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Research Article

A Newton Interpolation Approach to Generalized Stirling Numbers

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We employ the generalized factorials to define a Stirling-type pair $\{s(n,k;\boldsymbol{\alpha},\boldsymbol{\beta},r), S(n,k;\boldsymbol{\alpha},\boldsymbol{\beta},r)\}$ which unifies various Stirling-type numbers investigated by previous authors. We make use of the Newton interpolation and divided differences to obtain some basic properties of the generalized Stirling numbers including the recurrence relation, explicit expression, and generating function. The generalizations of the well-known Dobinski's formula are further investigated.

1. Introduction

Throughout this paper the following notations will be used. We denote by $\mathbb{R}$ the set of real numbers and by $\mathbb{C}$ the set of complex numbers. Let $\boldsymbol{\alpha} = (\alpha_0, \alpha_1, \ldots)$ be a vector. If $\alpha_i = i\theta (i = 0, 1, \ldots)$, we denote the vector by $\overline{\boldsymbol{\theta}}$. We further denote $(0, 1, \ldots)$ by $\overline{\mathbf{1}}$ and $(0, 0, \ldots)$ by $\mathbf{0}$. Moreover, let us denote the generalized kth falling factorial of x with increment θ by $(x)^{(k,\theta)} = x(x-\theta)\cdots(x-k\theta+\theta)$. Particularly, if $\theta = 1$, we write $(x)^{(k)} = x(x-1)\cdots(x-k+1)$.

In mathematics, Stirling numbers of the first and second kind, which are named after James Stirling, arise in a variety of combinatorics problems. They have played important roles in combinatorics. Stirling numbers of the first kind are the coefficients in the expansion $(x)^{(n)} = \sum_{k=0}^{n} s(n,k)x^k$, and Stirling numbers of the second kind are characterized by $x^n = \sum_{k=0}^{n} S(n,k)(x)^{(k)}$.

Over the past few decades, there has been an interest in generalizing and extending the Stirling numbers in mathematics literature. By starting with transformations between generalized factorial involving three arbitrary parameters α, β, and r, Hsu and Shiue [1] introduced the generalized numbers $S(n,k;\alpha,\beta,r)$ and unified those generalizations of the Stirling numbers due to Riordan [2], Carlitz [3, 4], Howard [5], Charalambides-Koutras [6], Gould-Hopper [7], Tsylova [8], and others. They define a Stirling-type pair $\{S(n,k;\alpha,\beta,r), s(n,k;\alpha,\beta,r)\}$ by

$$(x)^{(n,\alpha)} = \sum_{k=0}^{n} S(n,k;\alpha,\beta,r)(x-r)^{(k,\beta)},$$
$$(x)^{(n,\beta)} = \sum_{k=0}^{n} s(n,k;\alpha,\beta,r)(x+r)^{(k,\alpha)}. \tag{1.1}$$

They systematically investigated many basic properties including orthogonality relations, recurrence relations, generating function, and the Dobinski identity for their Stirling numbers. Recently, Comtet [9] defines $s_\alpha(n,k)$ and $S_\alpha(n,k)$, the generalized Stirling numbers of the first kind and second kind associate with $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$, by

$$(x-\alpha_0)(x-\alpha_1)\cdots(x-\alpha_{n-1}) = \sum_{k=0}^{n} s_\alpha(n,k)x^k,$$
$$x^n = \sum_{k=0}^{n} S_\alpha(n,k)(x-\alpha_0)(x-\alpha_1)\cdots(x-\alpha_{k-1}). \tag{1.2}$$

El-Desouky [10] modified the noncentral Stirling numbers of the first and second kind. He defined the multiparameter noncentral Stirling numbers of the first kind and second kind as follows:

$$(x-\alpha_0)(x-\alpha_1)\cdots(x-\alpha_{n-1}) = \sum_{k=0}^{n} S(n,k;\overline{\alpha})(x)^{(k)}, \tag{1.3}$$

$$(x)^{(n)} = \sum_{k=0}^{n} s(n,k;\overline{\alpha})(x-\alpha_0)(x-\alpha_1)\cdots(x-\alpha_{k-1}). \tag{1.4}$$

The recurrence relations, generating functions, and explicit forms for El-Desouky's Stirling numbers are obtained.

In another direction, Stirling numbers and their generalizations were investigated via differential operators. Carlitz and Klamkin [11] defined the Stirling numbers of the second kind by

$$(xD)^n = \sum_{k=1}^{n} S(n,k)x^k D^k, \tag{1.5}$$

where D is a differential operator d/dx. Actually, this can be traced back at least to Scherk [12]. In the physical literature, Katriel [13] discovered (1.5) was in connection with the normal ordering expressions in the boson creation operator $a^\dagger$ and annihilation a, satisfying the commutation relation $[a, a^\dagger] = 1$ of the Weyl algebra. Recently, Lang [14, 15] generalized the stirling numbers of the second kind by the following operator identity:

$$(x^r D)^n = \sum_{k=1}^{n} S(r; n, k) x^k D^k, \tag{1.6}$$

where r is a nonnegative integer. He further obtained many properties of these numbers. More recently, Blasiak et al. [16] defined $S_{r,s}(n, k)$, the generalized Stirling numbers of the second kind arising in the solution of the general normal ordering problem for a boson string, as follows

$$(x^r D^s)^n = x^{n(r-s)} \sum_{k=s}^{ns} S_{r,s}(n, k) x^k D^k. \tag{1.7}$$

These numbers were firstly defined by Carlitz [17]. More generally, given two sequences of nonnegative integers $\mathbf{r} = (r_1, r_2, \ldots, r_n)$ and $\mathbf{s} = (s_1, s_2, \ldots, s_n)$, Blasiak [18] generalized this formula by

$$x^{r_n} D^{s_n} \cdots x^{r_2} D^{s_2} x^{r_1} D^{s_1} = x^{d_n} \sum_{k=s_1}^{s_1+s_2+\cdots+s_n} S_{\mathbf{r},\mathbf{s}}(n, k) x^k D^k, \tag{1.8}$$

where $d_n = \sum_{k=1}^{n} (r_k - s_k)$. He gave an explicit formula for the generalized Stirling numbers $S_{\mathbf{r},\mathbf{s}}(n, k)$. In [19], a different explicit expression for these numbers was presented.

