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Research Article

Some Results on Equivalence Groups

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The comparison of two common types of equivalence groups of differential equations is discussed, and it is shown that one type can be identified with a subgroup of the other type, and a case where the two groups are isomorphic is exhibited. A result on the determination of the finite transformations of the infinitesimal generator of the larger group, which is useful for the determination of the invariant functions of the differential equation, is also given. In addition, the Levidecomposition of the Lie algebra associated with the larger group is found; the Levi factor of which is shown to be equal, up to a constant factor, to the Lie algebra associated with the smaller group.

1. Introduction

An invertible point transformation that maps every element in a family $\mathcal{F}$ of differential equations of a specified form into the same family is commonly referred to as an equivalence transformation of the equation [1–3]. Elements of the family $\mathcal{F}$ are generally labeled by a set of arbitrary functions, and the set of all equivalence transformations forms, in general, an infinite dimensional Lie group called the equivalence group of $\mathcal{F}$. One type of equivalence transformations usually considered [1, 4, 5] is that in which the arbitrary functions are also transformed. More specifically, if we denote by $A = (A_1, \dots, A_m)$ the arbitrary functions specifying the family element in $\mathcal{F}$, then for given independent variables $x = (x^1, \dots, x^p)$ and dependent variable y, this type of equivalence transformations takes the form

$$x = \varphi(z, w, B), \tag{1.1a}$$

$$y = \psi(z, w, B), \tag{1.1b}$$

$$A = \zeta(z, w, B), \tag{1.1c}$$

where $z = (z^1, \dots, z^p)$ is the new set of independent variables, $w = w(z)$ is the new dependent variable, and $B = (B_1, \dots, B_m)$ represents the new set of arbitrary functions. The original arbitrary functions A_i may be functions of x, y, and the derivatives of y up to a certain order, although quite often they arise naturally as functions of x alone, and for the equivalence transformations of the type (1.1a), (1.1b), and (1.1c), the corresponding equivalence group that we denote by G_S is simply the symmetry pseudogroup of the equation, in which the arbitrary functions are also considered as additional dependent variables.

The other type of equivalence transformations commonly considered [2, 6–8] involves only the ordinary variables of the equation, that is, the independent and the dependent variables, and thus with the notation already introduced, it consists of point transformations of the form

$$x = \varphi(z, w), \tag{1.2a}$$

$$y = \psi(z, w). \tag{1.2b}$$

If we let G denote the resulting equivalence group, then it follows from a result of Lie [9] that G induces another group of transformations G_c acting on the arbitrary functions of the equation. The invariants of the group G_c are what are referred to as the invariants of the family $\mathcal{F}$ of differential equations, and they play a crucial role in the classification and integrability of differential equations [1, 6, 10–14].

In the recent scientific literature, there has been a great deal of interest for finding infinitesimal methods for the determination of invariant functions of differential equations [2, 7, 15–17]. Some of these methods consist in finding the infinitesimal (generic) generator X of G_S, and then using it in one way or another [7, 18] to obtain the infinitesimal generator X^0 of G_c, which gives the determining equations for the invariant functions. Most of these methods remain computationally demanding and in some cases quite inefficient, perhaps just because the connection between the three groups G, G_c, and G_S does not seem to have been fully investigated.

We therefore present in this paper a comparison of the groups G and G_S and show in particular that G can be identified with a subgroup of G_S, and we exhibit a case where the two groups are isomorphic. We also show that the generator X of G_S admits a simple linear decomposition of the form $X = X^1 + X^2$, where X^1 is an operator uniquely associated with G, and we also give a very simple and systematic method for extracting X^1 from X. This decomposition also turns out to be intimately associated with the Lie algebraic structure of the equation, as we show that X^1 and X^2 each generate a Lie algebra, the two of which are closely related to the components of the Levi decomposition of the Lie algebra of G_S.

2. The Relationship between G and G_S

We will call type I the equivalence transformations of the form (1.2a) and (1.2b) and type II those of the form (1.1a), (1.1b), and (1.1c), whose equivalence groups we have denoted by G and G_S, respectively. When the coordinates system in which a vector field is expressed is clearly understood, it will be represented only by its components, so that a vector field

$$\omega = \xi \partial_x + \eta \partial_y + \phi \partial_A \tag{2.1}$$

will be represented simply by $\omega = \{\xi, \eta, \phi\}$. On the other hand, for a vector $a = (a^1, \dots, a^n)$ representing a subset of coordinates, the notation $f\partial_a$ will mean

$$f\partial_a = f^1\partial_{a^1} + \cdots + f^n\partial_{a^n}, \\ f = \left(f^1, \dots, f^n\right). \tag{2.2}$$

Hence, with the notation introduced in the previous section, we may represent the generator X of G_S as

$$X = \{\xi, \eta, \phi\} \equiv \xi\partial_x + \eta\partial_y + \phi\partial_A. \tag{2.3}$$

Let $V = \{\xi, \eta\}$ be the projection of this generator into the (x, y)-space, and let $V^0 = \{\xi^0, \eta^0\}$ be the infinitesimal generator of G. Elements of $\mathcal{F}$ may be thought of as differential equations of the form

$$\Delta\left(x, y_{(n)}; A_1, \dots, A_m\right) = 0, \tag{2.4}$$

where $y_{(n)}$ denotes y and all its derivatives up to the order n. We have the following result.

