Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2012, Article ID 421050, 11 pages
doi:10.1155/2012/421050

Research Article

Strong Convergence of Viscosity Approximation Methods for Nonexpansive Mappings in CAT(0) Spaces

Luo Yi Shi and Ru Dong Chen

Department of Mathematics, Tianjin Polytechnic University, Tianjin 300160, China

Correspondence should be addressed to Ru Dong Chen, chenrd@tjpu.edu.cn

Received 30 March 2012; Accepted 27 April 2012

Academic Editor: Yonghong Yao

Viscosity approximation methods for nonexpansive mappings in CAT(0) spaces are studied. Consider a nonexpansive self-mapping T of a closed convex subset C of a CAT(0) space X. Suppose that the set $\mathrm{Fix}(T)$ of fixed points of T is nonempty. For a contraction f on C and $t \in (0,1)$, let $x_t \in C$ be the unique fixed point of the contraction $x \mapsto tf(x) \oplus (1-t)Tx$. We will show that if X is a CAT(0) space satisfying some property, then $\{x_t\}$ converge strongly to a fixed point of T which solves some variational inequality. Consider also the iteration process $\{x_n\}$, where $x_0 \in C$ is arbitrary and $x_{n+1} = \alpha_n f(x_n) \oplus (1-\alpha_n)Tx_n$ for $n \geq 1$, where $\{\alpha_n\} \subset (0,1)$. It is shown that under certain appropriate conditions on α_n, $\{x_n\}$ converge strongly to a fixed point of T which solves some variational inequality.

1. CAT(0) Spaces

A metric space X is a CAT(0) space if it is geodesically connected and if every geodesic triangle in X is at least as thin as its comparison triangle in the Euclidean plane. The precise definition is given below. It is well known that any complete, simply connected Riemannian manifold having nonpositive sectional curvature is a CAT(0) space. Other examples include pre-Hilbert spaces [1], R-trees [2], Euclidean buildings [3], the complex Hilbert ball with a hyperbolic metric [4], and many others. For a thorough discussion of these spaces and of the fundamental role they play in geometry, we refer the reader to Bridson and Haefliger [1].

Fixed-point theory in CAT(0) spaces was first studied by Kirk (see [5, 6]). He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. Since then, the fixed-point theory for single-valued and multivalued mappings in CAT(0) spaces has been rapidly developed, and many papers have appeared [2, 7–17].

The purpose of this paper is to study the iterative scheme defined as follows. Consider a nonexpansive self-mapping T of a closed convex subset C of a CAT(0) space X. Suppose that the set $\mathrm{Fix}(T)$ of fixed points of T is nonempty. For a contraction f on C and $t \in (0,1)$, let $x_t \in C$ be the unique fixed point of the contraction $x \mapsto tf(x) \oplus (1-t)Tx$. Consider the iteration process $\{x_n\}$, where $x_0 \in C$ is arbitrary and

$$x_{n+1} = \alpha_n f(x_n) \oplus (1-\alpha_n) T x_n, \tag{1.1}$$

for $n \geq 1$, where $\{\alpha_n\} \subset (0,1)$. We show that $\{x_n\}$ converge strongly to a fixed point of T under certain appropriate conditions on α_n, and the fixed point of T solves some variational inequality.

Let (X,d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a map c from a closed interval $[0,l] \subset R$ to X such that $c(0)=x, c(l)=y$, and $d(c(t),c(t')) = |t-t'|$ for all $t,t \in [0,l]$. In particular, c is an isometry and $d(x,y)=l$. The image α of c is called a *geodesic* (or *metric*) *segment* joining x and y. When it is unique, this geodesic segment is denoted by $[x,y]$. The space (X,d) is said to be a *geodesic space* if every two points of X are joined by a geodesic, and X is said to be *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x,y \in X$. A subset $Y \subseteq X$ is said to be convex if Y includes every geodesic segment joining any two of its points.

