}z'^3 + \beta_{11}y'^2 + \beta_{12}y'z' + \beta_{13}z'^2 + \gamma_{11}y' + \gamma_{12}z' + \delta_1 = 0,$$

$$z'' + \alpha_{21}y'^3 + \alpha_{22}y'^2z' + \alpha_{23}y'z'^2 + \alpha_{24}z'^3 + \beta_{21}y'^2 + \beta_{22}y'z' + \beta_{23}z'^2 + \gamma_{21}y' + \gamma_{22}z' + \delta_2 = 0, \tag{5.15}$$

the coefficients being functions of the independent and dependent variables as well as the derivatives of the dependent variables. This is the most general candidate for two second-order ODEs that may be linearizable. Comparing the coefficients appearing in this system with those of a linearizable scalar complex second-order cubically semilinear ODE provides the conditions on the coefficients of (5.15). Writing out these conditions yields the theorem nextly mentioned.

Theorem 5.2. *Any system of the form* (5.15) *corresponds to a scalar complex second-order ODE if and only if the coefficients α_{ij}, β_{ik}, γ_{il}, and δ_i satisfy the conditions:*

$$\begin{gathered} 3\alpha_{11} = -\alpha_{13} = -\alpha_{22} = -\alpha_{24}, \qquad -\alpha_{12} = 3\alpha_{14} = 3\alpha_{21} = -\alpha_{23}, \\ 2\beta_{11} = \beta_{22} = -2\beta_{13}, \qquad 2\beta_{21} = -\beta_{12} = -2\beta_{23}, \\ \gamma_{11} = \gamma_{22}, \qquad \gamma_{21} = -\gamma_{12}, \end{gathered} \tag{5.16}$$

where $i = l = 1, 2,\ j = 1, .., 4$, and $k = 1, 2, 3$.

The simplest form for linear systems of two second-order ODEs corresponding to complex scalar ODEs is obtained by using the equivalence of scalar second-order linear ODEs and hence reduces the number by one. Starting with a general linear scalar complex second-order ODE

$$\ddot{w} = \zeta_1(x)\dot{w} + \zeta_2(x)w + \zeta_3(x), \tag{5.17}$$

one reduces to

$$\ddot{w} = \alpha(t)w, \tag{5.18}$$

where $\alpha(t) = \alpha_1(t) + \iota\alpha_2(t)$. This yields the system of two linear second-order ODEs

$$\ddot{y} = \alpha_1 y - \alpha_2 z, \qquad \ddot{z} = \alpha_2 y + \alpha_1 z. \tag{5.19}$$

The reason that we can reduce to just *two functions* is that we are dealing with the special class of linear systems of ODEs that correspond to scalar complex linearizable ODS. This gives the following theorem [49].

Theorem 5.3. *A linear system of two second-order ODEs,*

$$\ddot{y} = \alpha_1 y - \alpha_2 z, \qquad \ddot{z} = \alpha_2 y + \alpha_1 z, \tag{5.20}$$

can be mapped invertibly to a system of the form

$$Y'' = -\beta(x) Z, \qquad Z'' = \beta(x) Y, \tag{5.21}$$

where $\beta(x)$ is a complex function.

Since we have only one (complex) coefficient function involved; there are now the usual three cases: (a) $\beta(x)$ is an arbitrary function; (b) it is a constant function; (c) it is zero. On working through it we found that (a) gives 6 symmetry generators, (b) 7, and (c) 15. The cases of 5 and 8 symmetry generators were not accessed by complex symmetry analysis, but it *did* get the other two missing classes! One would really like to find some way of characterizing them.

I now give some illustrative examples taken from [49].

Example 5.4. The system corresponding to $\beta = 0$,

$$\begin{aligned} y'' &= -y'^2 + z'^2 - \frac{2}{x} y', \\ z'' &= -2y'z' - \frac{2}{x} z', \end{aligned} \tag{5.22}$$

can be mapped to the linear free particle system by the linearizing transformation

$$X = \frac{1}{x}, \qquad Y = e^y \cos z, \qquad Z = e^y \sin z. \tag{5.23}$$

It corresponds to the scalar linearizable ODE,

$$w'' = -w'^2 - \frac{2}{x} w', \tag{5.24}$$

which is linearizable and has a 15-dimensional symmetry algebra.

