Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2012, Article ID 436531, 12 pages
doi:10.1155/2012/436531

Research Article

Extended Precise Large Deviations of Random Sums in the Presence of END Structure and Consistent Variation

Shijie Wang[1] and Wensheng Wang[2]

[1] *School of Mathematic Sciences, Anhui University, Hefei, Anhui 230601, China*
[2] *Department of Mathematics, Hangzhou Normal University, Hangzhou 310036, China*

Correspondence should be addressed to Shijie Wang, ahuwsj@126.com

Received 6 November 2011; Accepted 30 December 2011

Academic Editor: Ying U. Hu

The study of precise large deviations of random sums is an important topic in insurance and finance. In this paper, extended precise large deviations of random sums in the presence of END structure and consistent variation are investigated. The obtained results extend those of Chen and Zhang (2007) and Chen et al. (2011). As an application, precise large deviations of the prospective-loss process of a quasirenewal model are considered.

1. Introduction

In the risk theory, heavy-tailed distributions are often used to model large claims. They play a key role in some fields such as insurance, financial mathematics, and queueing theory. We say that a distribution function F belongs to the class $\mathcal{C}$ if

$$\lim_{y\searrow 1}\liminf_{x\to\infty}\frac{\overline{F}(xy)}{\overline{F}(x)}=1 \quad \text{or equivalently} \quad \lim_{y\nearrow 1}\limsup_{x\to\infty}\frac{\overline{F}(xy)}{\overline{F}(x)}=1. \tag{1.1}$$

Such a distribution function F is usually said to have a consistently varying tail. The heavy-tailed subclass $\mathcal{C}$ was also studied by Cline and Samorodnitsky [1] who called it "intermediate regular variation." Another well-known class is called the dominated variation class (denoted by $\mathfrak{D}$). A distribution function F supported on $(-\infty,\infty)$ is in $\mathfrak{D}$ if and only if

$$\limsup_{x\to\infty}\frac{\overline{F}(xy)}{\overline{F}(x)}<\infty \tag{1.2}$$

for any $0 < y < 1$ (or equivalently for some $0 < y < 1$). For more details of other heavy-tailed subclasses (e.g., $\mathcal{R}, \mathcal{S}, \mathcal{L}$, and so on) and their relations, we refer the reader to [2] or [3].

Throughout this paper, let $\{X_k, k = 1, 2, \ldots\}$ be a sequence of real-valued random variables with common distribution function $1 - \overline{F}(x) = F(x) \in \mathcal{C}$ and finite mean μ. Let $\{N(t), t \geq 0\}$ be a nonnegative integer valued counting process independent of $\{X_k, k = 1, 2, \ldots\}$ with mean function $\lambda(t) = EN(t)$, which tends to infinity as $t \to \infty$. In insurance and finance, $\{X_k, k = 1, 2, \ldots\}$ and $\{N(t), t \geq 0\}$ always denote the claims and claim numbers respectively. Hence, randomly indexed sums (random sums), which denote the loss process of the insurer during the period $[0, t]$, can be written as

$$S_{N(t)} = \sum_{k=1}^{N(t)} X_k, \quad t \geq 0. \tag{1.3}$$

Recently, for practical reasons, precise large deviations of random sums with heavy tails have received a remarkable amount of attention. The study of precise large deviations is mainly to describe the deviations of a random sequence or a stochastic process away from its mean. The mainstream research of precise large deviations of $S_{N(t)}$ focuses on the study of the asymptotic relation

$$P\left(S_{N(t)} - \mu\lambda(t) > x\right) \sim \lambda(t)\overline{F}(x), \tag{1.4}$$

which holds uniformly for some x-region $\mathfrak{D}(t)$ as $t \to \infty$. The study of precise large deviations of random sums was initiated by Klüppelberg and Mikosch [4], who presented several applications in insurance and finance. For some latest works, we refer the reader to [2, 3, 5–11], among others.

In this paper, we are interested in the deviations of random sums $S_{N(t)}$ away from $m\lambda(t)$ with any fixed real number m. We aim at proving the following asymptotic relation:

$$P\left(S_{N(t)} - m\lambda(t) > x\right) \sim \lambda(t)\overline{F}\left(x + (m - \mu)\lambda(t)\right) \tag{1.5}$$

and the uniformity of (1.5). That is to say, in which x-region $\mathfrak{D}(t)$ as $t \to \infty$ (1.5) holds uniformly. It is interesting that (1.5) reduces to (1.4) with m replaced by μ. Hence, we call (1.5) the extended precise large-deviation probabilities. More interestingly, setting $m = 0$ and replacing X_k with $X_k - (1 + \delta)\mu$ in (1.5), where β denotes the safety loading coefficient, now (1.5) reduces to precise large-deviation probabilities for prospective-loss process. About precise large deviations for prospective-loss process, we refer the reader to [5].