A *geodesic triangle* $\triangle(x_1,x_2,x_3)$ in a geodesicmetric space (X,d) consists of three points x_1, x_2, and x_3 in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of $\triangle$). A *comparison triangle* for the geodesic triangle $\triangle(x_1,x_2,x_3)$ in (X,d) is a triangle $\overline{\triangle}(x_1,x_2,x_3) := \triangle(\overline{x_1},\overline{x_2},\overline{x_3})$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(\overline{x_i},\overline{x_j}) = d(x_i,x_j)$ for $i,j \in 1,2,3$.

A geodesic space is said to be a CAT(0) space if all geodesic triangles satisfy the following comparison axiom.

CAT(0): let $\triangle$ be a geodesic triangle in X, and let $\overline{\triangle}$ be a comparison triangle for $\triangle$. Then, $\triangle$ is said to satisfy the CAT(0) inequality if for all $x,y \in \triangle$ and all comparison points $\overline{x},\overline{y} \in \overline{\triangle}$,

$$d(x,y) \leq d_{\mathbb{E}^2}(\overline{x},\overline{y}). \tag{1.2}$$

Let $x,y \in X$, and by Lemma 2.1 (iv) of [18] for each $t \in [0,1]$, there exists a unique point $z \in [x,y]$ such that

$$d(x,z) = td(x,y), \qquad d(y,z) = (1-t)d(x,y). \tag{1.3}$$

From now on, we will use the notation $(1-t)x \oplus ty$ for the unique point z satisfying the above equation.

We now collect some elementary facts about CAT(0) spaces which will be used in the proofs of our main results.

Lemma 1.1. *Let X be a CAT(0) space. Then,*

(i) *(see [18, Lemma 2.4]) for each* $x,y,z \in X$ *and* $t \in [0,1]$, *one has*

$$d((1-t)x \oplus ty, z) \leq (1-t)d(x,z) + td(y,z), \tag{1.4}$$

(ii) *(see [7]) for each* $x, y, z \in X$ *and* $t, s \in [0,1]$*, one has*

$$d\big((1-t)x \oplus ty, (1-s)x \oplus sy\big) \leq |t-s| d(x,y), \tag{1.5}$$

(iii) *(see [6, Lemma 3]) for each* $x, y, z \in X$ *and* $t \in [0,1]$*, one has*

$$d\big((1-t)z \oplus tx, (1-t)z \oplus ty\big) \leq t d(x,y), \tag{1.6}$$

(iv) *(see [18, Lemma 2.5]) for each* $x, y, z \in X$ *and* $t \in [0,1]$*, one has*

$$d\big((1-t)x \oplus ty, z\big)^2 \leq (1-t)d(x,z)^2 + t d(y,z)^2 - t(1-t)d(x,y)^2. \tag{1.7}$$

Let X be a complete CAT(0) space, let $\{x_n\}$ be a bounded sequence in a complete X, and for $x \in X$, set

$$r(x, \{x_n\}) = \limsup_{n \to \infty} d(x, x_n). \tag{1.8}$$

The *asymptotic radius* $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\}, \tag{1.9}$$

and the *asymptotic center* $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}. \tag{1.10}$$

It is known (see, e.g., [11, Proposition 7]) that in a CAT(0) space, $A(\{x_n\})$ consists of exactly one point.

A sequence $\{x_n\}$ in X is said to $\triangle$*-converge* to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\triangle\text{-}\lim_n x_n = x$ and call x the $\triangle$-limit of $\{x_n\}$.

Lemma 1.2. *Assume that X is a CAT(0) space. Then,*

(i) *(see [14]) every bounded sequence in X has a* $\triangle$*-convergent subsequence,*

(ii) *(see [14, Proposition 3.7]) if* K *is a closed convex subset of* X*, and* $f : K \to X$ *is a nonexpansive mapping, then the conditions* $\{x_n\}$ $\triangle$*-converge to* x *and* $d(x_n, f(x_n)) \to 0$ *and imply* $x \in K$ *and* $f(x) = x$*.*