By considering powers $(VU)^n$ of the noncommuting variables U, V satisfying $UV = VU + hV^s$, Mansour and Schork [20] introduced a new family of generalized Stirling numbers $\mathfrak{S}_{s;h}(n, k)$ as

$$(VU)^n = \sum_{k=1}^{n} \mathfrak{S}_{s;h}(n, k) V^{s(n-k)+k} U^k, \tag{1.9}$$

which reduced to the conventional Stirling numbers of second kind and Bell numbers in the case $s = 0$, $h = 1$. As mentioned in [21], this type of generalized Stirling numbers is not a special case of Howard's degenerate weight Stirling numbers although they look very similar.

Moreover, for any sequence of real numbers $\overline{\boldsymbol{\alpha}} = (\alpha_0, \alpha_1, \ldots, \alpha_{n-1})$ and a sequence of nonnegative integers $\overline{\mathbf{r}} = (r_0, r_1, \ldots, r_{n-1})$, by using operational identity [22, 23] El-Desouky and Cakić [24] defined a generalized multiparameter noncentral Stirling numbers of the second kind $S(n, k; \overline{\boldsymbol{\alpha}}, \overline{\mathbf{r}})$ by

$$\prod_{j=0}^{n-1} \left(x^{\alpha_i} \delta x^{-\alpha_j}\right)^{r_j} = \prod_{j=0}^{n-1} \left(\delta - \alpha_j\right)^{r_j} = \sum_{k=0}^{|\overline{\mathbf{r}}|} S(n, k; \overline{\boldsymbol{\alpha}}, \overline{\mathbf{r}}) x^k D^k, \tag{1.10}$$

where $|\overline{\mathbf{r}}| = r_0 + r_1 + \cdots + r_{n-1}$. These numbers reduced to the multiparameter noncentral Stirling numbers of the second kind $S(n, k; \overline{\alpha})$ in (1.3) if all $r_i = 1$.

As a useful tool, the Newton interpolation with divided differences was utilized to obtain closed formulas for Dickson-Stirling numbers in the paper [25] provided by the referee. In this paper, we make use of the generalized factorials to define a Stirling-type pair $\{s(n,k;\boldsymbol{\alpha},\boldsymbol{\beta},r), S(n,k;\boldsymbol{\alpha},\boldsymbol{\beta},r)\}$ which unifies various Stirling-type numbers investigated by previous authors. By using the Newton interpolation and divided differences, we obtain the basic properties including recurrence relations, explicit expression, and generating function. The generalizations of the well-known Dobinski's formula are further investigated. This paper is organized as follows. In Section 2, we introduce the Newton interpolation and divided differences. Several important properties of divided differences are presented. In Section 3, the definitions of a new family of Stirling numbers are given. According to the definitions, the recurrence relation as well as an explicit formula is derived. Moreover, we also investigate the generating function for our generalized Stirling numbers. In views of our results, we rediscover many interesting special cases which are introduced in the above. Finally, in Section 4, the associated generalized Bell numbers and Bell polynomials are presented. Furthermore, a generalized Dobinski's formula is derived.

2. Divided Differences and Newton Interpolation

For a sequence of points $\boldsymbol{\alpha} = (\alpha_0, \alpha_1, \ldots)$ and all $\alpha_i \in \mathbb{R}$ or $\mathbb{C}$, we define

$$\omega_{0,\boldsymbol{\alpha}}(x) = 1, \qquad \omega_{n,\boldsymbol{\alpha}}(x) = \prod_{i=0}^{n-1}(x-\alpha_i), \quad n = 1,2,\ldots. \tag{2.1}$$

Let $N_n(x)$ be the Newton interpolating polynomial of degree at most n that interpolates a function $f(x)$ at the point $\alpha_0, \alpha_1, \ldots, \alpha_n$; then this polynomial is given as in

$$N_n(x) = \sum_{i=0}^{n} \Delta(\alpha_0,\ldots,\alpha_i)f \cdot \omega_{i,\boldsymbol{\alpha}}(x), \tag{2.2}$$

where $\Delta(\alpha_0,\ldots,\alpha_i)f$ is the divided difference of the ith order of the function f. As is well known, for the distinct points $\alpha_0, \alpha_1, \ldots, \alpha_n$, the divided differences of the function f are defined recursively by the following formula:

$$\Delta(\alpha_0)f = f(\alpha_0), \tag{2.3}$$

$$\Delta(\alpha_0,\ldots,\alpha_n)f = \frac{\Delta(\alpha_0,\ldots,\alpha_{n-1})f - \Delta(\alpha_1,\ldots,\alpha_n)f}{\alpha_0 - \alpha_n}. \tag{2.4}$$

Divided differences as the coefficients of the Newton interpolating polynomial have played an important role in numerical analysis, especially in interpolation and approximation by polynomials and in spline theory; see [26] for a recent survey. They also have many applications in combinatorics [27, 28]. The divided differences can be expressed by the explicit formula

$$\Delta(\alpha_0,\ldots,\alpha_n)f = \sum_{i=0}^{n} \frac{f(\alpha_i)}{\prod_{j=0,\neq i}^{n}(\alpha_i - \alpha_j)}. \tag{2.5}$$

From the above expression it is not difficult to find the divided differences are symmetric functions of their arguments. In particular, taking $\alpha_i = \alpha + i\theta(\theta \neq 0)$ we have

$$\Delta(\alpha, \ldots, \alpha + n\theta) f = \frac{1}{n!\theta^n} \Delta_\theta^n f(\alpha) = \frac{1}{n!\theta^n} \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} f(\alpha + i\theta), \tag{2.6}$$

where Δ_θ is the difference operator with step size θ.