The basic assumption of this paper is that $\{X_k, k = 1, 2, \ldots\}$ is extended negatively dependent (END). The END structure was firstly introduced by Liu [12].

Definition 1.1. One calls random variables $\{X_k, k = 1, 2, \ldots\}$ END if there exists a constant $M > 0$ such that

$$P(X_1 \leq x_1, \ldots, X_n \leq x_n) \leq M \prod_{i=1}^{n} P(X_i \leq x_i), \tag{1.6}$$

$$P(X_1 > x_1, \ldots, X_n > x_n) \leq M \prod_{i=1}^{n} P(X_i > x_i) \tag{1.7}$$

hold for each $n = 1, 2, \ldots$, and all $x_1, \ldots, x_n$.

Recall that $\{X_k, k = 1, 2, \ldots\}$ are called ND if both (1.6) and (1.7) hold with $M = 1$; they are called positively dependent (PD) if inequalities (1.6) and (1.7) hold both in the reverse direction with $M = 1$. According to Liu's [12] interpretation, an ND sequence must be an END sequence. On the other hand, for some PD sequences, it is possible to find a corresponding positive constant M such that (1.6) and (1.7) hold. Therefore, the END structure is substantially more comprehensive than the ND structure in that it can reflect not only a negative dependence structure but also a positive one to some extent.

Under the assumption that $\{X_k, k = 1, 2, \ldots\}$ is an ND sequence, Liu [6] and Chen and Zhang [7] investigated precise large deviations of random sums $S_{N(t)}$ of nonnegative random variables and real-valued random variables, respectively. For a slightly more general dependence of END structure, Chen et al. [11] obtained precise large deviations of random sums $S_{N(t)}$ of nonnegative random variables and random sums $S_{N_{t,c}} = \sum_{k=1}^{N(t)} (X_k + c)$ of real-valued random variables with mean zero centered by a constant c. Up to now, to the best of our knowledge, little is known about extended precise large deviations of random sums in the presence of END structure and heavy tails. Our obtained results extend those of Chen and Zhang [7] and Chen et al. [11].

The rest of this paper is organized as follows. Section 2 gives some preliminaries. Precise large deviations of random sums in the presence of END real-valued random variables are presented in Section 3. In Section 4 we consider precise large deviations of the prospective-
loss process of a quasirenewal model as an application of our main results.

2. Preliminaries

Throughout this paper, by convention, we denote $S_n = \sum_{k=1}^{n} X_k$. For two positive infinitesimals $f(\cdot)$ and $g(\cdot)$ satisfying

$$a \le \liminf \frac{f(\cdot)}{g(\cdot)} \le \limsup \frac{f(\cdot)}{g(\cdot)} \le b, \tag{2.1}$$

we write $f(\cdot) = O(g(\cdot))$ if $b < \infty$; $f(\cdot) = o(g(\cdot))$ if $b = 0$; $f(\cdot) \lesssim g(\cdot)$ if $b = 1$; $f(\cdot) \gtrsim g(\cdot)$ if $a = 1$; $f(\cdot) \sim g(\cdot)$ if both and write $f(\cdot) \asymp g(\cdot)$ if $0 < \liminf(f(\cdot)/g(\cdot)) \le \limsup(f(\cdot)/g(\cdot)) < \infty$. For theoretical and practical reasons, we usually equip them with certain uniformity. For instance, for two positive bivariate functions $f(t, x)$ and $g(t, x)$, we say that $f(t, x) \sim g(t, x)$ holds as $t \to \infty$ uniformly for all $x \in \mathfrak{D}(t) \neq \phi$ in the sense that

$$\lim_{t\to\infty} \sup_{x\in\mathfrak{D}(t)} \left| \frac{f(t, x)}{g(t, x)} - 1 \right| = 0. \tag{2.2}$$

For a distribution, set

$$J_F^+ := \inf\left\{ -\frac{\log \overline{F}_*(y)}{\log y}, y > 1 \right\}, \tag{2.3}$$

where $\overline{F}_*(y) = \liminf_{x\to\infty} (\overline{F}(xy))/\overline{F}(x))$. In the terminology of Tang and Tsitsiashvili [13],

J_F^+ is called the upper Matuszewska index of F. Clearly, if $F \in \mathcal{C}$, then $J_F^+ < \infty$. It holds for every $p > J_F^+$ that

$$x^{-p} = o\left(\overline{F}(x)\right), \quad x \longrightarrow \infty. \tag{2.4}$$

Moreover, $J_F^+ \geq 1$ if the distribution $F(x) = F(x)1_{\{x \geq 0\}}$ has a finite mean. See [11].