Lemma 1.3 (see [19, Proposition 3.5]). *Assume that X is a CAT(0) space, C is a closed convex subset of X. Then the metric (nearest point) projection $P_C : X \to C, P_C(x) := \inf\{d(x,y); y \in C\}$ is a nonexpansive mapping. one calls a CAT(0) space X satisfying property $\mathcal{P}$ if for $x, u, y_1, y_2 \in X$,*

$$d(x, P_{[x,y_1]}(u))d(x,y_1) \leq d(x, P_{[x,y_2]}(u))d(x,y_2) + d(x,u)d(y_1,y_2). \tag{1.11}$$

Remark 1.4. The property $\mathcal{P}$ in Hilbert space corresponds to the inequality

$$\left|\langle u - x, y_1 - x\rangle\right| \leq \left|\langle u - x, y_2 - x\rangle\right| + \|u - x\| \cdot \|y_1 - y_2\|. \tag{1.12}$$

Recall that a continuous linear functional μ on l^∞, the Banach space of bounded real sequences, is called a Banach limit if $\|\mu\| = \mu(1,1,\ldots) = 1$ and $\mu_n(a_n) = \mu_n(a_{n+1})$ for all $\{a_n\} \in l^\infty$.

Lemma 1.5 (see [20, Proposition 2]). *Let $(a_1, a_2, \ldots) \in l^\infty$ be such that $\mu_n(a_n) \leq 0$ for all Banach limits μ and $\limsup_n (a_{n+1} - a_n) \leq 0$. Then $\limsup_n a_n \leq 0$.*

Lemma 1.6 (see [21, Lemma 2.3]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ a sequence of real numbers in $[0,1]$ with $\sum_{n=1}^\infty \alpha_n = \infty$, $\{u_n\}$ a sequence of nonnegative real numbers with $\sum_{n=1}^\infty u_n < \infty$, and $\{t_n\}$ a sequence of real numbers with $\limsup_n t_n \leq 0$. Suppose that*

$$s_{n+1} = (1 - \alpha_n)s_n + \alpha_n t_n + u_n, \quad \forall n \in \mathbb{N}. \tag{1.13}$$

Then $\lim_{n\to\infty} s_n = 0$.

2. Viscosity Iteration for a Single Mapping

In this section, we prove the main results of this paper.

Lemma 2.1. *Let C be a closed convex subset of a complete CAT(0) space X, and let $T : C \to C$ be a nonexpansive mapping. Let f be a contraction on C with coefficient $\alpha < 1$. For each $t \in [0,1]$, the mapping $S_t : C \to C$ defined by*

$$S_t x = t f(x) \oplus (1 - t)Tx, \qquad \text{for } x \in C \tag{2.1}$$

has a unique fixed point $x_t \in C$, that is,

$$x_t = t f(x_t) \oplus (1 - t)Tx_t. \tag{2.2}$$

Proof. For $x, y \in C$, according to Lemma 1.1, we have the following:

$$\begin{aligned} d(S_t(x), S_t(y)) &= d(tf(x) \oplus (1-t)Tx, tf(y) \oplus (1-t)Ty) \\ &\leq d(tf(x) \oplus (1-t)Tx, tf(x) \oplus (1-t)Ty) \\ &\quad + d(tf(y) \oplus (1-t)Tx, tf(y) \oplus (1-t)Ty) \\ &\leq td(f(x), f(y)) + (1-t)d(Tx, Ty) \\ &\leq (1-t(1-\alpha))d(x, y). \end{aligned} \tag{2.3}$$

This implies that S_t is a contraction mapping, and hence, the conclusion follows. □

The following result is to prove that the net $\{x_t\}$ converge strongly to a fixed point of T.

Theorem 2.2. *Let C be a closed convex subset of a complete CAT(0) space X satisfying the property $\mathcal{P}$, and let $T : C \to C$ be a nonexpansive mapping. Let f be a contraction on C with coefficient $\alpha < 1$. For each $t \in [0, 1]$, let $\{x_t\}$ be given by*

$$x_t = tf(x_t) \oplus (1-t)Tx_t. \tag{2.4}$$

Then one has $\lim_{t\to 0} x_t =: \tilde{x}$ and $\tilde{x} = P_{\mathrm{Fix}(T)} f(\tilde{x})$.