Divided differences can be extended to the cases with repeated points. From the recursive formula (2.4), it is clear that if $\alpha_0 \neq \alpha_1$ the following holds:

$$\Delta(\alpha_0, \alpha_1) f = \frac{f(\alpha_1) - f(\alpha_0)}{\alpha_1 - \alpha_0}. \tag{2.7}$$

If repetitions are permitted in the arguments and the function f is smooth enough, then

$$\lim_{\alpha_1 \to \alpha_0} \Delta(\alpha_0, \alpha_1) f = \lim_{\alpha_1 \to \alpha_0} \frac{f(\alpha_1) - f(\alpha_0)}{\alpha_1 - \alpha_0} = f'(\alpha_0). \tag{2.8}$$

This gives the definition of first-order divided differences with repeated points

$$\Delta(\alpha_0, \alpha_0) f = f'(\alpha_0). \tag{2.9}$$

In general, let $\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n$. Then the divided differences with repeated points obey the following recursive formula:

$$\Delta(\alpha_0, \ldots, \alpha_n) f = \begin{cases} \dfrac{\Delta(\alpha_0, \ldots, \alpha_{n-1}) f - \Delta(\alpha_1, \ldots, \alpha_n) f}{\alpha_0 - \alpha_n}, & \text{if } \alpha_n \neq \alpha_0, \\ \dfrac{f^{(n)}(\alpha_0)}{n!}, & \text{if } \alpha_n = \alpha_0. \end{cases} \tag{2.10}$$

It is evident that divided differences can be viewed as a discrete analogue of derivatives. If $\alpha_n = \alpha_0$, then all the points $\alpha_0, \alpha_1, \ldots, \alpha_n$ are the same. In this case, $N_n(x)$ in (2.2) is the Taylor polynomial of the function f at the point α_0. More generally, if $\{\alpha_0, \alpha_1, \ldots, \alpha_n\} = \{\underbrace{\alpha'_0, \ldots, \alpha'_0}_{p_0}, \underbrace{\alpha'_1, \ldots, \alpha'_1}_{p_1}, \ldots, \underbrace{\alpha'_m, \ldots, \alpha'_m}_{p_m}\}$ and $p_0 + p_1 + \cdots + p_m = n + 1$ where $\alpha'_0, \alpha'_1, \ldots, \alpha'_m$ are distinct, we define

$$\Omega_i(x) = \prod_{k=0, \neq i}^{m} (x - \alpha'_k)^{p_k}, \qquad S_{li}(x) = \sum_{k=0, \neq i}^{m} \frac{p_k}{(\alpha'_k - x)^l}, \tag{2.11}$$

with $l \geq 1, 0 \leq i \leq m$. Recall that the cycle index of symmetric group

$$Z_n(x_k) = Z_n(x_1, x_2, \ldots, x_n) = \sum_{a_1 + 2a_2 + \cdots + na_n = n} \frac{1}{a_1!(1)^{a_1} a_2!(2)^{a_2} \cdots a_n!(n)^{a_n}} x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \tag{2.12}$$

is one of the essential tools in enumerative combinatorics [29]. Using the cycle index of symmetric group, the divided differences with repeated points can be expressed by the following explicit formula [30] (see also [31]):

$$\Delta(\alpha_0,\ldots,\alpha_n)f = \sum_{i=0}^{m}\Omega_i(\alpha_i')^{-1}\sum_{j=0}^{p_i-1}Z_{p_i-1-j}(S_{!i}(\alpha_i'))\frac{f^{(j)}(\alpha_i')}{j!}, \tag{2.13}$$

where

$$Z_{p_i-1-j}(S_{!i}(\alpha_i')) = Z_{p_i-1-j}(S_{1i}(\alpha_i'), S_{2i}(\alpha_i'),\ldots,S_{p_i-1-ji}(\alpha_i')). \tag{2.14}$$

It is well known that the Leibniz formula for higher derivatives is basic and important in calculus. A divided difference form of this formula given by [32] is stated as below. Let $h = fg$. If f and g are sufficiently smooth functions, then for arbitrary points $\alpha_0,\alpha_1,\ldots,\alpha_n$,

$$\Delta(\alpha_0,\ldots,\alpha_n)h = \sum_{i=0}^{n}\Delta(\alpha_0,\ldots,\alpha_i)f\cdot\Delta(\alpha_i,\ldots,\alpha_n)g. \tag{2.15}$$

This formula is called the Steffensen formula which is a generalization of the Leibniz formula. If $\alpha_0 = \alpha_1 = \cdots = \alpha_n$, then the Leibniz formula holds, namely,

$$h^{(n)}(\alpha_0) = \sum_{i=0}^{n}\binom{n}{i}f^{(i)}(\alpha_0)g^{(n-i)}(\alpha_0). \tag{2.16}$$

3. Generalized Stirling Numbers

Let $\boldsymbol{\alpha} = (\alpha_0,\alpha_1,\ldots)$ and $\boldsymbol{\beta} = (\beta_0,\beta_1,\ldots)$ be two vectors. We define two kinds of Stirling-type numbers as

$$\omega_{n,\boldsymbol{\alpha}}(x) = \sum_{k=0}^{n}S(n,k;\boldsymbol{\alpha},\boldsymbol{\beta},r)\omega_{k,\boldsymbol{\beta}}(x-r), \tag{3.1}$$

$$\omega_{n,\boldsymbol{\beta}}(x) = \sum_{k=0}^{n}s(n,k;\boldsymbol{\alpha},\boldsymbol{\beta},r)\omega_{k,\boldsymbol{\alpha}}(x+r), \tag{3.2}$$

where $S(n,k;\boldsymbol{\alpha},\boldsymbol{\beta},r)$ are called the generalized Stirling numbers of the second kind with the parameters $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, and r, and $s(n,k;\boldsymbol{\alpha},\boldsymbol{\beta},r)$ are called the generalized Stirling numbers of the first kind. It is obvious that $S(n,k;\boldsymbol{\alpha},\boldsymbol{\beta},r) = s(n,k;\boldsymbol{\beta},\boldsymbol{\alpha},-r)$. In particular, $S(n,k;\mathbf{0},\overline{\mathbf{1}},0)$ is the conventional Stirling number of the second kind, and $s(n,k;\mathbf{0},\overline{\mathbf{1}},0)$ is of the first kind.

In this section, making use of divided difference operator and the Newton interpolation in Section 2, we will investigate orthogonality relations, recurrences, explicit expressions, and generating functions for the generalized Stirling numbers.

