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*Research Article*

# First-Order Three-Point Boundary Value Problems at Resonance Part III

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The main purpose of this paper is to investigate the existence of solutions of BVPs for a very general case in which both the system of ordinary differential equations and the boundary conditions are nonlinear. By employing the implicit function theorem, sufficient conditions for the existence of three-point boundary value problems are established.

## 1. Introduction

We consider existence of solutions at resonance to first-order three-point BVPs with nonlinear boundary conditions using results developed in [1, 2].

Consider

$$x' - A(t)x = H(t, x, \varepsilon) = \varepsilon F(t, x, \varepsilon) + E(t), \quad 0 \le t \le 1, \tag{1.1}$$

$$Mx(0) + Nx(\eta) + Rx(1) = \ell + \varepsilon g(x(0), x(\eta), x(1)), \tag{1.2}$$

where $M$, $N$, and $R$ are constant square matrices of order $n$, $A(t)$ is an $n \times n$ matrix with continuous entries, $E : [0,1] \to \mathbb{R}$ is continuous, $F : [0,1] \times \mathbb{R}^n \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ is a continuous function where $\varepsilon_0 > 0$, $\ell \in \mathbb{R}^n$, $\eta \in (0,1)$, and $g : \mathbb{R}^{3n} \to \mathbb{R}^n$ is continuous.

Our existence theorem uses the implicit function theorem; see for example Nagle [3]. Nagle [3] extended the alternative method considered by Hale [4] for handling the periodic case of non-self-adjoint problems subject to homogeneous boundary conditions. These results extend the work of Feng and Webb [5] and Gupta [6] of three-point BVPs with linear boundary conditions for $\alpha = 1$ and $\alpha\eta = 1$ to nonlinear boundary conditions. Feng and Webb [5] studied the existence of solutions of the following BVPs (1.3) and (1.4):

$$\begin{gathered} y'' = f(t, y, y') + e(t), \quad 0 \le t \le 1, \\ y'(0) = 0, \qquad y(1) = \alpha y(\eta), \end{gathered} \tag{1.3}$$

$$\begin{gathered} y'' = f(t, y, y') + e(t), \quad 0 \le t \le 1, \\ y(0) = 0, \qquad y(1) = \alpha y(\eta), \end{gathered} \tag{1.4}$$

where $\eta \in (0,1)$, $\alpha \in \mathbb{R}$, $f : [0,1] \times \mathbb{R}^2 \to \times\mathbb{R}$ is a continuous function, and $e : [0,1] \to \mathbb{R}$ is a function in $L^1[0,1]$. Both of the problems are resonance cases under the assumption $\alpha = 1$ for the problem (1.3), and $\alpha\eta = 1$ for the problem (1.4). The problem for nonlinear boundary conditions for discrete systems has been studied by Rodriguez [7, 8]. Rodriguez [7] extended results of Halanay [9], who considered periodic boundary conditions and also extended those of Rodriguez [10] and Agarwal [11] who considered linear boundary conditions. To our knowledge there appears to be no research in the literature on multipoint BVPs for systems of first-order equations with nonlinear boundary conditions at resonance. The results of this paper fill this gap in the literature.

Our results are analogues for three-point boundary conditions of those periodic boundary conditions for perturbed systems of first-order equations at resonance considered by Coddington and Levinson [12] and Cronin [13, 14]. Moreover, our results extend the work of Urabe [15], Liu [16], and of Nagle [3], where he solved the two-point BVP using the Cesari-Hale alternative method.

## 2. Preliminaries

Now we state the following basic existence theorems for systems with a parameter and use them to formulate the existence results for problem (1.1) and (1.2).

**Theorem 2.1** (see Coppel [17, Page 19]).

(i) *Let $F(t, x, \varepsilon)$ be a continuous function of $(t, x, \varepsilon)$ for all points $(t, x)$ in an open set $D$ and all values $\varepsilon$ near $\overline{\varepsilon}$.*

(ii) *Let $x(t, c, \varepsilon)$ be any noncontinuable solution of the differential equation*

$$x' = F(t, x, \varepsilon), \quad \text{with } x(0) = c. \tag{2.1}$$

*If $x(t, \overline{c}, \overline{\varepsilon})$ is defined on the interval $[0,1]$ and is unique, then $x(t, c, \varepsilon)$ is defined on $[0,1]$ for all $(c, \varepsilon)$ sufficiently near $(\overline{c}, \overline{\varepsilon})$ and is a continuous function of its threefold arguments at any point $(t, \overline{c}, \overline{\varepsilon})$.*

**Theorem 2.2** (see Coppel [17, Page 22]).

(i) *Let $F(t, x, \varepsilon)$ be a continuous function of $(t, x, \varepsilon)$ for all points $(t, x)$ in a domain $D$ and all values of the vector parameter $\varepsilon$ near $\overline{\varepsilon}$.*

(ii) *Let* $x(t,\overline{c},\overline{\varepsilon})$ *be a solution of the differential equation*

$$x' = F(t,x,\overline{\varepsilon}), \quad \text{with } x(0) = \overline{c} \tag{2.2}$$

*defined on a compact interval* $[0,1]$.

(iii) *Suppose that* $F$ *has continuous partial derivatives* $F_x$, $F_\varepsilon$ *at all points* $(t,x(t,\overline{c},\overline{\varepsilon}),\overline{\varepsilon})$ *with* $t \in [0,1]$.

*Then for all* $(c,\varepsilon)$ *sufficiently near* $(\overline{c},\overline{\varepsilon})$ *the differential equation*

$$x' = F(t,x,\varepsilon), \quad \text{with } x(0) = c \tag{2.3}$$

*has a unique solution* $x(t,c,\varepsilon)$ *over* $[0,1]$ *that is close to the solution* $x(t,\overline{c},\overline{\varepsilon})$ *of* (ii). *The continuous differentiability of* $F$ *with respect to* $x$ *and* $\varepsilon$ *implies the additional property that the solution* $x(t,c,\varepsilon)$ *is differentiable with respect to* $(t,c,\varepsilon)$ *for* $(c,\varepsilon)$ *near* $(\overline{c},\overline{\varepsilon})$.

We recall the following results of [2].