Proof. We first show that $\{x_t\}$ is bounded. Indeed choose a $p \in \mathrm{Fix}(T)$, and using Lemma 1.1 and the nonexpansive of T, we derive that

$$\begin{aligned} d(x_t, p) &= d(tf(x_t) \oplus (1-t)Tx_t, p) \\ &\leq td(f(x_t), p) + (1-t)d(Tx_t, p) \\ &\leq td(f(x_t), p) + (1-t)d(x_t, p). \end{aligned} \tag{2.5}$$

It follows that

$$\begin{aligned} d(x_t, p) \leq d(f(x_t), p) &\leq d(f(x_t), f(p)) + d(f(p), p) \\ &\leq \alpha d(x_t, p) + d(f(p), p). \end{aligned} \tag{2.6}$$

Hence,

$$d(x_t, p) \leq \frac{1}{1-\alpha} d(f(p), p), \tag{2.7}$$

and $\{x_t\}$ is bounded, so are $\{Tx_t\}$ and $\{f(x_t)\}$. As a result, we can get that

$$\begin{aligned} \lim_{t\to 0} d(x_t, Tx_t) &= \lim_{t\to 0} d(tf(x_t) \oplus (1-t)Tx_t, Tx_t) \\ &= \lim_{t\to 0} td(f(x_t), Tx_t) = 0. \end{aligned} \tag{2.8}$$

Assume that $\{t_n\} \subseteq (0,1)$ is such that $t_n \to 0$ as $n \to \infty$. Put $x_n := x_{t_n}$. We will show that $\{x_n\}$ contains a subsequence converging strongly to $\tilde{x}$, where $\tilde{x} \in \mathrm{Fix}(T)$.

Since $\{x_n\}$ is bounded, by Lemma 1.2(i),(ii), we may assume that $\{x_n\}\Delta$-converges to a point $\tilde{x}$, and $\tilde{x} \in \mathrm{Fix}(T)$.

Next we will prove that $\{x_n\}$ converge strongly to $\tilde{x}$.

Indeed, according to Lemma 1.1 and the property of T and f, we can get that

$$\begin{aligned} d^2(x_n,\tilde{x}) &= d^2(t_n f(x_n) \oplus (1-t_n)Tx_n, \tilde{x}) \\ &\le t_n d^2(f(x_n),\tilde{x}) + (1-t_n)d^2(Tx_n,\tilde{x}) - t_n(1-t_n)d^2(f(x_n),Tx_n) \\ &\le t_n d^2(f(x_n),\tilde{x}) + (1-t_n)d^2(x_n,\tilde{x}) - t_n(1-t_n)d^2(f(x_n),Tx_n). \end{aligned} \tag{2.9}$$

It follows that

$$\begin{aligned} d^2(x_n,\tilde{x}) &\le d^2(f(x_n),\tilde{x}) - (1-t_n)d^2(f(x_n),Tx_n) \\ &= d^2(f(x_n),\tilde{x}) - d^2(f(x_n),Tx_n) + t_n d^2(f(x_n),Tx_n). \end{aligned} \tag{2.10}$$

Since $\lim_{t\to 0} d(x_t, Tx_t) = 0$, we can get that

$$\limsup_{n\to\infty} d^2(x_n,\tilde{x}) \le \limsup_{n\to\infty} d^2(f(x_n),\tilde{x}) - d^2(f(x_n),x_n). \tag{2.11}$$

Let $\Delta(\overline{\tilde{x}},\overline{x_n},\overline{f(x_n)})$ be a comparison triangle for $\Delta(\tilde{x},x_n,f(x_n))$ in $\mathbb{E}^2$. Then,