Theorem 2.1. (a) *The group G can be identified with a subgroup of G_S.*

(b) *The component functions ξ^0 and η^0 are particular values of the functions ξ and η, respectively.*

Proof. Suppose that the action of G_c induced by that of G on the arbitrary functions of the equation is given by the transformations

$$A_i = \gamma_i(z, w, B_1, \dots, B_m), \quad i = 1, \dots, m. \tag{2.5}$$

Then, since (1.2a) and (1.2b) leave the equation invariant except for the arbitrary functions, by also viewing the functions A_i as dependent variables, (1.2a) and (1.2b) together with (2.5) constitute a symmetry transformation of the equation. This is more easily seen if we consider the inverse transformations of (1.2a) and (1.2b) which may be put in the form

$$z = \widehat{\varphi}(x, y), \tag{2.6a}$$

$$w = \widehat{\psi}(x, y). \tag{2.6b}$$

If we now denote by

$$B_i = \widehat{\gamma}\left(x, y, A_1, \dots, A_m\right), \quad i = 1, \dots, m \tag{2.7}$$

the resulting arbitrary functions in the transformed equation, it follows that in terms of the new set of variables z, w, and B_i, any element of $\mathcal{F}$ is locally invariant under (2.6a), (2.6b), and (2.7), and this proves the first part of the theorem. The second part of the theorem is an immediate consequence of the first part, for we can associate with any element (φ, ψ) of G a triplet (φ, ψ, γ) in G_S, where γ is the action in (2.5) induced by (φ, ψ) on the arbitrary functions of the equation. The result thus follows by first recalling that G_S has generic generator $X = \{\xi, \eta, \phi\}$ and by considering the infinitesimal counterpart of the finite transformations (φ, ψ, γ), which must be of the form $\{\xi^0, \eta^0, \zeta^0\}$ for a certain function ζ^0. □

On the basis of Theorem 2.1, it is clear that one can obtain the generator $V^0 = \{\xi^0, \eta^0\}$ of G by imposing on the projection $V = \{\xi, \eta\}$ of X the set of minimum conditions Ω that reduces it to the infinitesimal generator of the equivalence group G of $\mathcal{F}$, so that $V^0 = V_{|\Omega}$. It was also observed (see [7]) that in case A is the function of x alone, if we let ϕ^0 denote the resulting value of ϕ when these minimum conditions are also imposed on $X = \{\xi, \eta, \phi\}$, then the generator X^0 of G_c can be obtained by setting $X^0 = \{\xi^0, \phi^0\}$. However, the problem that arises is that of finding the simplest and most systematic way of extracting $X^1 = X_{|\Omega} = \{\xi^0, \eta^0, \phi^0\}$ from $X = \{\xi, \eta, \phi\}$.

To begin with, we note that the coefficient ϕ^0 is an m-component vector that depends in general on $(p+1)+m$ variables, and finding its corresponding finite transformations by integrating the vector field $\{\xi^0, \eta^0, \phi^0\}$ can be a very complicated task. Fortunately, once the finite transformations of the generator V^0 of G which are easier to find are known, we can easily obtain those associated with ϕ^0 using the following result.

Lemma 2.2. *The finite transformations associated with the component ϕ^0 of $X^1 = \{\xi^0, \eta^0, \phi^0\}$ are precisely given by the action* (2.7) *of G_c induced by that of* (2.6a) *and* (2.6b).

Proof. Since $X^1 = X_{|\Omega}$, where Ω is the set of minimum conditions to be imposed on $V = \{\xi, \eta\}$ to reduce it into an infinitesimal generator $V^0 = \{\xi^0, \eta^0\}$ of G, it first follows that once the finite transformations (2.6a) and (2.6b) corresponding to V^0 are applied to the equation, the resulting equation is invariant, except for the expressions of the arbitrary functions which are now given by (2.7). Thus if (z, w, b) are the new variables generated by the symmetry operator X^1, where $b = (b_1, \dots, b_m)$, then the only way to have an invariant equation is to set

$$b_i = \widehat{\gamma}(x, y, A_1, \dots, A_m), \quad i = 1, \dots, m, \tag{2.8}$$

where $\widehat{\gamma}$ is the same function appearing in (2.7), and this readily proves the lemma. □