Example 5.5. The system corresponding to constant β,

$$\begin{aligned} y'' + y'^2 - z'^2 &= c_1 y' - c_2 z', \\ z'' + 2y'z' &= c_2 y' + c_1 z', \end{aligned} \tag{5.25}$$

can be invertibly mapped

$$\begin{aligned} Y'' &= c_1 Y' - c_2 Z', \\ Z'' &= c_2 Y' + c_1 Z', \end{aligned} \tag{5.26}$$

which has a 7-dimensional Lie algebra, by the point transformation

$$X = x, \qquad Y = e^{y}\cos z, \qquad Z = e^{y}\sin z. \tag{5.27}$$

Example 5.6. The system corresponding to variable β,

$$\begin{aligned} y'' + y'^2 - z'^2 &= c_1(1+x)y' - c_2(1+x)z', \\ z'' + 2y'z' &= c_2(1+x)y' + c_1(1+x)z', \end{aligned} \tag{5.28}$$

corresponds to the complex second-order ODE

$$w'' + w'^2 - cx = 0 \tag{5.29}$$

and can be linearized by the previously mentioned point transformation. It has 6 symmetry generators.

5.3. Beyond the Wafo Soh-Mahomed Theorem

Though we cannot get the 5-dimensional case of linearizable systems for two variables, we can go still further in a novel way. Here is an example from [50] that has a 4-dimensional Lie algebra only.

Example 5.7. We have

$$\begin{aligned} y'' - y'^3 + 3y'z'^2 &= 0, \\ z'' - 3y'^2z' + z'^3 &= 0, \end{aligned} \tag{5.30}$$

which corresponds to the complex scalar linearizable ODE

$$w'' - w'^3 = 0 \tag{5.31}$$

that is linearizable and has the solution

$$w(x) = \pm\sqrt{2x + 2C_1} + C_2. \tag{5.32}$$

Thus the solution of the system is

$$\begin{aligned} y(x) &= \pm\left[\sqrt{\frac{(x+a+b)}{2}} + \sqrt{\frac{(x+a-b)}{2}}\right] + c, \\ z(x) &= \pm\left[\sqrt{\frac{(x+a+b)}{2}} - \sqrt{\frac{(x+a-b)}{2}}\right] + d. \end{aligned} \tag{5.33}$$

Though this system could be solved by the normal (tedious) symmetry methods, as it does have the minimum number of symmetry generators required for the purpose, it could not be solved by *linearization*.

What happened to the Wafo Soh-Mahomed theorem which said that such systems cannot be linearized as that would require 5 generators?

There is worse (or perhaps one should say "better") to follow. It was found that one can even go down below the minimum number required for symmetry solutions of systems of ODEs. There are examples of 3 and 2 and even of *one* symmetry generator; the last of which is given here.

Example 5.8. We have

$$\begin{aligned} &y'' - xyy'^3 + 3xzy'^2z' + 3xyy'z'^2 - xzz'^3 = 0, \\ &z'' - xzy'^3 - 3xyy'^2z' + 3xzy'z'^2 + xyz'^3 = 0, \end{aligned} \tag{5.34}$$

with the single scaling symmetry generator $X_1 = x\partial_x$. It corresponds to the complex scalar linearizable ODE

$$w'' - xww'^3 = 0, \tag{5.35}$$

which has the implicit solution

$$x = \Re e\left[aAi(y + \imath z) + bBi(y + \imath z)\right], \qquad 0 = \Im m\left[aAi(y + \imath z) + bBi(y + \imath z)\right], \tag{5.36}$$

where Ai, Bi are the Airy A and B functions, and $a, b\ \varepsilon\mathbb{R}$.

Now what happened not only to the Wafo Soh-Mahomed theorem but also to all the limitations of the general (tedious or not) symmetry methods? They seem to have been *all* bypassed.

The Wafo Soh-Mahomed theorem continues to hold. *The system never got linearized*. It only corresponded to a complex scalar ODE that *was* linearizable. The point is that the linearizing transformation for the complex scalar ODE converts the real independent variable to a complex one. Now we no longer generally have a system of ODEs, it is a system of PDEs. It turns out that the solution of the original system does "go through the filter" of complexifying and then reducing to the real, but the rest need not. In the previously mentioned example with 4 generators we got 4 arbitrary constants, which could (in principle) have solved a linear 2-dimensional system, but in the example with only one symmetry generator there are only *two* arbitrary constants. This could not be the general solution of a system of 2 linear second order ODEs. This is not a complete answer to how the limitations got bypassed but may be regarded as a partial answer. It is an example of Penrose's "complex magic" [28].

6. Conditional Linearizability

A major problem with the geometric approach is that it is limited to the second order. For other orders it would appear that we have to forego the power of geometry. Recall that the connection to geometry is through the geodesic equations, which are second order and limited to those of geodesic type. The latter restriction was avoided by using projections. However, there is no geometrical quantity that comes naturally from the third or higher derivatives. How, then, can we go beyond this restriction?