**Lemma 2.3** (see [2]). *Consider the system*

$$x' = A(t)x, \tag{2.4}$$

*where* $A(t)$ *is an* $n\times n$ *matrix with continuous entries on the interval* $[0,1]$. *Let* $Y(t)$ *be a fundamental matrix of* (2.4). *Then the solution of* (2.4) *which satisfies the initial condition*

$$x(0) = c \tag{2.5}$$

*is* $x(t) = Y(t)Y^{-1}(0)c$ *where* $c$ *is a constant n-vector. Abbreviate* $Y(t)Y^{-1}(0)$ *to* $Y_0(t)$. *Thus* $x(t) = Y_0(t)c$.

**Lemma 2.4** (see [2]). *Let* $Y(t)$ *be a fundamental matrix of* (2.4). *Then any solution of* (1.1) *and* (2.5) *can be written as*

$$x(t,c,\varepsilon) = x(t) = Y_0(t)c + \int_0^t Y(t)Y^{-1}(s)H(s,x(s),\varepsilon)ds. \tag{2.6}$$

*The solution* (1.1) *satisfies the boundary conditions* (1.2) *if and only if*

$$\mathcal{L}c = \varepsilon\mathcal{N}(c,\alpha,\eta,\varepsilon) + d, \tag{2.7}$$

*where* $\mathcal{L} = M + NY_0(\eta) + RY_0(1)$, $\mathcal{N}(c,\alpha,\eta,\varepsilon) = -(\int_0^\eta NY(\eta)Y^{-1}(s)F(s,x(s,c,\varepsilon),\varepsilon)ds + \int_0^1 RY(1)Y^{-1}(s)F(s,x(s,c,\varepsilon),\varepsilon)ds - g(c,x(\eta),x(1)))$, $d = -(\int_0^\eta NY(\eta)Y^{-1}(s)E(s)ds + \int_0^1 RY(1)Y^{-1}(s)E(s)ds - \ell)$, *and* $x(t,c,\varepsilon)$ *is the solution of* (1.1) *given* $x(0) = c$.

Thus (2.7) is a system of $n$ real equations in $\varepsilon, c_1, \ldots, c_n$ where $c_1, \ldots, c_n$ are the components of $c$. The system (2.7) is sometimes called the branching equations.

Next we suppose that $\mathcal{L}$ is a singular matrix. This is sometimes called the resonance case or degenerate case. Now we consider the case rank $\mathcal{L} = n-r$, $0 < n-r < n$. Let $E_r$ denote the null space of $\mathcal{L}$, and let $E_{n-r}$ denote the complement in $\mathbb{R}^n$ of $E_r$; that is,

$$\mathbb{R}^n = E_{n-r} \oplus E_r \ \text{(direct sum)}. \tag{2.8}$$

Let $x_1, \ldots, x_n$ be a basis for $\mathbb{R}^n$ such that $x_1, \ldots, x_r$ is a basis for $E_r$ and $x_{r+1}, \ldots, x_n$ a basis for $E_{n-r}$.

Let $P_r$ be the matrix projection onto Ker $\mathcal{L} = E_r$, and $P_{n-r} = I - P_r$, where $I$ is the identity matrix. Thus $P_{n-r}$ is a projection onto the complementary space $E_{n-r}$ of $E_r$, and

$$P_r^2 = P_r, \qquad P_{n-r}^2 = P_{n-r}, \qquad P_{n-r}P_r = P_rP_{n-r} = 0. \tag{2.9}$$

Without loss of generality, we may assume

$$P_rc = (c_1, \ldots, c_r, 0, \ldots, 0), \qquad P_{n-r}c = (0, \ldots, 0, c_{r+1}, \ldots, c_n). \tag{2.10}$$

We will identify $P_rc$ with $c^r = (c_1, \ldots, c_r)$ and $P_{n-r}c$ with $c^{n-r} = (c_{r+1}, \ldots, c_n)$ whenever it is convenient to do so.

Let $H$ be a nonsingular $n \times n$ matrix satisfying

$$H\mathcal{L} = P_{n-r}. \tag{2.11}$$

Matrix $H$ can be computed easily. The nature of the solutions of the branching equations depends heavily on the rank of the matrix $\mathcal{L}$.

**Lemma 2.5** (see [2]). *The matrix $\mathcal{L}$ has rank $n-r$ if and only if the three-point BVP* (2.4) *and* $Mx(0) + Nx(\eta) + Rx(1) = 0$ *has exactly $r$ linearly independent solutions.*

Next we give a necessary and sufficient condition for the existence of solutions of $x(t, c, \varepsilon)$ of three-point BVPs for $\varepsilon > 0$ such that the solution satisfies $x(0) = c$ where $c = c(\varepsilon)$ for suitable $c(\varepsilon)$.

We need to solve (2.7) for $c$ when $\varepsilon$ is sufficiently small. The problem of finding solutions to (1.1) and (1.2) is reduced to that of solving the branching equations (2.7) for $c$ as function of $\varepsilon$ for $|\varepsilon| < \varepsilon_0$. So consider (2.7) which is equivalent to

$$\mathcal{L}(P_r + P_{n-r})c = \varepsilon\mathcal{N}\big((P_r + P_{n-r})c, \alpha, \eta, \varepsilon\big) + d. \tag{2.12}$$

Multiplying (2.7) by the matrix $H$ and using (2.11), we have

$$P_{n-r}c = \varepsilon H\mathcal{N}\big((P_r + P_{n-r})c, \alpha, \eta, \varepsilon\big) + Hd, \tag{2.13}$$

where $H\mathcal{N}((P_r+P_{n-r})c,\alpha,\eta,\varepsilon) = -H(\int_0^\eta NY(\eta)Y^{-1}(s)F(s,x(s,c,\varepsilon),\varepsilon)ds + \int_0^1 RY(1)Y^{-1}(s)F(s,x(s,c,\varepsilon),\varepsilon)ds - g(c,x(\eta),x(1)))$ and $Hd = -H(\int_0^\eta NY(\eta)Y^{-1}(s)E(s)ds + \int_0^1 RY(1)Y^{-1}(s)E(s)ds - \ell)$.

Since the matrix $H$ is nonsingular, solving (2.7) for $c$ is equivalent to solving (2.13) for $c$. The following theorem due to Cronin [13, 14] gives a necessary condition for the existence of solutions to the BVP (1.1) and (1.2).

**Theorem 2.6** (see [2]). *A necessary condition that* (2.13) *can be solved for* $c$, *with* $|\varepsilon| < \varepsilon_0$, *for some* $\varepsilon_0 > 0$ *is* $P_r H d = 0$.