3.1. Basic Properties of the Generalized Stirling Numbers

Firstly, let us consider orthogonality relations of the two kinds of the generalized Stirling numbers. By substituting (3.1) into (3.2) and (3.2) into (3.1), one may easily get the following orthogonality relations

$$\sum_{i=k}^{n} s(n,i;\boldsymbol{\alpha},\boldsymbol{\beta},r)S(i,k;\boldsymbol{\alpha},\boldsymbol{\beta},r) = \delta_{n,k}, \tag{3.3}$$

$$\sum_{i=k}^{n} S(n,i;\boldsymbol{\alpha},\boldsymbol{\beta},r)s(i,k;\boldsymbol{\alpha},\boldsymbol{\beta},r) = \delta_{n,k}, \tag{3.4}$$

respectively, where the Kronecker symbol $\delta_{n,k}$ is defined by $\delta_{n,k} = 1$ if $n = k$, and $\delta_{n,k} = 0$ if $n \neq k$. As a consequence, the inverse relations are immediately obtained:

$$f_n = \sum_{k=0}^{n} S(n,k;\boldsymbol{\alpha},\boldsymbol{\beta},r)g_k \Longleftrightarrow g_n = \sum_{k=0}^{n} s(n,k;\boldsymbol{\alpha},\boldsymbol{\beta},r)f_k. \tag{3.5}$$

Next, from the definition (3.1), one may see that $S(n,k;\boldsymbol{\alpha},\boldsymbol{\beta},r)$ can be viewed as the coefficients of the Newton interpolation of the function $\omega_{n,\boldsymbol{\alpha}}$ at the points $r+\beta_0,\ r+\beta_1,\dots,\ r+\beta_n$. Thus, we immediately have the following theorem.

Theorem 3.1. *For arbitrary parameters $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, and r, there holds*

$$S(n,k;\boldsymbol{\alpha},\boldsymbol{\beta},r) = \Delta\left(r+\beta_0,\dots,r+\beta_k\right)\omega_{n,\boldsymbol{\alpha}}. \tag{3.6}$$

In particular, if $\beta_0,\beta_1,\dots,\beta_n$ are distinct, we have

$$S(n,k;\boldsymbol{\alpha},\boldsymbol{\beta},r) = \sum_{i=0}^{k} \frac{\prod_{j=0}^{n-1}\left(r+\beta_i-\alpha_j\right)}{\prod_{j=0,\neq i}^{k}\left(\beta_i-\beta_j\right)}. \tag{3.7}$$

If $\beta_0 = \beta_1 = \cdots = \beta_n = 0$, then

$$S(n,k;\boldsymbol{\alpha},\mathbf{0},r) = \frac{1}{k!}\sum_{0\leq i_1<\cdots<i_{n-k}\leq n-1} \left(r-\alpha_{i_1}\right)\cdots\left(r-\alpha_{i_{n-k}}\right). \tag{3.8}$$

This theorem gives the explicit expressions for the generalized Stirling numbers. We can similarly get $s(n,k;\boldsymbol{\alpha},\boldsymbol{\beta},r) = \Delta(-r+\alpha_0,\dots,-r+\alpha_k)\omega_{n,\boldsymbol{\beta}}$. By (3.6), we can further get the recurrence relations as follows.

Theorem 3.2. *For arbitrary parameters $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, and r, there holds*

$$S(n,k;\boldsymbol{\alpha},\boldsymbol{\beta},r) = S(n-1,k-1;\boldsymbol{\alpha},\boldsymbol{\beta},r) + \left(r+\beta_k-\alpha_{n-1}\right)S(n-1,k;\boldsymbol{\alpha},\boldsymbol{\beta},r). \tag{3.9}$$

In particular, we have

$$S(n,0;\boldsymbol{\alpha},\boldsymbol{\beta},r) = \omega_{n,\boldsymbol{\alpha}}(r+\beta_0) = (r+\beta_0-\alpha_0)\cdots(r+\beta_0-\alpha_{n-1}). \tag{3.10}$$

Proof. According to (3.6), we have

$$S(n,k;\boldsymbol{\alpha},\boldsymbol{\beta},r) = \Delta(r+\beta_0,\ldots,r+\beta_k)(\omega_{n-1,\boldsymbol{\alpha}}(\cdot-\alpha_{n-1})). \tag{3.11}$$

By using the Steffensen formula for divided differences and the basic facts

$$\Delta(x_0,\ldots,x_i)\omega_{j,\boldsymbol{\alpha}} = \delta_{i,j}, \tag{3.12}$$

we have

$$\begin{aligned} S(n,k;\boldsymbol{\alpha},\boldsymbol{\beta},r) &= \Delta(r+\beta_0,\ldots,r+\beta_{k-1})\omega_{n-1,\boldsymbol{\alpha}} \\ &\quad + (r+\beta_k-\alpha_{n-1})\Delta(r+\beta_0,\ldots,r+\beta_k)\omega_{n-1,\boldsymbol{\alpha}}. \end{aligned} \tag{3.13}$$

This leads to (3.9), and the proof is complete. □

Finally, let us consider the generating function of the Stirling numbers $S(n,k;\boldsymbol{\alpha},\boldsymbol{\beta},r)$ denoted by $G(t;k,\boldsymbol{\alpha},\boldsymbol{\beta},r)$. Assume that $G(t;k,\boldsymbol{\alpha},\boldsymbol{\beta},r)$ is of the form:

$$G(t;k,\boldsymbol{\alpha},\boldsymbol{\beta},r) = \sum_{n=0}^{\infty} A_n S(n,k;\boldsymbol{\alpha},\boldsymbol{\beta},r)t^n, \tag{3.14}$$

where $A_0, A_1, \ldots$ is a reference sequence. In this way we treat at the same time the case of ordinary coefficients of $G(A_n = 1)$ and the case of Taylor coefficients ($A_n = 1/n!$). Let $\Phi(x,t) = \sum_{n=0}^{\infty} A_n\omega_{n,\boldsymbol{\alpha}}(x)t^n$. Making use of (3.6), we get the following:

$$G(t;k,\boldsymbol{\alpha},\boldsymbol{\beta},r) = \sum_{n=0}^{\infty} A_n\Delta(r+\beta_0,\ldots,r+\beta_k)\omega_{n,\boldsymbol{\alpha}}t^n = \Delta(r+\beta_0,\ldots,r+\beta_k)\Phi(\cdot,t). \tag{3.15}$$

This formula is essential and important for getting the generating function of the generalized Stirling numbers. If we get the analytic expression of $\Phi(x,t)$ by choosing special $\boldsymbol{\alpha}$, the analytic expression of $G(t;k,\boldsymbol{\alpha},\boldsymbol{\beta},r)$ is obtained as well.