One way is to differentiate the second-order ODE that can be dealt with geometrically but that seems to be trivial. Once again, the appearance of triviality may be misleading. Consider the symmetries of the free particle equation and its derivative. Instead of increasing the number of symmetry generators, we have *reduced* it! Further, the structure of the symmetry algebra is *totally* different. It is largely abelian, with a small nonabelian subgroup. Differentiating once again would restore the number of generators, but the symmetry algebra would be different. Again, something odd is happening. One also takes it for granted that the solution of the original equation will be the solution of the new equation. Implicit in there is the assumption that requiring that inserting the original equation into the differentiated equation will make no difference. However, the symmetry structure of the system of equations will be vastly different. As such, it is worthwhile to explicitly insert the original equation into the differentiated one to see what happens.

Differentiate the general second-order cubically semilinear scalar ODE (3.2) and insert the original equation in by replacing the second-derivative term using (3.2). One now has the third-order quintically semilinear ODE

$$\begin{aligned} &y''' - 3c^2y'^5 + (c_y - 5cg)y'^4 + \left(c_x + g_y - 4ch - 2g^2\right)y'^3 \\ &\quad + (g_x + h_y - 3cd - 2gh)y'^2 + \left(h_x + d_y - 2gd - h^2\right)y' + (d_x - hd) = 0. \end{aligned} \tag{6.1}$$

The number of symmetry generators here does not need to be the same as for the original second-order or the differentiated third-order ODE. It could have more or less symmetry generators than either. We have thoroughly "messed up" the symmetry structure of the ODEs. The new ODE is not a total derivative in general. This is easily seen by considering an ODE that had constant d and h. On differentiation it would not have a constant term in it, but on inserting back the original ODE it would again have a constant term in it. As such, it could not be a total derivative. To see how to use this procedure of differentiation and reinsertion, consider the general quintically semilinear third-order ODE

$$y''' - \alpha y'^5 + \beta y'^4 - \gamma y'^3 + \delta y'^2 - \epsilon y' + \phi = 0. \tag{6.2}$$

We can now compare coefficients to determine the second-order ODE from which it could have arisen. Of course, this equation has 6 coefficients, while a cubically semilinear second-order ODE could only have 4. Therefore, there is no guarantee that the third-order ODE could have arisen from the candidate second-order ODE. Consequently there have to be consisten criteria to be satisfied. These are given in detail in [51].

This procedure is called *conditional linearization*, and the second-order ODE from which it can be obtained is called the *root equation*. It can yield the solution of higher order ODEs that have only 2 arbitrary constants in them, coming from the second-order ODE that could be differentiated to obtain them by replacement. It may be that such equations are not amenable to the classical (Lie) linearization procedure. An illustrative example is presented.

Example 6.1. The third-order ODE taken is

$$y''' - \frac{3x^2y'^5}{y^4} - \frac{3xy'^4}{y^3} - \frac{6y'^3}{y^2} + \frac{6y'^2}{xy} - \frac{6y'}{x^2} = 0. \tag{6.3}$$

It does not satisfy the criteria given by Ibragimov and Meleshko [13] but *is* amenable to conditional linearizability. On identifying the coefficients and checking the consistency criteria one can construct the second-order ODE from which it can be obtained. That ODE is geometrically linearizable and its solution is

$$Axy + \frac{Bx}{y} = 1. \tag{6.4}$$

The previous procedure can be taken further [52]. One can differentiate the third-order ODE obtained by differentiating the root equation and then inserting the root equation in it, inserting the third-order ODE in, first inserting the root equation into the third-order and then differentiating again. or even inserting the root equation in twice. If we follow the last mentioned procedure, we get a fourth-order septically semilinear ODE. Thus, given such an ODE in general, one can check if it can come from a second-order ODE. If so, we can construct the root equation, and if one can solve it one has the solution of the fourth-order ODE. A couple of examples are provided that do not satisfy the Ibragimov-Meleshko-Suksern criteria for fourth-order ODEs [14].

Example 6.2. The fourth-order ODE

$$y'''' - \frac{15x^3y'^7}{y^6} - \frac{15xy'^6}{y^5} + \frac{39y'^5}{y^4} + \frac{39y'^4}{xy^3} - \frac{36y'^3}{xy^2} - \frac{36y'^2}{x^2y} + \frac{24y'}{x^3} = 0 \tag{6.5}$$

is septically semilinear with only the first derivative appearing in it and turns out to be conditionally linearizable with the same second-order root equation as in the previous example and hence has the same solution.