Next we will need some lemmas in the proof of our theorems. From Lemma 2.3 of Chen et al. [11] with a slight modification, we have the following lemma.

Lemma 2.1. *Let $\{X_k, k = 1, 2, \ldots\}$ be a sequence of real-valued END random variables with common distribution function F. If $0 < \mu_+ = EX_1 1_{\{X_1 \geq 0\}} < \infty$, then, for every fixed $\nu > 0$ and some $C = C(\nu) > 0$, the inequality*

$$P(S_n > x) \leq n\overline{F}\left(\frac{x}{\nu}\right) + C\left(\frac{n}{x}\right)^{\nu} \tag{2.5}$$

holds for all $n = 1, 2, \ldots,$ and $x > 0$.

Lemma 2.2 below is a reformulation of Theorem 2.1 of [12], which is one of the main results in [12].

Lemma 2.2. *Let $\{X_k, k = 1, 2, \ldots\}$ be a sequence of real-valued END random variables with common distribution function $F(x) \in \mathcal{C}$ and finite mean μ, satisfying*

$$F(-x) = o\left(\overline{F}(x)\right) \quad \text{as } x \longrightarrow \infty, \qquad E|X_1|^r 1_{\{X_1 \leq 0\}} < \infty \quad \text{for some } r > 1. \tag{2.6}$$

Then, for any fixed $\gamma > 0$, relation

$$P(S_n - n\mu > x) \sim n\overline{F}(x), \quad \text{as } n \longrightarrow \infty, \tag{2.7}$$

holds uniformly for all $x \geq \gamma n$.

3. Main Results and Their Proofs

In this sequel, all limiting relationships, unless otherwise stated, are according to $t \to \infty$. To state the main results, we need the following two basic assumptions on the counting process $\{N(t), t \geq 0\}$.

Assumption 3.1. For any $\delta > 0$ and some $p > J_F^+$,

$$EN^p(t)1_{(N(t) > (1+\delta)\lambda(t))} = O(\lambda(t)). \tag{3.1}$$

Assumption 3.2. The relation

$$P(N(t) \leq (1-\delta)\lambda(t)) = o\left(\lambda(t)\overline{F}(\lambda(t))\right) \tag{3.2}$$

holds for all $0 < \delta < 1$.

Remark 3.3. One can easily see that Assumption 3.1 or Assumption 3.2 implies that

$$\frac{N(t)}{\lambda(t)} \xrightarrow{P} 1. \tag{3.3}$$

See [5, 11].

Theorem 3.4. *Let $\{X_k, k = 1,2,\ldots\}$ be a sequence of END real-valued random variables with common distribution function $F(x) \in \mathcal{C}$ having finite mean $\mu \geq 0$ and satisfying* (2.6)*, and let $\{N(t), t \geq 0\}$ be a nonnegative integer-valued counting process independent of $\{X_k, k = 1,2,\ldots\}$ satisfying Assumption 3.1. Let m be a real number; then, for any $\gamma > (\mu - m) \vee 0$, the relation* (1.5) *holds uniformly for $x \geq \gamma\lambda(t)$.*

Theorem 3.5. *Let $\{X_k, k = 1,2,\ldots\}$ be a sequence of END real-valued random variables with common distribution function $F(x) \in \mathcal{C}$ having finite mean $\mu < 0$ and satisfying* (2.6)*, and let $\{N(t), t \geq 0\}$ be a nonnegative integer valued counting process independent of $\{X_k, k = 1,2,\ldots\}$.*

(i) *Assume that $\{N(t), t \geq 0\}$ satisfies Assumption 3.1 and m is a real number (regardless of $m \geq 0$ or $m < 0$), then for any fixed $\gamma > -m \vee 0$, the relation* (1.5) *holds uniformly for $x \geq \gamma\lambda(t)$.*

(ii) *Assume that $\{N(t), t \geq 0\}$ satisfies Assumption 3.2 and m is a negative real number; then, for any fixed $\gamma \in (\mu - m \vee 0, -m]$, the relation* (1.5) *holds uniformly for $x \geq \gamma\lambda(t)$.*

Remark 3.6. One can easily see that Theorem 3.4 extends Theorem 3.1 of [11] with m replaced by μ. On the other hand, replacing X_k with $X_k - \mu + c$, setting $m = 0$, and noticing that $E(X_k - \mu + c) = c$, (3.4) yields Theorem 4.1(i) of [11].