3.2. Special Cases

Because the parameters $\boldsymbol{\alpha},\boldsymbol{\beta}$, and r are arbitrary, our results contain many interesting special cases. In this part we will investigate these special cases. Some results have been derived and some are new.

Let $\overline{\boldsymbol{\theta}} = (0,\theta,\ldots)$ and all β_i be distinct. According to Theorems 3.1 and 3.2, we have the explicit expressions and the recurrence relations for new generalized Stirling numbers $S(n,k;\overline{\boldsymbol{\theta}},\boldsymbol{\beta},r)$.

Corollary 3.3. *The numbers $S(n,k;\overline{\boldsymbol{\theta}},\boldsymbol{\beta},r)$ have the following explicit expression*

$$S\left(n,k;\overline{\boldsymbol{\theta}},\boldsymbol{\beta},r\right)=\sum_{i=0}^{k}\frac{\prod_{j=0}^{n-1}(r+\beta_i-j\theta)}{\prod_{j=0,\neq i}^{k}(\beta_i-\beta_j)}. \tag{3.16}$$

Corollary 3.4. *The numbers $S(n,k;\boldsymbol{\theta},\boldsymbol{\beta},r)$ satisfy the following recurrence relation*

$$S\left(n,k;\overline{\boldsymbol{\theta}},\boldsymbol{\beta},r\right)=S\left(n-1,k-1;\overline{\boldsymbol{\theta}},\boldsymbol{\beta},r\right)+(r+\beta_k-(n-1)\theta)S\left(n-1,k;\overline{\boldsymbol{\theta}},\boldsymbol{\beta},r\right). \tag{3.17}$$

For $\overline{\boldsymbol{\theta}}=(0,\theta,\ldots)$ and $A_n=1/n!$, if $\theta\neq 0$ we have

$$\Phi(x,t)=\sum_{n=0}^{\infty}(x)^{(n,\theta)}\frac{t^n}{n!}=(1+\theta t)^{x/\theta}, \tag{3.18}$$

and if $\theta=0$ we have

$$\Phi(x,t)=\sum_{n=0}^{\infty}x^n\frac{t^n}{n!}=e^{xt}. \tag{3.19}$$

Thus, by (3.15) one easily obtain the following theorem.

Theorem 3.5. *The sequence $\{S(n,k;\overline{\boldsymbol{\theta}},\boldsymbol{\beta},r)\}$ has the following exponential generating function:*

$$\sum_{n=0}^{\infty}S\left(n,k;\overline{\boldsymbol{\theta}},\boldsymbol{\beta},r\right)\frac{t^n}{n!}=\begin{cases}(1+\theta t)^{r/\theta}\displaystyle\sum_{i=0}^{k}\frac{(1+\theta t)^{\beta_i/\theta}}{\prod_{j=0,\neq i}^{k}(\beta_i-\beta_j)}, & \theta\neq 0,\\ e^{rt}\displaystyle\sum_{i=0}^{k}\frac{e^{\beta_i t}}{\prod_{j=0,\neq i}^{k}(\beta_i-\beta_j)}, & \theta=0.\end{cases} \tag{3.20}$$

Our generalized Stirling numbers $S(n,k;\overline{\boldsymbol{\theta}},\boldsymbol{\beta},r)$ include the Stirling numbers due to Hsu and Shiue [1], EI-Desouky's multiparameter noncentral Stirling numbers [10], and the so-called Comtet numbers [9] as special cases. Now, let us discuss these special cases as follows.

Example 3.6. Let $\boldsymbol{\beta}=\overline{\boldsymbol{\theta}'}:=(0,\theta',\ldots)$ and $\theta'\neq 0$. This implies the points β_i are equally spaced with step size θ'. By Corollaries 3.3 and 3.4, we immediately get the explicit expression for the generalized Stirling numbers

$$S\left(n,k;\overline{\boldsymbol{\theta}},\overline{\boldsymbol{\theta}'},r\right)=\frac{1}{k!\theta'^k}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}\prod_{j=0}^{n-1}(r+j\theta'-i\theta) \tag{3.21}$$

and the recurrence relation

$$S\left(n+1,k;\overline{\boldsymbol{\theta}},\overline{\boldsymbol{\theta}'},r\right)=S\left(n,k-1;\overline{\boldsymbol{\theta}},\overline{\boldsymbol{\theta}'},r\right)+(r+k\theta'-n\theta)S\left(n,k;\overline{\boldsymbol{\theta}},\overline{\boldsymbol{\theta}'},r\right). \tag{3.22}$$

For $\theta \neq 0$, the following holds

$$\begin{aligned}\sum_{i=0}^{k} \frac{(1+\theta t)^{(r+i\theta')/\theta}}{\prod_{j=0,\neq i}^{k}(i\theta' - j\theta')} &= \frac{1}{k!\theta'^{k}}(1+\theta t)^{r/\theta} \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} (1+\theta t)^{i\theta'/\theta} \\ &= \frac{1}{k!\theta'^{k}}(1+\theta t)^{r/\theta} \left((1+\theta t)^{\theta'/\theta} - 1\right)^{k}.\end{aligned} \tag{3.23}$$

In a similar manner, we can also get the generating function for $\theta = 0$. Thus, we have

$$\sum_{n=0}^{\infty} S\left(n,k;\overline{\boldsymbol{\theta}},\overline{\boldsymbol{\theta}'},r\right) \frac{t^n}{n!} = \begin{cases} \dfrac{1}{k!}(1+\theta t)^{r/\theta} \left(\dfrac{(1+\theta t)^{\theta'/\theta} - 1}{\theta'}\right)^{k}, & \theta \neq 0, \\ \dfrac{1}{k!} e^{rt} \left(\dfrac{e^{\theta' t} - 1}{\theta'}\right)^{k}, & \theta = 0. \end{cases} \tag{3.24}$$