Example 6.3. The fourth-order ODE,

$$\begin{aligned} y'''' &- \left(\frac{6xy'}{y^2} + \frac{2}{y}\right)y''^2 - \left(\frac{9x^2y'^4}{y^4} - \frac{2xy'^3}{y^3} - \frac{7y'^2}{y^2} - \frac{8y'}{xy} + \frac{8}{x^2}\right)y'' \\ &+ \frac{6x^2y'^6}{y^5} - \frac{2xy'^5}{y^4} - \frac{2y'^4}{y^3} - \frac{6y'^3}{xy^2} - \frac{4y'^2}{x^2y} + \frac{8y'}{x^3} = 0, \end{aligned} \tag{6.6}$$

is quadratically semilinear in the second derivative, with the allowed polynomials of firstderivatives as coefficients and is again conditionally linearizable with the same second-order root equation as in the previous example and hence has the same solution.

7. Proposal for Classification of Systems of ODEs

How far can we take this procedure? In principle, we can go to any higher order ODE with a second-order root equation and retain the power of geometry. There are, of course, many more possibilities for the replacement of the derivatives here. Though the calculations get extremely complicated and messy; in principle one could do the same for any system of ODEs [53]. However, if one wants to take a different root equation one gets restricted to scalar equations only as the power of geometry is lost. That is not to say that the concept of conditional linearizability will not apply. Rather, the procedure for directly writing down the solution of the equation is now lacking.

At the moment ignoring this problem, it is clear that one can say something nontrivial about the original motivation of Lie in introducing Lie groups. The key point was not to provide solutions but to classify equations according to their solvability by point transformations a'la Galois. A handle has been provided by conditional linearizability [54] as I now explain.

One can start with a root equation of any order m, including the first. In fact, the first-order ODEs are all linearizable, and hence we are not restricted to any special class there. For the second order we have seen how Lie established linearizability. For higher order scalar ODEs the linearizability criteria can, in principle, be obtained by algebraic computer codes. One can then differentiate to any other (higher) order, m, and go through some replacement procedure. For example, the fifth-order ODE that only depends on the first derivative and is conditionally linearizable with a second-order root equation has a ninth-order polynomial dependence on the first derivative. Similarly, starting with one of the Ibragimov-Meleshko classes yields a fourth-order conditionally linearizable ODE. The resulting equation will have at least one solution, that is, the general solution of the root equation. That must have m arbitrary constants. Depending on the replacement procedure, the number of arbitrary constants, p, appearing in the general solution will lie somewhere between m and n. It can be hoped that we would be able to determine p by the replacement procedure. As such, there would be p arbitrary initial conditions that could be required for the ODE. We can then give the following definition.

Definition 7.1. An mth order ODE ($m > 1$) will be said to be *conditionally classifiable* by a symmetry algebra $\mathcal{A}$ with respect to a pth order root ODE ($p < m$) if and only if the mth order ODE jointly with the pth order ODE forms an overdetermined compatible system (so the solutions of the mth order ODE reduce to the solutions of the pth order ODE), and the pth order ODE has symmetry algebra $\mathcal{A}$. The most general class of linearizable, conditionally linearizable, or conditionally classifiable ODEs will be denoted by $\mathcal{L}_p^m$.

In the context of this definition the following conjecture was stated [54].

Conjecture 7.2. *All ODEs, or systems of ODEs, of order $p \geq 2$ are linearizable, conditionally linearizable, symmetry classifiable, or conditionally classifiable by symmetry, that is in terms of $\mathcal{L}_p^m$.*

This leads to the proposal that Lie's programme could, perhaps, be completed by using linearizability and classifiability presented previously. Of course, this would not apply to the singular case $p = 1$, which is, why the conjecture was stated for $p \geq 2$.

8. Summary and Discussion

Inspired by Galois' success with algebraic (polynomial) equations, Lie tried to replicate it for differential equations. While Galois developed groups, Lie developed Lie groups. In one sense, then, Galois' development seems more fundamental than Lie's. It led to more definitive results, namely, the non solvability of quintic and higher order equations by means of radicals. However, Lie's attempt was far more ambitious, given the much greater complexity of the problem of dealing with DEs as a whole. The attempt paid rich dividends. Not only did he provides a systematic procedure for solving DEs by transformation of variables, his groups led to a much deeper understanding of differential geometry and provided it with much greater power. In fact, Lie's original aim seems to have got lost in the process. Recently Lie's symmetry analysis of DEs has gained popularity because it has made it possible to solve nonlinear problems arising in engineering, such as those in non-Newtonian fluid dynamics. Those problems had been trivialized so as to make them solvable. With the advent of electronic computing the tendency was to find numerical solutions. However, these could be misleading as one may appear to get a solution that does not exist. Also, the approximations involved could often "throw the baby out with the bath water." Now it has begun to appear that geometry can pay back for its benefits with interest. In this paper the "pay-back" has been reviewed.