Remark 3.7. Under the conditions of Theorem 3.5, choosing $m = \mu$, one can easily see that the relation (1.4) holds uniformly over the x-region $x \geq \gamma\lambda(t)$ for arbitrarily fixed $\gamma > 0$. Hence, Theorem 3.5 extends Theorem 1.2 of [7].

Proof of Theorem 3.4. We use the commonly used method with some modifications to prove Theorem 3.4. The starting point is the following standard decomposition:

$$\begin{aligned}
P\left(S_{N(t)} - m\lambda(t) > x\right) &= \sum_{n=1}^{\infty} P(S_n - m\lambda(t) > x) P(N(t) = n) \\
&= \left(\sum_{n<(1-\delta)\lambda(t)} + \sum_{(1-\delta)\lambda(t)\leq n\leq(1+\delta)\lambda(t)} + \sum_{n>(1+\delta)\lambda(t)} \right) \\
&\quad \times P(S_n - m\lambda(t) > x) P(N(t) = n) \\
&:= I_1(x,t) + I_2(x,t) + I_3(x,t),
\end{aligned} \tag{3.4}$$

where we choose $0 < \delta < 1$ such that $(\gamma + m)/(1 + \delta) - \mu > 0$.

We first deal with $I_1(x,t)$. Note that $x + m\lambda(t) - n\mu \ge ((\gamma+m)/(1-\delta) - \mu)n$. Thus, as $t \to \infty$ and uniformly for $x \ge \gamma\lambda(t)$, it follows from Lemma 2.2 that

$$\begin{aligned}
I_1(x,t) &\sim \sum_{n<(1-\delta)\lambda(t)} n\overline{F}(x + m\lambda(t) - n\mu)P(N(t)=n) \\
&\le \sum_{n<(1-\delta)\lambda(t)} \overline{F}(x + m\lambda(t) - (1-\delta)\mu\lambda(t))nP(N(t)=n) \\
&\le (1-\delta)\lambda(t)\overline{F}(x + (m-\mu)\lambda(t))P(N(t) < (1-\delta)\lambda(t)) \\
&= o\Big(\lambda(t)\overline{F}(x + (m-\mu)\lambda(t))\Big).
\end{aligned} \tag{3.5}$$

Next, for $I_2(x,t)$, noticing that $x+m\lambda(t)-n\mu \ge ((\gamma+m)/(1+\delta)-\mu)n$, as $t \to \infty$ and uniformly for $x \ge \gamma\lambda(t)$, Lemma 2.2 yields that

$$\begin{aligned}
I_2(x,t) &\sim \sum_{(1-\delta)\lambda(t)\le n\le(1+\delta)\lambda(t)} n\overline{F}(x + m\lambda(t) - n\mu)P(N(t)=n) \\
&\le (1+\delta)\lambda(t)\overline{F}(x + m\lambda(t) - \mu(1+\delta)\lambda(t))P\left(\left|\frac{N(t)}{\lambda(t)} - 1\right| < \delta\right) \\
&\sim (1+\delta)\lambda(t)\overline{F}(x + (m-\mu)\lambda(t) - \delta\mu\lambda(t)) \\
&\le (1+\delta)\lambda(t)\overline{F}\left(\left(1 - \frac{\delta\mu}{\gamma+m-\mu}\right)(x + (m-\mu)\lambda(t))\right).
\end{aligned} \tag{3.6}$$

On the other hand,

$$\begin{aligned}
I_2(x,t) &\sim \sum_{(1-\delta)\lambda(t)\le n\le(1+\delta)\lambda(t)} n\overline{F}(x + m\lambda(t) - n\mu)P(N(t)=n) \\
&\le (1-\delta)\lambda(t)\overline{F}(x + m\lambda(t) - \mu(1-\delta)\lambda(t))P\left(\left|\frac{N(t)}{\lambda(t)} - 1\right| < \delta\right) \\
&\sim (1-\delta)\lambda(t)\overline{F}(x + (m-\mu)\lambda(t) + \delta\mu\lambda(t)) \\
&\le (1-\delta)\lambda(t)\overline{F}\left(\left(1 + \frac{\delta\mu}{\gamma+m-\mu}\right)(x + (m-\mu)\lambda(t))\right).
\end{aligned} \tag{3.7}$$