Here $S(n,k;\overline{\boldsymbol{\theta}},\overline{\boldsymbol{\theta}'},r)$ is equivalent to $S(n,k;\alpha,\beta,r)$ in [1]. As mentioned in [1], the generalized Stirling numbers $S(n,k;\overline{\boldsymbol{\theta}},\overline{\boldsymbol{\theta}'},r)$ contain serval special cases, for example, two kinds of the classical Stirling numbers, the binomial coefficients, the Lah numbers, Carlitz's two kinds of weighted Stirling numbers [4], Carlitz's two kinds of degenerate Stirling numbers [3], Howard's weighted degenerate Stirling numbers [5], Gould-Hopper's noncentral Lah numbers [7], Riordan's noncentral Stirling numbers [2], the noncentral C numbers extensively studied by Charalambides and Koutras [6], Tsylova's numbers [8], Todorov's numbers [33], Nandi and Dutta's associated Lah numbers [34], and the r-Stirling numbers of the first kind fully developed by Broder [35]. Hsu and Shiue obtained the recurrence relation for the generalized Stirling numbers $S(n,k;\overline{\boldsymbol{\theta}},\overline{\boldsymbol{\theta}'},r)$, and they also found the generating function by solving a difference-differential equation. However, the formula (3.21) was new and not given by [1]. Obviously, in the present paper we follow a very different approach to rediscover the recurrence relation and the generating function. In the case $r = 0$, one may also refer to [36].

It is remarkable that Mansour and Schork [20] recently considered $UV - VU = hV^s$ to generalize the commutation relation $UV - VU = 1$. They defined generalized Stirling numbers $\mathfrak{S}_{s;h}(n,k)$ by (1.9). The explicit expressions of these generalized Stirling numbers are given by [20] (see also [21]), and they are very closely related to the numbers considered by Lang [14]. In [21], the authors exploited many properties of these generalized Stirling numbers. It is interesting that observing our generalized Stirling numbers $S(n,k;\overline{\boldsymbol{\theta}},\boldsymbol{\beta},r)$ by $\theta = -sh, \beta_k = kh(1-s)$ and $r = 0$, we find that the Stirling numbers due to Mansour and Schork are actually a special case of ours and Hsu-Shiue's, and they are equivalent to the numbers due to [36]. Thus, by (3.20) we get the exponential generating function of the generalized Stirling numbers due to Mansour and Schork:

$$\frac{1}{k!}\left(\frac{(1-hst)^{(s-1)/s} - 1}{h(1-s)}\right)^{k} = \sum_{n=0}^{\infty} \mathfrak{S}_{s;h}(n,k) \frac{t^n}{n!}. \tag{3.25}$$

In [21], the authors gave the generating function for $k = 1$.

Example 3.7. Let $\theta = 1$, $r = 0$, and β_i be arbitrary but distinct, and one can get the multiparameter noncentral Stirling numbers of the first kind introduced by El-Desouky [10]. Here we denote the numbers by $S(n,k;\overline{\mathbf{1}},\boldsymbol{\beta},0)$. Using Corollaries 3.3 and 3.4 and Theorem 3.5, we rediscover the explicit expression, recurrence relation, and the generating function, namely,

$$S\left(n,k;\overline{\mathbf{1}},\boldsymbol{\beta},0\right) = \sum_{i=0}^{k} \frac{\prod_{j=0}^{n-1}(\beta_i - j)}{\prod_{j=0,\neq i}^{k}(\beta_i - \beta_j)},$$
$$S\left(n+1,k;\overline{\mathbf{1}},\boldsymbol{\beta},0\right) = S\left(n,k-1;\overline{\mathbf{1}},\boldsymbol{\beta},0\right) + (\beta_k - n) S\left(n,k;\overline{\mathbf{1}},\boldsymbol{\beta},0\right), \tag{3.26}$$
$$\sum_{n=0}^{\infty} S\left(n,k;\overline{\mathbf{1}},\boldsymbol{\beta},0\right)\frac{t^n}{n!} = \sum_{i=0}^{k} \frac{(1+t)^{\beta_i}}{\prod_{j=0,\neq i}^{k}(\beta_i - \beta_j)}.$$

Example 3.8. Let us consider the case $\overline{\boldsymbol{\theta}} = \mathbf{0}$. In this case, there holds

$$x^n = \sum_{k=0}^{n} S(n,k;\mathbf{0},\boldsymbol{\beta},r)\omega_{k,\boldsymbol{\beta}}(x-r), \tag{3.27}$$

which is equivalent to

$$(x+r)^n = \sum_{k=0}^{n} S(n,k;\mathbf{0},\boldsymbol{\beta},r)\omega_{k,\boldsymbol{\beta}}(x). \tag{3.28}$$

Especially, for $r = 0$ the Comtet numbers [9] (see also [10]) are defined associated with the sequence $\boldsymbol{\beta}$ by

$$x^n = \sum_{k=0}^{n} S_{\boldsymbol{\beta}}(n,k)(x-\beta_0)(x-\beta_1)\cdots(x-\beta_{k-1}). \tag{3.29}$$

This implies $S(n,k;\mathbf{0},\boldsymbol{\beta},0) = S_{\boldsymbol{\beta}}(n,k)$. Thus, it is not difficult to obtain

$$S(n,k;\mathbf{0},\boldsymbol{\beta},0) = \sum_{i=0}^{k} \frac{\beta_i^n}{\prod_{j=0,\neq i}^{k}(\beta_i - \beta_j)},$$
$$S(n+1,k;\mathbf{0},\boldsymbol{\beta},0) = S(n,k-1;\mathbf{0},\boldsymbol{\beta},0) + \beta_k S(n,k;\mathbf{0},\boldsymbol{\beta},0), \tag{3.30}$$

and the exponential generating function

$$\sum_{n=0}^{\infty} S(n,k;\mathbf{0},\boldsymbol{\beta},0)\frac{t^n}{n!} = \sum_{i=0}^{k} \frac{e^{\beta_i t}}{\prod_{j=0,\neq i}^{k}(\beta_i - \beta_j)}. \tag{3.31}$$

It is worth noting that the Comtet numbers $S(n,k;\mathbf{0},\boldsymbol{\beta},0)$ can be rewritten as an alternate form

$$S(n,k;\mathbf{0},\boldsymbol{\beta},0) = \sum_{0\le i_1\le\cdots\le i_n\le k} \beta_{i_1}\beta_{i_2}\cdots\beta_{i_n}, \tag{3.32}$$

which is really the complete symmetric function of nth order with respect to the variables β_0, $\beta_1,\ldots,\beta_k$. By (3.15), they have the ordinary generating function:

$$\sum_{n=0}^{\infty} S(n,k;\mathbf{0},\boldsymbol{\beta},0)t^n = \Delta(\beta_0,\ldots,\beta_k)\frac{1}{1-(\cdot)t} = \frac{t^k}{\prod_{i=0}^{k}(1-\beta_i t)}. \tag{3.33}$$

Moreover, if we let $r=-a, \boldsymbol{\beta}=\overline{\mathbf{1}}$, then we get the noncentral Stirling numbers of the second kind defined by Koutras [37] (see also [24]). For more details one refers to [37].