If $\mathcal{L}$ is a nonsingular matrix then the implicit function theorem is applicable to solve (2.7) uniquely for $c$ as a function of $\varepsilon$ in a neighborhood of the initial solution $c$ (see Cronin [14]). The implicit function theorem may be stated as in Voxman and Goetschel [18, page 222].

**Theorem 2.7** (the implicit function theorem). *Let* $\Omega \subset \mathbb{R}^n \times \mathbb{R}^m$ *be an open set, and let* $F : \Omega \to \mathbb{R}^m$ *be function of class* $C^1$. *Suppose* $(x_0, y_0) = 0$. *Assume that*

$$\det\begin{pmatrix} \dfrac{\partial F_1}{\partial y_1} & \cdots & \dfrac{\partial F_1}{\partial y_m} \\ & \cdots & \\ \dfrac{\partial F_m}{\partial y_1} & \cdots & \dfrac{\partial F_m}{\partial y_m} \end{pmatrix} \neq 0 \quad \text{evaluated at } (x_0, y_0), \tag{2.14}$$

*where* $F = (F_1, \ldots, F_m)$. *Then there are open sets* $U \subset \mathbb{R}^n$ *and* $V \subset \mathbb{R}^m$, *with* $x_0 \in U$ *and* $y_0 \in U$, *and a unique function* $f : U \to V$ *such that*

$$F(x, f(x)) = 0 \tag{2.15}$$

*for all* $x \in U$ *with* $y_0 = f(x_0)$. *Furthermore,* $f$ *is of class* $C^1$.

## 3. Main Results

In this section sufficient conditions are introduced for the existence of solutions to the BVP (1.1), (1.2). We recall the following Definition 1 of [2] to develop our main results.

*Definition 3.1* (see [2]). Let $E_r$ denote the null space of $\mathcal{L}$, and let $E_{n-r}$ denote the complement in $\mathbb{R}^n$ of $E_r$. Let $P_r$ be the matrix projection onto $\operatorname{Ker}\mathcal{L} = E_r$, and $P_{n-r} = I - P_r$, where $I$ is the identity matrix. Thus $P_{n-r}$ is a projection onto the complementary space $E_{n-r}$ of $E_r$. If $E_{n-r}$ is properly contained in $\mathbb{R}^n$, then $E_r$ is an $r$-dimensional vector space where $0 < r < n$. If $c = (c_1, \ldots, c_n)$, let $P_r c = c^r$ and $P_{n-r} = c^{n-r}$, then define a continuous mapping $\Phi_\varepsilon : \mathbb{R}^r \to \mathbb{R}^r$, given by

$$\Phi_\varepsilon(c_1, \ldots, c_r) = P_r H\mathcal{N}(c^r \oplus c^{n-r}(c^r, \varepsilon), \alpha, \eta, \varepsilon), \tag{3.1}$$

where $c^{n-r}(c^r, \varepsilon) = c^{n-r}$ is a differentiable function of $c^r$ and $\varepsilon$. By abuse of notation we will identify $P_r c$ and $c^r$ when convenient and where the meaning is clear from the context so that in defining $\Phi_\varepsilon$ above from the context we interpreted $P_r H\mathcal{N}$ as $(H\mathcal{N}_1, \ldots, H\mathcal{N}_r)$. Similarly we will sometimes identify $P_{n-r}c$ and $c^{n-r}$. Setting $\varepsilon = 0$, we have

$$\Phi_0(c_1, \ldots, c_r) = P_r H\mathcal{N}(c^r \oplus P_{n-r}Hd, \alpha, \eta, 0), \tag{3.2}$$

where $c^{n-r}(c^r, 0) = P_{n-r}Hd$; note that from the context $c^{n-r}(c^r, 0) = P_{n-r}Hd$ is interpreted as $c^{n-r}(c^r, 0) = (Hd_{r+1}, \ldots, Hd_n)$.

If $E_r = \mathbb{R}^n$ and $P_r = I$, then $P_{n-r} = 0$. Since $P_{n-r} = 0$, it follows that the matrix $H$ is the identity matrix. Thus define a continuous mapping $\Phi_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n$, given by $\Phi_\varepsilon(c) = \mathcal{N}(c, \alpha, \eta, \varepsilon)$. Setting $\varepsilon = 0$, we have $\Phi_0(c) = \mathcal{N}(c, \alpha, \eta, 0)$.

The following theorem is the main theorem of this paper and gives sufficient conditions for the existence of solutions of (1.1), (1.2) for $|\varepsilon| < \varepsilon_0$, for some $\varepsilon_0 > 0$. The existence theorem can be established using the implicit function theorem; see Theorem 2.7.

**Theorem 3.2.** *If $c = (c_1, \ldots, c_n) \in \mathbb{R}^n$, let $c^r = (c_1, \ldots, c_r)$. Let the conditions* (i), (ii), *and* (iii) *of Theorem 2.2 hold, and let $k_1 > 0$, $k > 0$ and $\varepsilon_0 > 0$ be small enough so that* (1.1) *has a unique $n$-vector $x(t, c, \varepsilon)$ defined on $[0,1] \times \overline{B}_{k_1} \times [-\varepsilon_0, \varepsilon_0]$. Let $\Phi_\varepsilon : \overline{B}_k \subseteq \mathbb{R}^r \to \mathbb{R}^r$, given by*

$$\Phi_\varepsilon(c_1, \ldots, c_r) = P_r H\mathcal{N}(c^r \oplus c^{n-r}(c^r, \varepsilon), \alpha, \eta, \varepsilon), \tag{3.3}$$

*where $c^{n-r}(c^r, \varepsilon) = c^{n-r}$ is a differentiable function of $c^r$ and $\varepsilon$, and*

$$\Phi_0(c_1, \ldots, c_r) = P_r H\mathcal{N}(c^r \oplus P_{n-r}Hd, \alpha, \eta, 0) \tag{3.4}$$

*for $(c^r \oplus P_{n-r}Hd) \in \overline{B}_k \times \{P_{n-r}Hd\} \subseteq \overline{B}_{k_1}$. If $\Phi_0(\overline{c_1}, \ldots, \overline{c_r}) = 0$ and*

$$\det \frac{\partial \Phi_0^i(c_1, \ldots, c_r)}{\partial c_j}\Big|_{(c_1, \ldots, c_r) = (\overline{c_1}, \ldots, \overline{c_r})} \neq 0, \tag{3.5}$$