$$\begin{aligned} d^2(f(x_n),\tilde{x}) - d^2(f(x_n),x_n) &= d^2\left(\overline{f(x_n)},\overline{\tilde{x}}\right) - d^2\left(\overline{f(x_n)},\overline{x_n}\right) \\ &= \left\langle \overline{f(x_n)} - \overline{\tilde{x}}, \overline{f(x_n)} - \overline{\tilde{x}} \right\rangle - \left\langle \overline{f(x_n)} - \overline{x_n}, \overline{f(x_n)} - \overline{x_n} \right\rangle \\ &= 2\left\langle \overline{f(x_n)} - \overline{\tilde{x}}, \overline{x_n} - \overline{\tilde{x}} \right\rangle - \left\langle \overline{x_n} - \overline{\tilde{x}}, \overline{x_n} - \overline{\tilde{x}} \right\rangle \\ &= 2\left\langle \overline{f(x_n)} - \overline{\tilde{x}}, \overline{x_n} - \overline{\tilde{x}} \right\rangle - d^2\left(\overline{x_n},\overline{\tilde{x}}\right) \\ &= 2\left\langle \overline{f(x_n)} - \overline{\tilde{x}}, \overline{x_n} - \overline{\tilde{x}} \right\rangle - d^2(x_n,\tilde{x}). \end{aligned} \tag{2.12}$$

Hence,

$$\limsup_{n\to\infty} d^2(x_n,\tilde{x}) \le \limsup_{n\to\infty} \left\langle \overline{f(x_n)} - \overline{\tilde{x}}, \overline{x_n} - \overline{\tilde{x}} \right\rangle. \tag{2.13}$$

Let $\Delta(\overline{\tilde{x}},\overline{x_n},\overline{f(\tilde{x})})$ be a comparison triangle for $\Delta(\tilde{x},x_n,f(\tilde{x}))$ in $\mathbb{E}^2$. For each n, let $\overline{u_n}$ be the point of the segment $[\overline{\tilde{x}},\overline{f(\tilde{x})}]$ which is nearest to $\overline{x_n}$, and let u_n be the point of the segment $[\tilde{x},f(\tilde{x})]$ for which $d(u_n,\tilde{x}) = d(\overline{u_n},\overline{\tilde{x}})$.

By passing to subsequences again, we may suppose that $\{\overline{u_n}\}$ converges to $\overline{u} \in [\overline{\tilde{x}},\overline{f(\tilde{x})}]$, $\{u_n\}$ converges to $u \in [\tilde{x},f(\tilde{x})]$.

Since $\{x_n\}$ $\triangle$-converges to a point $\widetilde{x}$, we have

$$\begin{aligned} r(\{x_n\}) &= \limsup_n d(x, x_n) \\ &= \limsup_n d(\overline{x}, \overline{x_n}) \\ &\geq \limsup_n d(\overline{u_n}, \overline{x_n}) \\ &= \limsup_n d(\overline{u}, \overline{x_n}) \\ &\geq \limsup_n d(u, x_n). \end{aligned} \tag{2.14}$$

Thus, $r(u, \{x_n\}) \leq r(\{x_n\})$. This implies that $u = x$ by uniqueness of the asymptotic center. Hence, $\overline{u} = \overline{x}$. That is to say, $\{\overline{u_n}\}$ converges to $\overline{\widetilde{x}}$, and $\{u_n\}$ converges to $\widetilde{x}$.

Moreover, since X satisfies the property $\mathcal{P}$, we can get that

$$\begin{aligned} \left|\left\langle \overline{f(x_n)} - \overline{\widetilde{x}}, \overline{x_n} - \overline{\widetilde{x}} \right\rangle\right| &= d\left(\overline{\widetilde{x}}, P_{[\overline{\widetilde{x}}, \overline{f(x_n)}]}(\overline{x_n})\right) \cdot d\left(\overline{\widetilde{x}}, \overline{f(x_n)}\right) \\ &= d\left(\widetilde{x}, P_{[\widetilde{x}, f(x_n)]}(x_n)\right) \cdot d\left(\widetilde{x}, f(x_n)\right) \\ &\leq d\left(\widetilde{x}, P_{[\widetilde{x}, f(\widetilde{x})]}(x_n)\right) \cdot d\left(\widetilde{x}, f(\widetilde{x})\right) \\ &\quad + d(\widetilde{x}, x_n) \cdot d\left(f(x_n), f(\widetilde{x})\right) \\ &= d\left(\overline{\widetilde{x}}, P_{[\overline{\widetilde{x}}, \overline{f(\widetilde{x})}]}(\overline{x_n})\right) \cdot d\left(\overline{\widetilde{x}}, \overline{f(\widetilde{x})}\right) \\ &\quad + d(\widetilde{x}, x_n) \cdot d\left(f(x_n), f(\widetilde{x})\right) \\ &\leq d\left(\overline{u_n}, \overline{\widetilde{x}}\right) d\left(\overline{\widetilde{x}}, \overline{f(\widetilde{x})}\right) + \alpha d^2(x_n, \widetilde{x}). \end{aligned} \tag{2.15}$$