Finally, to deal with $I_3(x,t)$, we formulate the remaining proof into two parts according to $m \ge 0$ and $m < 0$. In the case of $m \ge 0$, setting $v = p$ in Lemma 2.1 with $p > J_F^+ \ge 1$, for sufficiently large t and $x \ge \gamma\lambda(t)$, there exists some constant $C_1 > 0$ such that

$$\begin{aligned}
I_3(x,t) &\le \sum_{n>(1+\delta)\lambda(t)} P(S_n > x)P(N(t)=n) \\
&\le \sum_{n>(1+\delta)\lambda(t)} \left(n\overline{F}\left(\frac{x}{p}\right) + C_1\left(\frac{n}{x}\right)^p\right)P(N(t)=n).
\end{aligned} \tag{3.8}$$

In the case of $m < 0$, note that $1 + m/\gamma > 0$ since $\gamma > \mu - m \geq -m > 0$. Similar to (3.8), for sufficiently large t and $x \geq \gamma\lambda(t)$, there exists some constant $C_2 > 0$ such that

$$\begin{aligned} I_3(x,t) &\leq \sum_{n>(1+\delta)\lambda(t)} P(S_n > (1+m/\gamma)x)P(N(t)=n) \\ &\leq \sum_{n>(1+\delta)\lambda(t)} \left(n\overline{F}\left(\frac{(1+m/\gamma)x}{p}\right) + C_2\left(\frac{n}{(1+m/\gamma)x}\right)^p \right) P(N(t)=n). \end{aligned} \tag{3.9}$$

As a result, by (2.4), as $t \to \infty$ and uniformly for $x \geq \gamma\lambda(t)$, both (3.8) and (3.9) yield that

$$\begin{aligned} I_3(x,t) &\lesssim \overline{F}(x)EN(t)1_{\{N(t)>(1+\delta)\lambda(t)\}} + x^{-p}EN^p(t)1_{\{N(t)>(1+\delta)\lambda(t)\}} \\ &\leq o(1)\overline{F}(x)EN^p(t)1_{\{N(t)>(1+\delta)\lambda(t)\}} \\ &\leq o(1)\lambda(t)\overline{F}(x+(m-\mu)\lambda(t)), \end{aligned} \tag{3.10}$$

where in the last step, we used

$$\overline{F}(x+(m-\mu)\lambda(t)) \geq \overline{F}\left(\left(1+\frac{|m-\mu|}{\gamma}\right)x\right) \gtrsim \overline{F}(x). \tag{3.11}$$

Substituting (3.5), (3.6), (3.7), and (3.10) into (3.4), one can see that relation (1.5) holds by the condition $F \in \mathcal{C}$ and the arbitrariness of δ. □

Proof of Theorem 3.5. (i) We also start with the decomposition (3.4).

For $I_1(x,t)$ and $I_2(x,t)$, note that $x + m\lambda(t) - n\mu \geq ((\gamma+m)/(1+\delta) - \mu)n$ since $n \leq (1+\delta)\lambda(t)$. Hence, mimicking the proof of Theorem 3.4, we obtain, as $t \to \infty$ and uniformly for $x \geq \gamma\lambda(t)$,

$$\begin{aligned} I_1(x,t) &\sim \sum_{n<(1-\delta)\lambda(t)} n\overline{F}(x+m\lambda(t)-n\mu)P(N(t)=n) \\ &\leq \overline{F}(x+m\lambda(t)) \sum_{n<(1-\delta)\lambda(t)} nP(N(t)=n) \\ &= o\Big(\lambda(t)\overline{F}(x+(m-\mu)\lambda(t))\Big), \end{aligned} \tag{3.12}$$

where, in the last step, we used the relation

$$\overline{F}(x+m\lambda(t)) \leq \overline{F}\left(\left(1+\frac{m}{\gamma}\right)x\right) \lesssim \overline{F}(x), \tag{3.13}$$

$$\overline{F}(x+(m-\mu)\lambda(t)) \geq \overline{F}\left(\left(1+\frac{|m-\mu|}{\gamma}\right)x\right) \gtrsim \overline{F}(x). \tag{3.14}$$