*for some $(\overline{c_1}, \ldots, \overline{c_r}) \in \overline{B}_k$, then there is $\overline{\varepsilon}$, $0 < \overline{\varepsilon} \leq \varepsilon_0$, and $\delta > 0$ such that* (1.1), (1.2) *has a unique solution $x(t, c(\varepsilon), \varepsilon)$ for all $|\varepsilon| < \overline{\varepsilon}$ such that $c(0) = \overline{c} = (\overline{c}^r \oplus P_{n-r}Hd)$ and $|c(\varepsilon) - \overline{c}| < \delta$.*

*Proof.* The existence and uniqueness of a solution $x(t, c, \varepsilon)$ for $|\varepsilon| < \varepsilon_0$ with $x(0, c, \varepsilon) = c \in \mathbb{R}^n$ follows directly from conditions (i), (ii), and (iii) of Theorem 2.2. Now

$$\begin{gathered} \Phi_0(\overline{c_1}, \ldots, \overline{c_r}) = P_r H\mathcal{N}(\overline{c}^r \oplus P_{n-r}Hd, \alpha, \eta, 0) = 0, \\ \det \frac{\partial \Phi_0^i(c_1, \ldots, c_r)}{\partial c_j}\Big|_{(c_1, \ldots, c_r) = (\overline{c_1}, \ldots, \overline{c_r})} \neq 0, \end{gathered} \tag{3.6}$$

for some $(\overline{c_1}, \ldots, \overline{c_r}) \in \overline{B}_k$, thus it follows from the implicit function theorem that there is $\overline{\varepsilon}$, $0 < \overline{\varepsilon} \leq \varepsilon_0$ such that (3.3) has a unique solution $(c_1, \ldots, c_r) = (c_1(\varepsilon), \ldots, c_r(\varepsilon))$, with $|(c_1(\varepsilon), \ldots, c_r(\varepsilon)) - (\overline{c_1}, \ldots, \overline{c_r})| < \delta$, for all $\varepsilon$, $|\varepsilon| < \overline{\varepsilon}$. From this it follows that $x(t, c(\varepsilon), \varepsilon)$ is a unique solution of the BVP (1.1), (1.2) which satisfies the initial value $x(0, c(\varepsilon), \varepsilon) = c(\varepsilon)$ and $c(0) = \overline{c} = (\overline{c^r} \oplus P_{n-r}Hd)$ and $|c(\varepsilon) - \overline{c}| < \delta$, where $c(\varepsilon) = (\overline{c}^r(\varepsilon) \oplus c^{n-r}(\overline{c}^r(\varepsilon), \varepsilon))$. □

We now consider the BVP (1.1), (1.2) in the case $r = n$; that is, $\mathcal{L}$ is the zero matrix, which is sometimes called the totally degenerate case.

**Theorem 3.3** (compare with Theorem 3.8, page 69 of Cronin [14]). *If $r = n$, a necessary condition in order that* (2.7) *has a solution for each $\varepsilon$ with $|\varepsilon| < \varepsilon_0$ for some $\varepsilon_0 > 0$ is $d = 0$; that is,*

$$\int_0^\eta NY(\eta)Y^{-1}(s)E(s)ds + \int_0^1 RY(1)Y^{-1}(s)E(s)ds = \ell. \tag{3.7}$$

**Theorem 3.4.** *Let the conditions* (i), (ii), *and* (iii) *of Theorem 2.2 hold, and let $k_1 > 0$, $k > 0$ and $\varepsilon_0 > 0$ be small enough so that* (1.1) *has a unique solution $x(t,c,\varepsilon)$ defined on $[0,1] \times \overline{B}_{k_1} \times [-\varepsilon_0, \varepsilon_0]$. If $r = n$, $d = 0$, and*

$$\begin{aligned}\Phi_\varepsilon(c) = &-\int_0^\eta NY(\eta)Y^{-1}(s)F(s,x(s,c,\varepsilon),\varepsilon)ds \\ &-\int_0^1 RY(1)Y^{-1}(s)F(s,x(s,c,\varepsilon),\varepsilon)ds + g(c,x(\eta),x(1)),\end{aligned} \tag{3.8}$$

*then there is $\overline{\varepsilon}$, $0 < \overline{\varepsilon} \le \varepsilon_0$, and $\delta > 0$ such that* (1.1), (1.2) *has a unique solution $x(t,c(\varepsilon),\varepsilon)$ for all $|\varepsilon| < \overline{\varepsilon}$ such that $c(0) = \overline{c}$ and $|c(\varepsilon) - \overline{c}| < \delta$.*

*Proof.* If $r = n$ and $d = 0$, then $P_{n-\mathrm{r}} = 0$. This implies $P_r = I$. Since $P_{n-r} = 0$, it follows that $H = I$, the identity matrix. □

The existence and uniqueness of a solution $x(t,c(\varepsilon),\varepsilon)$ for $|\varepsilon| < \overline{\varepsilon} < \varepsilon_0$ with $x(0,c,\varepsilon) = c \in \mathbb{R}^n$ follows directly from conditions (i), (ii) and (iii) of Theorem 2.2. Now

$$\begin{aligned}\Phi_0(c) = &-\int_0^\eta NY(\eta)Y^{-1}(s)F(s,x(s,c,0),0)ds \\ &-\int_0^1 RY(1)Y^{-1}(s)F(s,x(s,c,0),0)ds + g(c,x(\eta),x(1)).\end{aligned} \tag{3.9}$$

If $\Phi_0(\overline{c}) = 0$,

$$\det \frac{\partial \Phi_0^i(c)}{\partial c_j}\Big|_{c=\overline{c}} \neq 0, \tag{3.10}$$

for some $\overline{c} = (\overline{c_1}, \ldots, \overline{c_n}) \in \overline{B}_k$; thus it follows from the implicit function theorem that there is $\overline{\varepsilon}$, $0 < \overline{\varepsilon} \le \varepsilon_0$ such that (3.8) has a unique solution $c = c(\varepsilon)$, with $|c - \overline{c}| < \delta$, for all $\varepsilon$, $|\varepsilon| < \overline{\varepsilon}$. From this it follows that $x(t,c(\varepsilon),\varepsilon)$ is a unique solution of the BVP (1.1), (1.2) which satisfies the initial values $x(0,c(\varepsilon),\varepsilon) = c(\varepsilon) \in \mathbb{R}^n$ for all $\varepsilon$, $|\varepsilon| < \overline{\varepsilon}$ such that $c(0) = \overline{c}$ and $|c(\varepsilon) - \overline{c}| < \delta$.