It follows that

$$\limsup_{n\to\infty} d^2(x_n, \widetilde{x}) \leq \frac{1}{1-\alpha} \limsup_{n\to\infty} d\left(\overline{u_n}, \overline{\widetilde{x}}\right) d\left(\overline{f(\widetilde{x})}, \overline{\widetilde{x}}\right). \tag{2.16}$$

Since $\{\overline{u_n}\}$ converges to $\overline{\widetilde{x}}$, we obtain that $\limsup_{n\to\infty} d^2(x_n, \widetilde{x}) = 0$, that is, $\{x_n\}$ converge strongly to $\widetilde{x}$. Since $\{t_n\} \subseteq (0,1)$ is such that $t_n \to 0$ as $n \to \infty$ is arbitrarily selected, we can get that $\lim_{t\to 0} x_t = \widetilde{x}$.

Finally, we will prove that $\widetilde{x}$ satisfy the equation $\widetilde{x} = P_{\mathrm{Fix}(T)} f(\widetilde{x})$.

Indeed, for any $y \in \mathrm{Fix}(T)$,

$$\begin{aligned} d(x_t, y) &= d\left(t f(x_t) \oplus (1-t) T x_t, y\right) \\ &= t d\left(f(x_t), y\right) + (1-t) d\left(T x_t, y\right) \\ &\leq t d\left(f(x_t), y\right) + (1-t) d\left(x_t, y\right). \end{aligned} \tag{2.17}$$

It follows that

$$d(x_t, y) \leq d\left(f(x_t), y\right). \tag{2.18}$$

Since $\lim_{t\to 0} x_t = \widetilde{x}$, we can get that

$$d(\widetilde{x}, y) \leq d(f(\widetilde{x}), y). \tag{2.19}$$

Hence,

$$\begin{aligned} &d(f(\widetilde{x}), y) \geq \left| d(\widetilde{x}, y) - d(\widetilde{x}, f(\widetilde{x})) \right|, \\ &d(\widetilde{x}, f(\widetilde{x})) \geq d(f(\widetilde{x}), y). \end{aligned} \tag{2.20}$$

That is to say, $\widetilde{x} = P_{\mathrm{Fix}(T)} f(\widetilde{x})$. □

Consider now the iteration process

$$\begin{gathered} x_0 \in C, \\ x_{n+1} = \alpha_n f(x_n) \oplus (1-\alpha_n) T x_n, \quad n \geq 0, \end{gathered} \tag{2.21}$$

where $\{\alpha_n\} \subseteq (0,1)$ satisfies

(H1) $\alpha_n \to 0$,

(H2) $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(H3) either $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ or $\lim_{n\to\infty}(\alpha_{n+1}/\alpha_n) = 1$.

Theorem 2.3. *Let X be a CAT(0) space satisfying the property $\mathcal{P}$, C a closed convex subset of X, $T : C \to C$ a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$, and $f : C \to C$ a contraction with coefficient $\alpha < 1$. Let $x_0 \in C$, $\{x_n\}$ be generated by $x_{n+1} = \alpha_n f(x_n) \oplus (1-\alpha_n) T x_n, n \geq 0$. Then under the hypotheses (H1)–(H3), $x_n \to \widetilde{x}$, where $\widetilde{x} = P_{\mathrm{Fix}(T)} f(\widetilde{x})$.*

Proof. We first show that the sequence $\{x_n\}$ is bounded. Let $p \in \mathrm{Fix}(T)$. Then,

$$\begin{aligned} d(x_{n+1}, p) &= d(\alpha_n f(x_n) \oplus (1-\alpha_n) T x_n, p) \\ &\leq \alpha_n d(f(x_n), p) + (1-\alpha_n) d(T x_n, p) \\ &\leq \alpha_n \left[d(f(x_n), f(p)) + d(f(p), p) \right] + (1-\alpha_n) d(x_n, p) \\ &\leq \max\left\{ d(x_n, p), \frac{1}{1-\alpha} d(f(p), p) \right\}. \end{aligned} \tag{2.22}$$