Again, as $t \to \infty$ and uniformly for $x \geq \gamma\lambda(t)$,

$$\begin{aligned}
I_2(x,t) &\sim \sum_{(1-\delta)\lambda(t)\leq n\leq(1+\delta)\lambda(t)} n\overline{F}(x+m\lambda(t)-n\mu)P(N(t)=n)\\
&\leq (1+\delta)\lambda(t)\overline{F}(x+m\lambda(t)-\mu(1-\delta)\lambda(t))P\left(\left|\frac{N(t)}{\lambda(t)}-1\right|<\delta\right)\\
&\lesssim (1+\delta)\lambda(t)\overline{F}\left(\left(1+\frac{\delta\mu}{\gamma+m-\mu}\right)(x+(m-\mu)\lambda(t))\right),\\
I_2(x,t) &\gtrsim (1-\delta)\lambda(t)\overline{F}\left(\left(1-\frac{\delta\mu}{\gamma+m-\mu}\right)(x+(m-\mu)\lambda(t))\right).
\end{aligned} \tag{3.15}$$

Finally, in $I_3(x,t)$, setting $v = p$ in Lemma 2.1 with $p > J_F^+ \geq 1$, by (2.4) and Assumption 3.1, as $t \to \infty$ and uniformly for $x \geq \gamma\lambda(t)$, there exists a constant $C_2 > 0$ such that

$$\begin{aligned}
I_3(x,t) &\leq \sum_{n>(1+\delta)\lambda(t)} P(S_n > (1+m/\gamma)x)P(N(t)=n)\\
&\leq \sum_{n>(1+\delta)\lambda(t)} \left(n\overline{F}\left(\frac{(1+m/\gamma)x}{p}\right)+C_2\left(\frac{n}{(1+m/\gamma)x}\right)^p\right)P(N(t)=n)\\
&\asymp \overline{F}(x)EN(t)1_{\{N(t)>(1+\delta)\lambda(t)\}}+x^{-p}EN^p(t)1_{\{N(t)>(1+\delta)\lambda(t)\}}\\
&\leq o(1)\lambda(t)\overline{F}(x+(m-\mu)\lambda(t)),
\end{aligned} \tag{3.16}$$

where in the last step we also used (3.14). Combining (3.12), (3.15), and (3.16), relation (1.5) holds by the condition $F \in \mathcal{C}$ and the arbitrariness of δ.

(ii) We also start with the representation (3.4) in which we choose $0 < \delta < 1$ such that $(\gamma+m)/(1-\delta)-\mu > 0$.

To deal with $I_1(x,t)$, arbitrarily choosing $\gamma_1 > -m$, we split the x-region into two disjoint regions as

$$[\gamma\lambda(t),\infty) = [\gamma_1\lambda(t),\infty) \cup [\gamma\lambda(t),\gamma_1\lambda(t)). \tag{3.17}$$

For the first x-region $x \geq \gamma_1\lambda(t)$, noticing that $x+m\lambda(t)-n\mu > |\mu|n$, by Lemma 2.2, it holds uniformly for all $x \geq \gamma_1\lambda(t)$ that

$$\begin{aligned}
I_1(x,t) &\sim \sum_{n<(1-\delta)\lambda(t)} n\overline{F}(x+m\lambda(t)-n\mu)P(N(t)=n)\\
&\leq (1-\delta)\lambda(t)\overline{F}(x+m\lambda(t))P(N(t)\leq(1-\delta)\lambda(t))\\
&\leq (1-\delta)\lambda(t)\overline{F}(x)P(N(t)\leq(1-\delta)\lambda(t))
\end{aligned}$$

$$\begin{aligned}
&\leq (1-\delta)\lambda(t)\overline{F}(x(m-\mu)\lambda(t))P(N(t)\leq(1-\delta)\lambda(t)) \\
&= o\Big(\lambda(t)\overline{F}(x+(m-\mu)\lambda(t))\Big),
\end{aligned} \tag{3.18}$$

where the second step and the before the last before the step can be verified, respectively, as

$$\begin{aligned}
\overline{F}(x+m\lambda(t)) &\leq \overline{F}\left(\left(1+\frac{m}{\gamma_1}\right)x\right) \lesssim \overline{F}(x), \\
\overline{F}(x+(m-\mu)\lambda(t)) &\geq \overline{F}\left(\left(1+\frac{|m-\mu|}{\gamma_1}\right)x\right) \gtrsim \overline{F}(x).
\end{aligned} \tag{3.19}$$

For the second x-region $\gamma\lambda(t)\leq x<\gamma_1\lambda(t)$, note that

$$\overline{F}(x+(m-\mu)\lambda(t)) \geq \overline{F}\left(\left(1+\frac{|m-\mu|}{\gamma}\right)x\right) \asymp \overline{F}(x) \geq \overline{F}(\gamma_1\lambda(t)) \asymp \overline{F}(\lambda(t)). \tag{3.20}$$

Hence, by Assumption 3.2, we still obtain

$$I_1(x,t) \leq P(N(t)\leq(1-\delta)\lambda(t)) = o(1)\lambda(t)\overline{F}(\lambda(t)) \lesssim o\Big(\lambda(t)\overline{F}(x+(m-\mu)\lambda(t))\Big) \tag{3.21}$$

uniformly for all $\gamma\lambda(t)\leq x<\gamma_1\lambda(t)$. As a result, the relation

$$I_1(x,t) = o\Big(\lambda(t)\overline{F}(x+(m-\mu)\lambda(t))\Big) \tag{3.22}$$

holds uniformly for all $x\geq\gamma\lambda(t)$.