## 4. Some Examples

To find $c$ for $\varepsilon$ small using Theorem 2.6, we need to compute $\Phi_0(c)$ from (3.3). We apply Theorem 3.2 to show the existence of solutions.

*Example 4.1.* $\alpha = 1$, rank $\mathcal{L}_{\alpha=1} = 1 < 2$, $\ell_i \equiv 0$ for $i = 1, 2$.

Consider the BVP

$$\begin{gathered} y'' = \varepsilon f(t, y, y', \varepsilon) + e(t), \\ y'(0) = \varepsilon g_1\left(y(0), y'(0), y\left(\frac{1}{2}\right), y'\left(\frac{1}{2}\right), y(1), y'(1)\right), \\ y(1) - y\left(\frac{1}{2}\right) = \varepsilon g_2\left(y(0), y'(0), y\left(\frac{1}{2}\right), y'\left(\frac{1}{2}\right), y(1), y'(1)\right), \end{gathered} \tag{4.1}$$

where $f \in C([0,1] \times \mathbb{R}^2 \times (-\varepsilon_0, \varepsilon_0); \mathbb{R})$, $e \in C[0,1]$, $g \in C(\mathbb{R}^6; \mathbb{R}^2)$. Then the BVP (4.1) is equivalent to

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \varepsilon\begin{pmatrix} 0 \\ f(t, x_1, x_2, \varepsilon) \end{pmatrix} + \begin{pmatrix} 0 \\ e(t) \end{pmatrix}, \tag{4.2}$$

$$\begin{aligned} &\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -\alpha & 0 \end{pmatrix}\begin{pmatrix} x_1\left(\frac{1}{2}\right) \\ x_2\left(\frac{1}{2}\right) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1(1) \\ x_2(1) \end{pmatrix} \\ &\quad = \begin{pmatrix} \varepsilon g_1\left(c_1, c_2, x_1\left(\frac{1}{2}\right), x_2\left(\frac{1}{2}\right), x_1(1), x_2(1)\right) \\ \varepsilon g_2\left(c_1, c_2, x_1\left(\frac{1}{2}\right), x_2\left(\frac{1}{2}\right), x_1(1), x_2(1)\right) \end{pmatrix}, \end{aligned} \tag{4.3}$$

where

$$\begin{gathered} M = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad N = \begin{pmatrix} 0 & 0 \\ -\alpha & 0 \end{pmatrix}, \qquad R = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \\ E(t) = \begin{pmatrix} 0 \\ e(t) \end{pmatrix}, \qquad F(t, x, \varepsilon) = \begin{pmatrix} 0 \\ f(t, x_1, x_2, \varepsilon) \end{pmatrix}, \\ g(c_1, c_2, x_1(\eta), x_2(\eta), x_1(1), x_2(1)) = \begin{pmatrix} g_1(c_1, c_2, x_1(\eta), x_2(\eta), x_1(1), x_2(1)) \\ g_2(c_1, c_2, x_1(\eta), x_2(\eta), x_1(1), x_2(1)) \end{pmatrix}, \\ Y(t) = e^{At} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \qquad Y_0(t) = Y(t)Y^{-1}(0) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}. \end{gathered} \tag{4.4}$$

By Lemma 2.4, we find $\mathcal{L}$:

$$\begin{aligned}\mathcal{L} &= M + NY_0(\eta) + RY_0(1)\\ &= \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0\\ -\alpha & 0\end{pmatrix}\begin{pmatrix}1 & \eta\\ 0 & 1\end{pmatrix} + \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\\ &= \begin{pmatrix}0 & 1\\ 1-\alpha & 1-\alpha\eta\end{pmatrix}.\end{aligned} \tag{4.5}$$

The resonance happens if $\det(\mathcal{L}) = -1+\alpha = 0$; that is the case where $\alpha = 1$. For $\alpha = 1$, rank $\mathcal{L}_{\alpha=1} = 1$; that is,

$$\mathcal{L}_{\alpha=1} = \begin{pmatrix}0 & 1\\ 0 & 1-\eta\end{pmatrix}. \tag{4.6}$$

Let $E_1$ denote the null space of $\mathcal{L}_{\alpha=1}$. Thus $e_1 = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ is a basis for $\operatorname{Ker}(\mathcal{L}_{\alpha=1})$, and $\operatorname{Ker}(\mathcal{L}_{\alpha=1}) = \operatorname{Span} e_1$. Let $P_1$ be the matrix projection onto $\operatorname{Ker}(\mathcal{L}_{\alpha=1})$. $P_1 = \left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right)$. $P_2 = I - P_1 = \left(\begin{smallmatrix}0 & 0\\ 0 & 1\end{smallmatrix}\right)$. Set $H = (1/(1-\eta))\left(\begin{smallmatrix}1-\eta & -1\\ 0 & 1\end{smallmatrix}\right)$ so that $H\mathcal{L}_{\alpha=1} = P_2$. In system (4.2), (4.3) let $f(t,x_1,x_2,\varepsilon) = x_1x_2^2$, $e(t) = \cos 4\pi t g_1(c_1,c_2,x_1(1/2),x_2(1/2),x_1(1),x_2(1)) = -x_1^2(1)$, and let $g_2(c_1,c_2,x_1(1/2),x_2(1/2),x_1(1),x_2(1)) = 2x_1(1/2)/256\pi^4$. We need to show that $P_1Hd = 0$ which is a necessary condition in order to apply Theorem 2.6:

$$\begin{aligned}P_1Hd &= 2\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\frac{1}{2} & -1\\ 0 & 1\end{pmatrix}\begin{pmatrix}0\\ \int_0^{1/2}\left(\frac{1}{2}-s\right)\cos 4\pi s\,ds - \int_0^1(1-s)\cos 4\pi s\,ds\end{pmatrix}\\ &= 2\begin{pmatrix}-\int_0^{1/2}\left(\frac{1}{2}-s\right)\cos 4\pi s\,ds + \int_0^1(1-s)\cos 4\pi s\,ds\\ 0\end{pmatrix}.\end{aligned} \tag{4.7}$$