By induction, we have

$$d(x_n, p) \leq \max\left\{ d(x_0, p), \frac{1}{1-\alpha} d(f(p), p) \right\}, \tag{2.23}$$

for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ is bounded and so is the sequence $\{T x_n\}$ and $\{f(x_n)\}$.

We claim that $d(x_{n+1}, x_n) \to 0$. Indeed, we have

$$
\begin{aligned}
d(x_{n+1}, x_n) &= d\big(\alpha_n f(x_n) \oplus (1-\alpha_n)Tx_n, \alpha_{n-1} f(x_{n-1}) \oplus (1-\alpha_{n-1})Tx_{n-1}\big) \\
&\le d\big(\alpha_n f(x_n) \oplus (1-\alpha_n)Tx_n, \alpha_n f(x_n) \oplus (1-\alpha_n)Tx_{n-1}\big) \\
&\quad + d\big(\alpha_n f(x_n) \oplus (1-\alpha_n)Tx_{n-1}, \alpha_n f(x_{n-1}) \oplus (1-\alpha_n)Tx_{n-1}\big) \\
&\quad + d\big(\alpha_n f(x_{n-1}) \oplus (1-\alpha_n)Tx_{n-1}, \alpha_{n-1} f(x_{n-1}) \oplus (1-\alpha_{n-1})Tx_{n-1}\big) \\
&\le (1-\alpha_n)d(Tx_n, Tx_{n-1}) + \alpha_n d\big(f(x_n), f(x_{n-1})\big) + |\alpha_n - \alpha_{n-1}| d\big(f(x_{n-1}), Tx_{n-1}\big) \\
&\le (1-\alpha_n)d(x_n, x_{n-1}) + \alpha_n \alpha d(x_n, x_{n-1}) + |\alpha_n - \alpha_{n-1}| d\big(f(x_{n-1}), Tx_{n-1}\big).
\end{aligned}
\tag{2.24}
$$

By the conditions H2 and H3, we have

$$
d(x_{n+1}, x_n) \longrightarrow 0. \tag{2.25}
$$

Consequently, by the condition H1,

$$
\begin{aligned}
d(x_n, Tx_n) &\le d(x_n, x_{n+1}) + d(x_{n+1}, Tx_n) \\
&= d(x_n, x_{n+1}) + d\big(\alpha_n f(x_n) \oplus (1-\alpha_n)Tx_n, Tx_n\big) \\
&= d(x_n, x_{n+1}) + \alpha_n d\big(f(x_n), Tx_n\big) \longrightarrow 0.
\end{aligned}
\tag{2.26}
$$

Since $\{x_n\}$ is bounded, we may assume that $\{x_n\}\triangle$-converges to a point $\widehat{x}$. By Lemma 1.2, we have $\widehat{x} \in \mathrm{Fix}(T)$.

Next we will prove that $\{x_n\}$ converge strongly to $\widehat{x}$ and $\widehat{x} = \widetilde{x}$. Indeed, according to Lemma 1.1 and the property of T and f, we can get that

$$
\begin{aligned}
d^2(x_{n+1}, \widehat{x}) &= d^2\big(t_n f(x_n) \oplus (1-t_n)Tx_n, \widehat{x}\big) \\
&\le \alpha_n d^2\big(f(x_n), \widehat{x}\big) + (1-\alpha_n)d^2(Tx_n, \widehat{x}) - \alpha_n(1-\alpha_n)d^2\big(f(x_n), Tx_n\big) \\
&\le (1-\alpha_n)d^2(x_n, \widehat{x}) + \alpha_n\Big[d^2\big(f(x_n), \widehat{x}\big) - (1-\alpha_n)d^2\big(f(x_n), T_n\big)\Big].
\end{aligned}
\tag{2.27}
$$