For $I_2(x,t)$, since $\gamma\in(\mu-m\vee 0,-m]$, it holds that

$$x+m\lambda(t)-n\mu \geq (\gamma+m)\lambda(t)-\mu n \geq \left(\frac{\gamma+m}{1-\delta}-\mu\right)n. \tag{3.23}$$

It follows from Lemma 2.2 that for all $x\geq\gamma\lambda(t)$

$$\begin{aligned}
I_2(x,t) &\sim \sum_{(1-\delta)\lambda(t)\leq n\leq(1+\delta)\lambda(t)} n\overline{F}(x+m\lambda(t)-n\mu)P(N(t)=n) \\
&\leq (1+\delta)\lambda(t)\overline{F}(x+m\lambda(t)-\mu(1-\delta)\lambda(t))P\left(\left|\frac{N(t)}{\lambda(t)}-1\right|<\delta\right) \\
&\sim (1+\delta)\lambda(t)\overline{F}(x+(m-\mu)\lambda(t)+\delta\mu\lambda(t)) \\
&\leq (1+\delta)\lambda(t)\overline{F}\left(\left(1+\frac{\delta\mu}{\gamma+m-\mu}\right)(x+(m-\mu)\lambda(t))\right).
\end{aligned} \tag{3.24}$$

Symmetrically,

$$I_2(x,t) \gtrsim (1-\delta)\lambda(t)\overline{F}\left(\left(1-\frac{\delta\mu}{\gamma+m-\mu}\right)(x+(m-\mu)\lambda(t))\right). \tag{3.25}$$

Finally, in $I_3(x,t)$, note that $x+m\lambda(t)-n\mu \ge (\gamma+m)\lambda(t)-\mu n \ge ((\gamma+m)/(1+\delta)-\mu)n$. Therefore, Lemma 2.2 implies, as $t \to \infty$ and uniformly for $x \ge \gamma\lambda(t)$, that

$$\begin{aligned} I_3(x,t) &\sim \sum_{n>(1+\delta)\lambda(t)} n\overline{F}(x+m\lambda(t)-n\mu)P(N(t)=n) \\ &\le \overline{F}(x+m\lambda(t)-\mu(1+\delta)\lambda(t)) \sum_{n>(1+\delta)\lambda(t)} nP(N(t)=n) \\ &\le \overline{F}(x+(m-\mu)\lambda(t))EN(t)1_{(N(t)>(1+\delta)\lambda(t))} \\ &= o\left(\lambda(t)\overline{F}(x+(m-\mu)\lambda(t))\right). \end{aligned} \tag{3.26}$$

Substituting (3.22), (3.24), (3.25), and (3.26) into (3.4) and letting $\delta \downarrow 0$, the proof of (ii) is now completed. □

4. Precise Large Deviations of the Prospective-Loss Process of a Quasirenewal Model

In this section we consider precise large deviations of the prospective-loss process of a quasi-renewal model, where the quasi-renewal model was first introduced by Chen et al. [11]. It is a nonstandard renewal model in which innovations, modeled as real-valued random variables, are END and identically distributed, while their interarrival times are also END, identically distributed, and independent of the innovations.

Let $\{X_k, k=1,2,\ldots\}$ be a sequence of END real-valued random variables with common distribution function $F(x) \in \mathcal{C}$ and finite mean μ, satisfying (2.7). Let $\{N(t), t \ge 0\}$ be a quasi-renewal process defined by

$$N(t) = \#\left\{n=1,2,\ldots : \sum_{k=1}^{n} Y_k \le t\right\}, \quad t \ge 0, \tag{4.1}$$

where $\{Y_k, k=1,2,\ldots\}$, independent of $\{X_k, k=1,2,\ldots\}$, form a sequence of END nonnegative random variables with common distribution G nondegenerate at zero and finite mean $1/\lambda > 0$. By Theorem 4.2 of [11], as $t \to \infty$,

$$\frac{N(t)}{\lambda t} \longrightarrow 1, \quad \text{almost surely}. \tag{4.2}$$