Since $\int_0^{1/2}(1/2-s)\cos 4\pi s\,ds = \int_0^1(1-s)\cos 4\pi s\,ds = 0$, it follows that $P_1Hd = 0$. From the boundary condition (4.3), we have $x_2(0) = c_2 = 0$. Then, by the variation of constants formula, we obtain

$$x(t,c,0) = \begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}\begin{pmatrix}c_1\\ 0\end{pmatrix} + \int_0^t\begin{pmatrix}1 & t-s\\ 0 & 1\end{pmatrix}\begin{pmatrix}0\\ \cos 4\pi s\end{pmatrix}ds. \tag{4.8}$$

Thus the BVP (4.2), (4.3) has a solution if $\alpha = 1$, $\varepsilon = 0$; namely, $x_1(t,c,0) = c_1 + ((1-\cos 4\pi t)/16\pi^2)$, $x_2(t,c,0) = \sin 4\pi t/4\pi$, $x_1(0) = x_1(1/2) = x_1(1) = c_1$, $x_2(0) = x_2(1/2) =$

$x_2(1) = 0$. Setting $\varepsilon = 0$, thus $f(t, c_1 + (1 - \cos 4\pi t)/16\pi^2, \sin 4\pi t/4\pi, 0) = \sin^2 4\pi t/16\pi^2(c_1 + (1 - \cos 4\pi t)/16\pi^2)$, $g_1(c_1, x_1(1/2), x_1(1)) = -c_1^2$ and $g_2(c_1, x_1(1/2), x_1(1)) = 2c_1/256\pi^4$. Hence

$$\begin{aligned}\Phi_0(c_1) &= \int_0^{1/2} f\left(s, c_1 + \frac{1 - \cos 4\pi s}{16\pi^2}, \frac{\sin 4\pi s}{4\pi}, 0\right) ds \\ &\quad + \int_{1/2}^{1} \left\{2(1-s) f\left(s, c_1 + \frac{1 - \cos 4\pi s}{16\pi^2}, \frac{\sin 4\pi s}{4\pi}, 0\right)\right\} ds \\ &\quad + g_1\left(c_1, x_1\left(\frac{1}{2}\right), x_1(1)\right) - 2g_2\left(c_1, x_1\left(\frac{1}{2}\right), x_1(1)\right) \\ &= -c_1^2 + \frac{c_1}{128\pi^2} + \frac{3}{2048\pi^4}.\end{aligned} \tag{4.9}$$

If $c_1 \approx -3.5023 \times 10^{-3}$ or $c_1 \approx 4.2938 \times 10^{-3}$, then $\Phi_0(c_1) = 0$ and

$$\frac{\partial \Phi_0(c_1)}{\partial c_1}\Big|_{(c_1 \approx -3.5023 \times 10^{-3})} \neq 0, \qquad \frac{\partial \Phi_0(c_1)}{\partial c_1}\Big|_{(c_1 \approx 4.2938 \times 10^{-3})} \neq 0. \tag{4.10}$$

Hence by Theorem 3.2 there is $\overline{\varepsilon}$, $0 < \overline{\varepsilon} \leq \varepsilon_0$ and $\delta > 0$ such that the BVP (4.2), (4.3) has a unique solution $x(t, c(\varepsilon), \varepsilon)$ which satisfies the initial values $x(0, c(\varepsilon), \varepsilon) = c(\varepsilon) \in \mathbb{R}^2$ for all $|\varepsilon| < \overline{\varepsilon}$ such that $c(0) = (c_1, 0)$ and $|c(\varepsilon) - c(0)| < \delta$.

*Example 4.2.* Rank $\mathcal{L} = 2 < 3$.

Consider the BVP

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \varepsilon \begin{pmatrix} f_1(t, x_1, x_2, x_3, \varepsilon) \\ f_2(t, x_1, x_2, x_3, \varepsilon) \\ f_3(t, x_1, x_2, x_3, \varepsilon) \end{pmatrix}, \tag{4.11}$$

$$\begin{aligned}&\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} + \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1(\pi) \\ x_2(\pi) \\ x_3(\pi) \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1(2\pi) \\ x_2(2\pi) \\ x_3(2\pi) \end{pmatrix} \\ &\quad = \begin{pmatrix} \ell_1 + \varepsilon g_1(c_1, c_2, c_3, x_1(\pi), x_2(\pi), x_3(\pi), x_1(2\pi), x_2(2\pi), x_3(2\pi)) \\ \ell_2 + \varepsilon g_2(c_1, c_2, c_3, x_1(\pi), x_2(\pi), x_3(\pi), x_1(2\pi), x_2(2\pi), x_3(2\pi)) \\ \ell_3 + \varepsilon g_3(c_1, c_2, c_3, x_1(\pi), x_2(\pi), x_3(\pi), x_1(2\pi), x_2(2\pi), x_3(2\pi)) \end{pmatrix},\end{aligned} \tag{4.12}$$

where $f_i \in C([0,1] \times \mathbb{R}^3 \times (-\varepsilon_0, \varepsilon_0); \mathbb{R})$, $i = 1,2,3$, $\ell = (\ell_1, \ell_2, \ell_3) \in \mathbb{R}^3$, $g \in C(\mathbb{R}^9; \mathbb{R}^3)$,

$$e(t) \equiv 0, \qquad A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad N = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$R = \begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad Y(t) = \begin{pmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad Y^{-1}(t) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$Y_0(\pi) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad Y_0(2\pi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$F(t, x_1, x_2, x_3, \varepsilon) = \begin{pmatrix} f_1(t, x_1, x_2, x_3, \varepsilon) \\ f_2(t, x_1, x_2, x_3, \varepsilon) \\ f_3(t, x_1, x_2, x_3, \varepsilon) \end{pmatrix},$$

$$\begin{aligned} &g(c_1, c_2, c_3, x_1(\pi), x_2(\pi), x_3(\pi), x_1(2\pi), x_2(2\pi), x_3(2\pi)) \\ &\quad = \begin{pmatrix} g_1(c_1, c_2, c_3, x_1(\pi), x_2(\pi), x_3(\pi), x_1(2\pi), x_2(2\pi), x_3(2\pi)) \\ g_2(c_1, c_2, c_3, x_1(\pi), x_2(\pi), x_3(\pi), x_1(2\pi), x_2(2\pi), x_3(2\pi)) \\ g_3(c_1, c_2, c_3, x_1(\pi), x_2(\pi), x_3(\pi), x_1(2\pi), x_2(2\pi), x_3(2\pi)) \end{pmatrix}. \end{aligned} \tag{4.13}$$