The benefit of geometry rests on the connection between the Lie symmetries of systems of geodesic equations and the isometries of the manifold in which they lie. The connection seems trivial if one is only looking at geodesic equations. However, it becomes a nontrivial method for solving systems of ODEs that could be regarded as geodesic equations. Further, it becomes thoroughly nontrivial when one extends to systems of ODEs that correspond to *projected* systems of geodesic equations. One then gets a method for directly writing down solutions of linearizable systems based on the beautiful result that the linearizability criteria are equivalent to the requirement that the underlying manifold is to be flat. There is a caveat here. For systems the requirements are not generally equivalent but only for the maximally symmetric linearizable cases. The geometrization of the other classes is a problem.

The problem was partially overcome by using the development of complex symmetry analysis. For a system of two ODEs one obtains 3 of the 5 linearizable classes. At present that is a very minor part of the total. However, one can split a system of 2 complex ODEs to obtain a system of 4 real ODEs, a system of 3 to obtain a system of 6, and so on. This remains limited to only even dimensional systems. How about the odd ones? There is a development to obtain odd dimensional systems, but that is not adequately explored to discuss here. The bigger problem is that even for the 2-dimensional case we are missing 2 classes. Where did they go? As yet there is no answer. One can speculate that the procedure of projection has limited the number of classes. If my original idea of embedding was used we may be able to bypass the restriction of only shifting by one step. One could perhaps embed *twice*. If so, we may recover the two missing classes. The level of computational complication that arises makes it difficult to actually apply the methods being talked of. With improved algebraic computing one may be able to routinely use these methods with a fast computer. As a bonus the complex methods provided a means of solving systems of ODEs that were not amenable to solution by symmetry methods. An interesting point arises in connection with complex linearization. We have found examples of systems of two ODEs that can be solved if they have even one symmetry generator. Does there exist a system with *no symmetry generator*. That would be worth looking for. Ali, Safdar, and I conjecture that no such systems of ODEs exists. However, we have no proof of this conjecture. It would be worth either finding one or finding a counter example to it.

The extension from ODEs of geodesic type has gone fairly far with the use of projection and complexification. However, all said and done it is *only* second-order ODEs that we are talking of. The geometric method cannot be used to linearize higher order systems of ODEs. An inroad is made with conditional linearization, by using a second-order root equation to obtain a higher system of ODEs. This does not necessarily provide a linearizable higher order system but does provide a (limited) solution. As a bonus it provides a possible path to the "holy grail" of classification of ODEs according to their solvability that had been Lie's original motivation. We might be able to classify them according to the nature of the initial value problem that can be solved as regards to the number of initial conditions that could be met in general. There is a caveat here as well. It has been found that the method used by Meleshko [32] for linearizing third-order autonomous ODEs does not fall in the Ibragimov-Meleshko classes or the conditionally linearizable classes. In fact, the method can be generalized to higher orders [55], and it is generally found that other classes of this type emerge. As such, there is at least one other method for linearization. There again, it is not linearization in the sense of Lie but will provide another general classifiable class. How many such classes are there? If the number is finite, the proposal can be meaningfully completed, but if there are infinitely many the proposal must fail.

Even with all of the previously mentioned pious hopes being met, we have still only scratched the surface of the problem. ODEs are a very limited class of DEs. What about PDEs? One needs to find a connection between geometry and PDEs. The natural expectation would be to look for a generalization of geodesics for higher dimensional subspaces. (Bear in mind that a geodesic is a 1-dimensional subspace.) They arise by solving the problem of minimizing the arc length. The natural extension would appear to be minimal surfaces. However, so far attempts to use them have not led to any great success. This is perhaps the most important development in this direction is needed.

Acknowledgments

The author gratefully acknowledges useful comments and suggestions by Sajid Ali, Hina Dutt, and Fazal Mahomed. He is also grateful to DECMA and the School of Computational and Applied Mathematics, University of the Witwatersrand, Johannesburg, South Africa, for support at the Galois Bicentenary Conference (SDEA-I), where this work was presented and to the Abdus Salam International Centre for Theoretical Physics, Trieste, Italy, for support at Trieste where this paper was written.

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