With a minor modification of the proof of the analogous statement in Theorem 2.2, we can get that

$$
\begin{aligned}
d^2\big(f(x_n), \widehat{x}\big) - d^2\big(f(x_n), x_n\big) &= 2\Big\langle \overline{f(x_n)} - \overline{\widehat{x}}, \overline{x_n} - \overline{\widehat{x}} \Big\rangle - d^2(x_n, \widehat{x}) \\
&\le 2d\Big(\overline{u_n}, \overline{\widehat{x}}\Big) d\Big(\overline{f(\widehat{x})}, \overline{\widehat{x}}\Big) + 2\alpha d^2(x_n, \widehat{x}) - d^2(x_n, \widehat{x}),
\end{aligned}
\tag{2.28}
$$

and $d(\overline{u_n}, \overline{\widehat{x}}) \to 0$.

Thus,

$$
\begin{aligned}
d^2(x_{n+1}, \hat{x}) \le{}& (1-\alpha_n)d^2(x_n, \hat{x}) + \alpha_n\Big[d^2(f(x_n), \hat{x}) - (1-\alpha_n)d^2(f(x_n), x_n) \\
&\quad +(1-\alpha_n)d^2(f(x_n), x_n) - (1-\alpha_n)d^2(Tx_n, x_n)\Big] \\
\le{}& (1-2(1-\alpha)\alpha_n)d^2(x_n, \hat{x}) \\
&+ 2(1-\alpha)\alpha_n\left[\frac{1}{(1-\alpha)}d\left(\overline{u_n}, \overline{\hat{x}}\right)d\left(\overline{f(\hat{x})}, \overline{\hat{x}}\right) + \frac{1}{(1-\alpha)}\beta_n\right],
\end{aligned}
\tag{2.29}
$$

where $\beta_n = (1-\alpha_n)d^2(f(x_n), x_n) - (1-\alpha_n)d^2(Tx_n, x_n)]$. Since $d(\overline{u_n}, \overline{\hat{x}}) \to 0$ and $d(x_n, Tx_n) \to 0$, we obtain that

$$
\limsup_n \frac{1}{(1-\alpha)}d\left(\overline{u_n}, \overline{\hat{x}}\right)d\left(\overline{f(\hat{x})}, \overline{\hat{x}}\right) + \frac{1}{(1-\alpha)}\beta_n \le 0.
\tag{2.30}
$$

According to Lemma 1.6, we can get $d^2(x_n, \hat{x}) \to 0$.

Finally, we prove that $\hat{x} = \tilde{x}$.

Indeed, for any $z \in \mathrm{Fix}(T)$,

$$
\begin{aligned}
d^2(x_{n+1}, z) &\le \alpha_n d^2(z, f(x_n)) + (1-\alpha_n)d^2(z, Tx_n) - \alpha_n(1-\alpha_n)d^2(f(x_n), Tx_n) \\
&\le \alpha_n d^2(z, f(x_n)) + (1-\alpha_n)d^2(z, x_n) - \alpha_n(1-\alpha_n)d^2(f(x_n), Tx_n).
\end{aligned}
\tag{2.31}
$$

Let μ be a Banach limit. Then,

$$
\mu_n d^2(x_{n+1}, z) \le \mu_n d^2(z, f(x_n)) - \mu_n d^2(f(x_n), Tx_n).
\tag{2.32}
$$

Since $x_n \to \hat{x}$, we obtain that

$$
d^2(\hat{x}, z) \le d^2(z, f(\hat{x})) - d^2(f(\hat{x}), \hat{x}).
\tag{2.33}
$$

It follows that

$$
d^2(f(\hat{x}), \hat{x}) \le d^2(z, f(\hat{x})),
\tag{2.34}
$$

that is to say, $\hat{x} = P_{\mathrm{Fix}(T)} f(\hat{x})$. Since $P_{\mathrm{Fix}(T)} f$ is a contraction and $\tilde{x} = P_{\mathrm{Fix}(T)} f(\tilde{x})$, we know that $\hat{x} = \tilde{x}$. □

Acknowledgment

This paper was supported by NSFC Grant no. 11071279.

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