By Lemma 2.4, the problem of solving (4.11), (4.12) is reduced to that of solving $\mathcal{L}c = \varepsilon\mathcal{N}(c, \alpha, \eta, \varepsilon) + d$ for $c$ provided solutions $x(t, c, \varepsilon)$ of initial value problems exist on $[0,1]$ for each $(c, \varepsilon)$. Thus we find $\mathcal{L}$:

$$\begin{aligned} \mathcal{L} &= M + NY_0(\pi) + RY_0(2\pi) \\ &= \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{4.14}$$

Since rank $\mathcal{L} = 2$, it follows that the matrix $\mathcal{L}$ is singular. Let $E_3$ denote the null space of $\mathcal{L}$. Thus $e_3 = \left(\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right)$ is a basis for $\mathrm{Ker}(\mathcal{L})$, and $\mathrm{Ker}(\mathcal{L}) = \mathrm{Span}\, e_3$. Let $P_3$ be the matrix projection onto $\mathrm{Ker}(\mathcal{L})$. $P_3 = \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$. So $P_2 = I - P_3 = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$. Set $H = \left(\begin{smallmatrix} 1/2 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$ so that $H\mathcal{L} = P_2$.

$$\begin{aligned} \mathcal{N}(c, \alpha, \eta, \varepsilon) = &-\int_0^{\pi} NY(\pi)Y^{-1}(s) \begin{pmatrix} f_1(s, x_1(s, c, \varepsilon), x_2(s, c, \varepsilon), x_3(s, c, \varepsilon), \varepsilon) \\ f_2(s, x_1(s, c, \varepsilon), x_2(s, c, \varepsilon), x_3(s, c, \varepsilon), \varepsilon) \\ f_3(s, x_1(s, c, \varepsilon), x_2(s, c, \varepsilon), x_3(s, c, \varepsilon), \varepsilon) \end{pmatrix} ds \\ &-\int_0^{2\pi} RY(2\pi)Y^{-1}(s) \begin{pmatrix} f_1(s, x_1(s, c, \varepsilon), x_2(s, c, \varepsilon), x_3(s, c, \varepsilon), \varepsilon) \\ f_2(s, x_1(s, c, \varepsilon), x_2(s, c, \varepsilon), x_3(s, c, \varepsilon), \varepsilon) \\ f_3(s, x_1(s, c, \varepsilon), x_2(s, c, \varepsilon), x_3(s, c, \varepsilon), \varepsilon) \end{pmatrix} ds \end{aligned}$$

$$
\begin{aligned}
&+\begin{pmatrix} g_1(c_1,c_2,c_3,x_1(\pi),x_2(\pi),x_3(\pi),x_1(2\pi),x_2(2\pi),x_3(2\pi)) \\ g_2(c_1,c_2,c_3,x_1(\pi),x_2(\pi),x_3(\pi),x_1(2\pi),x_2(2\pi),x_3(2\pi)) \\ g_3(c_1,c_2,c_3,x_1(\pi),x_2(\pi),x_3(\pi),x_1(2\pi),x_2(2\pi),x_3(2\pi)) \end{pmatrix} \\
&= \begin{pmatrix} \mathcal{N}_1(c,\alpha,\eta,\varepsilon) \\ \mathcal{N}_2(c,\alpha,\eta,\varepsilon) \\ \mathcal{N}_3(c,\alpha,\eta,\varepsilon) \end{pmatrix}.
\end{aligned} \tag{4.15}
$$

Since $d = 0$, it follows that $P_3Hd = 0$. Thus a necessary condition of Theorem 2.6 holds. We also have $P_2Hd = 0$. To obtain $\Phi_0(c)$ we must first calculate $x(t,c,0)$; that is the solution of $x' = A(t)x + e(t)$. By Lemma 2.3, and boundary condition (4.12), $x' = A(t)x$ has a solution $x(t)$ with $x(0) = c = (c_1,c_2,c_3)^T$. We note that at $\varepsilon = 0$, $P_2Hd = P_2c$, where $P_2c = \left(\begin{smallmatrix} 1&0&0\\0&1&0\\0&0&0 \end{smallmatrix}\right)\left(\begin{smallmatrix} c_1\\c_2\\c_3 \end{smallmatrix}\right)$ and $P_2Hd = 0$. Hence $c_1 = 0$ and $c_2 = 0$. Thus

$$
x(t,c,0) = \begin{pmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ c_3 \end{pmatrix}. \tag{4.16}
$$

Thus the BVP (4.11), (4.12) has a solution if $\varepsilon = 0$; namely, $x_1(t,c,0) = x_2(t,c,0) = 0$ and $x_3(t,c,0) = c_3$, and thus $\ell_i = 0$, $i = 1,2$, $x_3(\pi) = c_3 = -\ell_3$, and $x_3(2\pi) = c_3$:

$$
\begin{aligned}
P_3H\mathcal{N}(c,\alpha,\eta,\varepsilon) &= \begin{pmatrix} 0&0&0 \\ 0&0&0 \\ 0&0&1 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \mathcal{N}_1(c,\alpha,\eta,\varepsilon) \\ \mathcal{N}_2(c,\alpha,\eta,\varepsilon) \\ \mathcal{N}_3(c,\alpha,\eta,\varepsilon) \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ \mathcal{N}_3(c,\alpha,\eta,\varepsilon) \end{pmatrix},
\end{aligned} \tag{4.17}
$$

where

$$
\begin{aligned}
\mathcal{N}_3(c,\alpha,\eta,\varepsilon) = &\int_0^{\pi} f_3(s,x_1(s,c,\varepsilon),x_2(s,c,\varepsilon),x_3(s,c,\varepsilon),\varepsilon)ds \\
&+ g_3(c_1,c_2,c_3,x_1(\pi),x_2(\pi),x_3(\pi),x_1(2\pi),x_2(2\pi),x_3(2\pi)).
\end{aligned} \tag{4.18}
$$