What has been discussed above in this subsection is relevant to the generalized Stirling numbers with equidistance parameters α_i. However, we are also interested in the other cases. In a recent year, many authors [14, 16, 18, 19, 24] were devoted to the generalized Stirling numbers by differential operator. We here rediscover these generalized Stirling numbers by the Newton interpolation.

Example 3.9. Our generalized Stirling numbers $S(n,k;\boldsymbol{\alpha},\boldsymbol{\beta},r)$ also contain the numbers due to Blasiak [18] as a special case. Here we let $\mathbf{r}=(r_1,r_2,\ldots,r_m)$, $\mathbf{s}=(s_1,s_2,\ldots,s_m)$ and let $d_0=0$ and $d_m=\sum_{i=1}^{m}(r_i-s_i)$ for $m\ge 1$. Moreover, we let

$$\begin{aligned}\widehat{\mathbf{d}} &= -(d_0, d_0-1,\ldots,d_0-s_1+1, d_1, d_1-1,\ldots,d_1-s_2+1,\ldots,\\ &\qquad d_{m-1}, d_{m-1}-1,\ldots,d_{m-1}-s_m+1),\\ \widehat{\mathbf{s}} &= (0,1,\ldots,s_1+s_2+\cdots+s_m),\end{aligned} \tag{3.34}$$

where $s_1+s_2+\cdots+s_m=n$. By using (3.7) we immediately have the explicit expression of $S(n,k;\widehat{\mathbf{d}},\widehat{\mathbf{s}},0)$ as follows

$$S\left(n,k;\widehat{\mathbf{d}},\widehat{\mathbf{s}},0\right) = \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\prod_{i=1}^{m}(d_{i-1}+j)^{(s_i)}, \tag{3.35}$$

which is in accordance with the generalized Stirling numbers $S_{\mathbf{r},\mathbf{s}}(m,k)$ introduced by Blasiak [18]. Blasiak got this explicit formula by using the operator $x^{r_m}D^{s_m}\cdots x^{r_2}D^{s_2}x^{r_1}D^{s_1}$ to act on e^x. His proof is very different from ours. Recently, El-Desouky et al. [19] found a new expression by successive application of Leibniz formula. The special case $\mathbf{r}=(r,r,\ldots,r)$ and $\mathbf{s}=(s,s,\ldots,s)$ is investigated by Blasiak et al. [16], and they gave us Lang's result [14] as a special case for $\mathbf{s}=(1,1,\ldots,1)$.

Example 3.10. By operating with (1.10) on e^x and using Cauchy rule of multiplication of series, El-Desouky and Cakić [24] obtain the explicit formula

$$S(n,k;\overline{\boldsymbol{\alpha}},\overline{\mathbf{r}}) = \frac{1}{k!}\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\prod_{j=0}^{n-1}(k-\alpha_j)^{r_j}. \tag{3.36}$$

In fact, let $\widehat{\boldsymbol{\alpha}} = \{\underbrace{\alpha_0,\ldots,\alpha_0}_{r_0},\underbrace{\alpha_1,\ldots,\alpha_1}_{r_1},\ldots,\underbrace{\alpha_{n-1},\ldots,\alpha_{n-1}}_{r_{n-1}}\}$. It is not difficult to find $S(n,k;\overline{\boldsymbol{\alpha}},\overline{\mathbf{r}}) = S(r_0+r_1+\cdots+r_{n-1},k;\widehat{\boldsymbol{\alpha}},\overline{\mathbf{1}},0)$ holds. In particular, setting $n=2$, $\alpha_0=0$, $\alpha_1=1$, $r_0=l$, and $r_1=m-l$ in $S(r_0+r_1+\cdots+r_{n-1},k;\widehat{\boldsymbol{\alpha}},\overline{\mathbf{1}},0)$, we get the explicit expression of the number of partitions of $M=\{x_1,x_2,\ldots,x_m\}$ into n nonempty parts such that the distance of any two members in the same part differs from l denoted by $T_l(m,k)$; see [38].

4. Generalized Bell Polynomials and Dobinski-Type Formulas

Recall that the Bell numbers B_n and the exponential polynomials $B_n(x)$ are defined, respectively, by the sums

$$B_n = \sum_{k=0}^{n} S(n,k), \qquad B_n(x) = \sum_{k=0}^{n} S(n,k)x^k. \tag{4.1}$$

The Bell polynomials $B_n(x)$ have the generating function

$$\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} = e^{x(e^t-1)}. \tag{4.2}$$

They also satisfy the following remarkable Dobinski-type formula

$$B_n(x) = e^{-x}\sum_{i=0}^{\infty}\frac{i^n}{i!}x^i, \tag{4.3}$$

which reduces to the Dobinski formula when $x=1$. It is worth noting that $B_n(x)$ is represented as an infinite series in i.

As we know, the Dobinski-type formulas have been the subject of much combinatorial interest. Thus, it is worth looking for a general Dobinski-type formula.