By Chen et al. [11], for any $\delta > 0$, $p > 0$, and some $b > 1$,

$$\begin{aligned} EN^p(t)1_{(N(t)>(1+\delta)\lambda t)} &= \sum_{n>(1+\delta)\lambda t} n^p P(N(t) = n) \\ &= o(1) \sum_{n>(1+\delta)\lambda t} b^n P(N(t) \geq n) = o(1), \end{aligned} \tag{4.3}$$

where in the last step we use (4.10) in [11]. Thus, one can easily see that $\{N(t), t \geq 0\}$ satisfies Assumption 3.1. Assume that $\{N(t), t \geq 0\}$ also satisfies Assumption 3.2. Let $\beta > 0$ be the safety loading coefficient. Replacing X_k with $X_k - (1+\beta)\mu$ and setting $m = 0$ in Theorems 3.4 and 3.5, then, for any fixed $\gamma > 0$, the relation

$$P\left(\sum_{k=1}^{N(t)} \left(X_k - (1+\beta)\mu\right) > x\right) \sim \lambda t \overline{F}(x + \beta\mu\lambda t) \tag{4.4}$$

holds uniformly for $x \geq \gamma\lambda t$.

Acknowledgments

The authors would like to thank an anonymous referee for his/her constructive and insightful comments and suggestions that greatly improved the paper. This work was partially supported by NSFC Grant 11071076, the Talents Youth Fund of Anhui Province Universities (2011SQRL012ZD), the Project Sponsored by the Doctoral Scientific Research Foundation of Anhui University, and the 211 Project of Anhui University (2009QN020B).

References

[1] D. B. H. Cline and T. H. Samorodnitsky, “Subexponentiality of the product of independent random variables,” *Stochastic Processes and Their Applications*, vol. 49, no. 1, pp. 75–98, 1994.

[2] K. W. Ng, Q. H. Tang, J.-A. Yan, and H. L. Yang, “Precise large deviations for sums of random variables with consistently varying tails,” *Journal of Applied Probability*, vol. 41, no. 1, pp. 93–107, 2004.

[3] S. J. Wang and W. S. Wang, “Precise large deviations for sums of random variables with consistently varying tails in multi-risk models,” *Journal of Applied Probability*, vol. 44, no. 4, pp. 889–900, 2007.

[4] C. Klüppelberg and T. Mikosch, “Large deviations of heavy-tailed random sums with applications in insurance and finance,” *Journal of Applied Probability*, vol. 34, no. 2, pp. 293–308, 1997.

[5] K. W. Ng, Q. H. Tang, J.-A. Yan, and H. L. Yang, “Precise large deviations for the prospective-loss process,” *Journal of Applied Probability*, vol. 40, no. 2, pp. 391–400, 2003.

[6] Y. Liu, “Precise large deviations for negatively associated random variables with consistently varying tails,” *Statistics & Probability Letters*, vol. 77, no. 2, pp. 181–189, 2007.

[7] Y. Chen and W. P. Zhang, “Large deviations for random sums of negatively dependent random variables with consistently varying tails,” *Statistics & Probability Letters*, vol. 77, no. 5, pp. 530–538, 2007.

[8] J. Lin, “The general principle for negatively associated random variables with consistently varying tails,” *Statistics & Probability Letters*, vol. 78, no. 6, pp. 181–189, 2008.

[9] A. Baltrūnas, R. Leipus, and J. Šiaulys, “Precise large deviation results for the total claim amount under subexponential claim sizes,” *Statistics & Probability Letters*, vol. 78, no. 10, pp. 1206–1214, 2008.

[10] X. Shen and Z. Lin, “Precise large deviations for randomly weighted sums of negatively dependent random variables with consistently varying tails,” *Statistics & Probability Letters*, vol. 78, no. 18, pp. 3222–3229, 2008.

[11] Y. Q. Chen, K. C. Yuen, and K. W. Ng, "Precise large deviations of random sums in the presence of negatively dependence and consistent variation," *Methodology and Computing in Applied Probability*, vol. 13, no. 4, pp. 821–833, 2011.

[12] L. Liu, "Precise large deviations for dependent random variables with heavy tails," *Statistics & Probability Letters*, vol. 79, no. 9, pp. 1290–1298, 2009.

[13] Q. H. Tang and G. Tsitsiashvili, "Precise estimates for the ruin probability in finite horizon in a discrete-time model with heavy-tailed insurance and financial risks," *Stochastic Processes and Their Applications*, vol. 108, no. 2, pp. 299–325, 2003.