Thus $\Phi_\varepsilon(c_3) = \mathcal{N}_3(c^3 \oplus c^2(c^3,\varepsilon),\alpha,\eta,\varepsilon)$, where $c^3 = P_3c = (0,0,c_3)$ and $c^2 = P_2c = (c_1,c_2,0)$. Setting $\varepsilon = 0$, we have $\Phi_0(c_3) = \mathcal{N}_3(c^3,\alpha,\eta,0)$, where $c^2(c^3,0) = P_2Hd = 0$. Writing out the components and setting $\varepsilon = 0$, we obtain $x_1(t,c,0) = x_2(t,c,0) = 0$ and $x_3(t,c,0) = c_3$. Hence

$$
\begin{aligned}
\Phi_0(c_3) = &\int_0^{\pi} f_3(s,x_1(s,0,0,c_3,0),x_2(s,0,0,c_3,0),x_3(s,0,0,c_3,0),0)ds \\
&+ g_3(c_3,x_3(\pi),x_3(2\pi)),
\end{aligned} \tag{4.19}
$$

where $x_i(\pi) = x_i(2\pi) = 0$, $i = 1,2$, $x_3(\pi) = c_3 = -\ell_3$, and $x_3(2\pi) = c_3$. Let $f_3(t, x_1(t,0,0,c_3,0), x_2(t,0,0,c_3,0), x_3(t,0,0,c_3,0),0) = -c_3^2 \sin t$, and $g_3(c_3, x_3(\pi), x_3(2\pi)) = c_3^4$. Hence

$$\begin{aligned}\Phi_0(c_3) &= -\int_0^{\pi} \left\{ c_3^2 \sin s \right\} ds + c_3^4 \\ &= c_3^2 \cos t|_0^{\pi} + c_3^4.\end{aligned} \tag{4.20}$$

If $c_3 = \pm\sqrt{2}$, then $\Phi_0(c_3) = 0$ and

$$\frac{\partial \Phi_0(c_3)}{\partial c_3}\Big|_{(c_3=\pm\sqrt{2})} \neq 0. \tag{4.21}$$

Hence by Theorem 3.2 there is $\overline{\varepsilon}$, $0 < \overline{\varepsilon} \leq \varepsilon_0$ and $\delta > 0$ such that the BVP (4.11), (4.12) has a unique solution $x(t, c(\varepsilon), \varepsilon)$ which satisfies the initial values $x(0, c(\varepsilon), \varepsilon) = c(\varepsilon) \in \mathbb{R}^3$ for all $|\varepsilon| < \overline{\varepsilon}$ such that $c(0) = (0,0,c_3)$ and $|c(\varepsilon) - c(0)| < \delta$.

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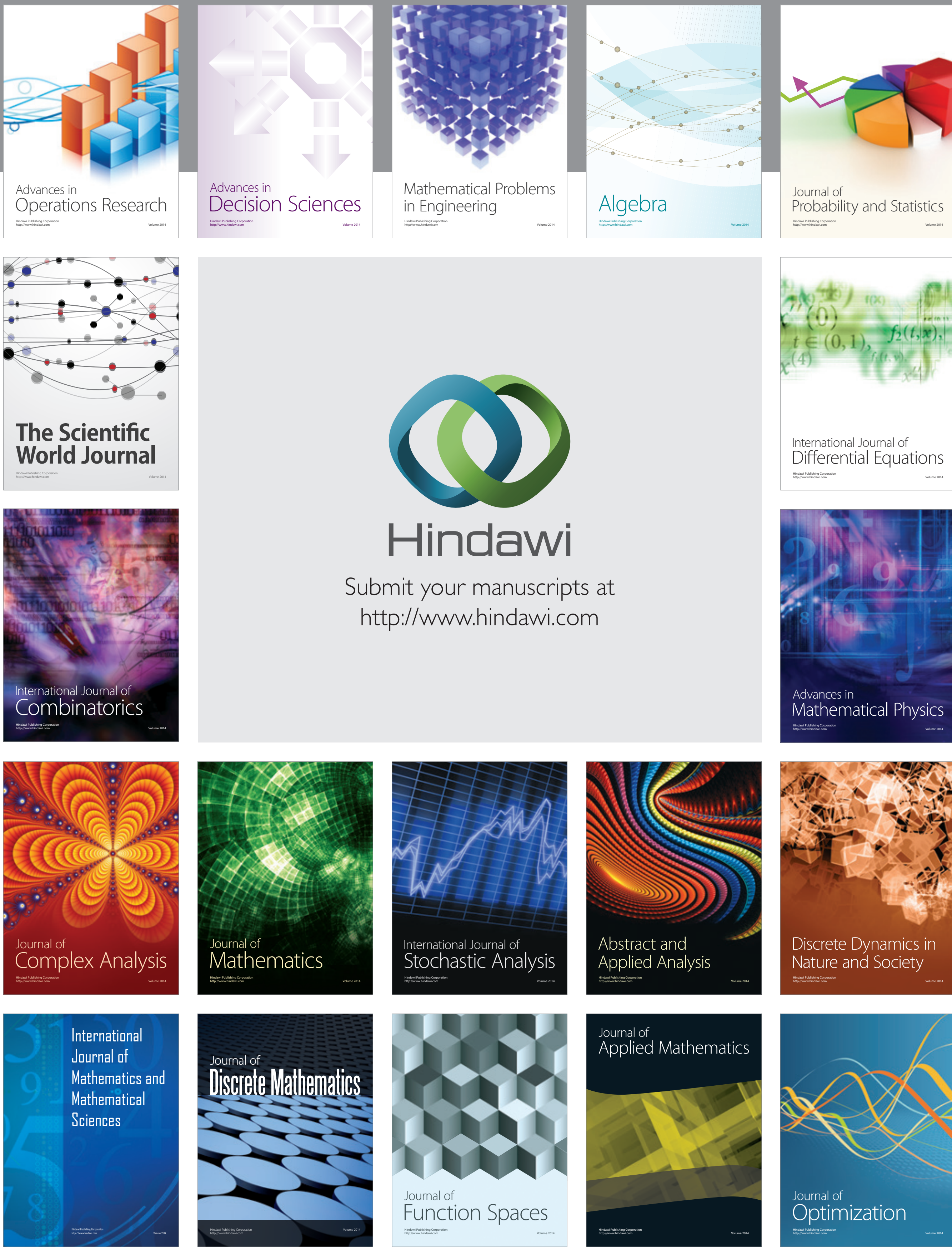
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