In this section, we define a generalized Bell polynomials by

$$B_{n;\boldsymbol{\alpha},\overline{\boldsymbol{\theta}'},r}(x) = \sum_{k=0}^{n} S\left(n,k;\boldsymbol{\alpha},\overline{\boldsymbol{\theta}'},r\right)x^k, \tag{4.4}$$

where $\overline{\boldsymbol{\theta}'} = (0, \theta', \ldots)$ and $\theta' \neq 0$. Naturally, one get an extended definition of generalized Bell numbers as follow:

$$B_{n;\boldsymbol{\alpha},\overline{\boldsymbol{\theta}'},r} = \sum_{k=0}^{n} S\left(n, k; \boldsymbol{\alpha}, \overline{\boldsymbol{\theta}'}, r\right). \tag{4.5}$$

Note that $B_{n;\mathbf{0},\overline{\mathbf{1}},0}(x) = B_n(x)$ and $B_{n;\mathbf{0},\overline{\mathbf{1}},0} = B_n$. We can make use of (3.6) to obtain the following Dobinski-type formula.

Theorem 4.1. *For $\overline{\boldsymbol{\theta}'} = (0, \theta', \ldots)$ and arbitrary $\boldsymbol{\alpha}, r$, we have the Dobinski-type formula*

$$B_{n;\boldsymbol{\alpha},\overline{\boldsymbol{\theta}'},r}(x) = e^{-x/\theta'} \sum_{i=0}^{\infty} \frac{\omega_{n,\boldsymbol{\alpha}}(r + i\theta')}{i!} \left(\frac{x}{\theta'}\right)^{i}, \tag{4.6}$$

where $\omega_{n,\boldsymbol{\alpha}}$ is defined by (2.1).

Proof. By (4.4) we have

$$\sum_{n=0}^{\infty} B_{n;\boldsymbol{\alpha},\overline{\boldsymbol{\theta}'},r}(x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{k=0}^{n} S\left(n, k; \boldsymbol{\alpha}, \overline{\boldsymbol{\theta}'}, r\right) x^k = \sum_{k=0}^{\infty} x^k \sum_{n \geq k} S\left(n, k; \boldsymbol{\alpha}, \overline{\boldsymbol{\theta}'}, r\right) \frac{t^n}{n!}. \tag{4.7}$$

Replacing $\boldsymbol{\beta}$ by $\overline{\boldsymbol{\theta}'}$ in (3.6) yields

$$\begin{aligned} \sum_{n=0}^{\infty} B_{n;\boldsymbol{\alpha},\overline{\boldsymbol{\theta}'},r}(x) \frac{t^n}{n!} &= \sum_{k=0}^{\infty} x^k \sum_{n \geq k} \Delta(r, r + \theta', \ldots, r + k\theta') \omega_{n,\boldsymbol{\alpha}} \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{x}{\theta'}\right)^{k} \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} \omega_{n,\boldsymbol{\alpha}}(r + i\theta'). \end{aligned} \tag{4.8}$$

By equating the coefficient of $t^n/n!$ within the first and last expressions, we arrive at

$$B_{n;\boldsymbol{\alpha},\overline{\boldsymbol{\theta}'},r}(x) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{x}{\theta'}\right)^{k} \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} \omega_{n,\boldsymbol{\alpha}}(r + i\theta'). \tag{4.9}$$

Using the Cauchy product rule gives

$$B_{n;\boldsymbol{\alpha},\overline{\boldsymbol{\theta}'},r}(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \left(\frac{x}{\theta'}\right)^{j} \sum_{i=0}^{\infty} \frac{\omega_{n,\boldsymbol{\alpha}}(r + i\theta')}{i!} \left(\frac{x}{\theta'}\right)^{i}. \tag{4.10}$$

This implies (4.6) is true and completes the proof. □

Letting $x = 1$ we directly obtain the generalized Dobinski formula.

Corollary 4.2. *For* $\overline{\boldsymbol{\theta}'} = (0, \theta', \ldots)$ *and arbitrary* $\boldsymbol{\alpha}, \boldsymbol{r}$, *we have*

$$B_{n;\boldsymbol{\alpha},\overline{\boldsymbol{\theta}'},r} = e^{-1/\theta'} \sum_{i=0}^{\infty} \frac{\omega_{n,\boldsymbol{\alpha}}(r + i\theta')}{i!} \left(\frac{1}{\theta'}\right)^{i}, \tag{4.11}$$

where $\omega_{n,\boldsymbol{\alpha}}$ *is defined by* (2.1).

It is clear that (4.3) is a special case of (4.6) with $\theta' = 1, \boldsymbol{\alpha} = \mathbf{0}, r = 0$.

Let $\phi_{\boldsymbol{\alpha},\theta'}(t) = \sum_{i=0}^{\infty}(i\theta' - \alpha_0) \cdots (i\theta' - \alpha_{n-1})(t^i/i!)$. It is worth noting that the formula (4.6) can be used to obtain a closed sum formula for this type of infinite series. As mentioned in [1], such a type of series cannot be summed by using the hypergeometric series method. Let $t = x/\theta',\ r = 0$; then according to (4.6) we have

$$\phi_{\boldsymbol{\alpha},\theta'}(t) = B_{n;\boldsymbol{\alpha},\overline{\boldsymbol{\theta}'},0}(\theta' t)e^{t} = e^{t} \sum_{k=0}^{n} S\left(n, k; \boldsymbol{\alpha}, \overline{\boldsymbol{\theta}'}, 0\right) \theta'^{k} t^{k}. \tag{4.12}$$

Example 4.3. Letting $\boldsymbol{\alpha} = \overline{\boldsymbol{\theta}} = (0, \theta, \ldots)$ we immediately obtain the Dobinski-type formula due to Hsu and Shiue [1] as follows:

$$B_{n;\overline{\boldsymbol{\theta}},\overline{\boldsymbol{\theta}'},r}(x) = e^{-x/\theta'} \sum_{i=0}^{\infty} \frac{\theta^{n}((r + i\theta')/\theta)^{(n)}}{i!} \left(\frac{x}{\theta'}\right)^{i}. \tag{4.13}$$

Example 4.4. Letting $\boldsymbol{\alpha} = \widehat{\mathbf{d}},\ \theta' = 1,\ n = s_1 + s_2 + \cdots + s_m$ we have the following Dobinski-type formula due to Blasiak [18]:

$$B_{n;\widehat{\mathbf{d}},\overline{\mathbf{1}},0}(x) = e^{-x} \sum_{i=s_1}^{\infty} \prod_{j=1}^{m} (d_{j-1} + i)^{(s_j)} \frac{x^{i}}{i!}. \tag{4.14}$$

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