3. Case of the General Third Order Linear ODE

We will look more closely at the connection between the two operators X and X^1 by considering the case of the family of third-order linear ordinary differential equations (ODEs) of the form

$$y^{(3)} + a^1(x)y' + a^0(x)y = 0, \tag{3.1}$$

which is said to be in its normal reduced form. Here, the arbitrary functions A_i of the previous section are simply the coefficients a^j of the equation. This form of the equation is in no way restricted, for any general linear third order ODE can be transformed into (3.1) by a simple change of the dependent variable [8, 16]. If we consider the arbitrary functions a^j as additional dependent variables, then by applying known procedures for finding Lie point symmetries [19–21], the infinitesimal generator X of the symmetry group G_S in the coordinates system (x, y, a^1, a^0) is found to be of the form

$$X = \left\{ f, (k_1 + f')y + g, -2\left(a^1 f' + f^{(3)}\right), C_4 \right\}, \tag{3.2a}$$

where

$$C_4 = -\frac{1}{y}\left(a^0 g + a^1 g' + g^{(3)}\right) - \left(3a^0 f' + a^1 f'' + f^{(4)}\right), \tag{3.2b}$$

and where f and g are arbitrary functions of x. The projection of X in the (x, y)-space is therefore

$$V = \{f, (k_1 + f')y + g\}, \tag{3.3}$$

and a simple observation of this expression shows that due to the homogeneity of (3.1), (3.3) may represent an infinitesimal generator of the equivalence group G only if $g = 0$. A search for the one-parameter subgroup $\exp(tW)$, satisfying $\exp(tW)(x, y) = (z, w)$ and generated by the resulting reduced vector field $W = \{f, (k_1 + f')y\}$, readily gives

$$\begin{aligned} \dot{z} &= f(z), \\ \dot{w} &= \left(k_1 + f'(z)\right)w, \end{aligned} \tag{3.4}$$

where

$$\dot{z} = \frac{dz}{dt}, \qquad \dot{w} = \frac{dw}{dt}. \tag{3.5}$$

Integrating these last two equations while taking into account the initial conditions gives

$$J(z) = t + J(x), \tag{3.6a}$$

$$w = e^{k_1 t}\frac{f(z)}{f(x)}y, \tag{3.6b}$$

where

$$J(z) = \int \frac{dz}{f(z)}. \tag{3.6c}$$

Differentiating both sides of (3.6a) with respect to x shows that $dz/dx = f(z)/f(x)$. Thus, if we assume that z is explicitly given by

$$z = F_t(x) \equiv F(x), \tag{3.7}$$

for some function F, then this leads to

$$z = F(x), \tag{3.8a}$$

$$w = e^{k_1 t} F'(x) y, \tag{3.8b}$$

and we thus recover the well-known equivalence transformation [6, 8, 14] of (3.1). Therefore, the condition $g = 0$ is the necessary and sufficient condition for the vector V in (3.3) to represent the infinitesimal generator of G. In other words, the set Ω of necessary and sufficient conditions to be imposed on X to obtain X^1 is reduced in this case to setting $g = 0$. More explicitly, we have

$$V^0 = \{f, (k_1 + f')y\} \equiv \left\{\xi^0, \eta^0\right\}, \tag{3.9a}$$

$$\phi^0 = \left\{-2\left(a^1 f' + f'''\right), -\left(3a^0 f' + a^1 f'' + f^{(4)}\right)\right\}, \tag{3.9b}$$

$$X^1 = \left\{f, (k_1 + f')y, -2\left(a^1 f' + f'''\right), -\left(3a^0 f' + a^1 f'' + f^{(4)}\right)\right\}, \tag{3.9c}$$

where $X^1 = X_{|_{g=0}}$.

We would now like to derive some results on the algebraic structure of L_S, the Lie algebra of the group G_S related to (3.1), and its connection with that for the corresponding group G. Thus, for any generator X of G_S, set $X^2 = X - X^1$, where $X^1 = X_{|_{g=0}}$ is given by (3.9c), while X^2 takes the form

$$X^2 = \left\{0, g, 0, \frac{-1}{y}\left(a^0 g + a^1 g' + g^{(3)}\right)\right\}. \tag{3.10}$$

Since X^1 depends on f and k_1 while X^2 depends on g, we set

$$X^1(f, k_1) = \left\{f, (k_1 + f')y, -2\left(a^1 f' + f'''\right), -\left(3a^0 f' + a^1 f'' + f^{(4)}\right)\right\}, \tag{3.11a}$$

$$X^2(g) = \left\{0, g, 0, \frac{-1}{y}\left(a^0 g + a^1 g' + g^{(3)}\right)\right\}, \tag{3.11b}$$

for any arbitrary functions f and g and arbitrary constant k_1. Let L_0, L_1, and L_2 be the vector spaces generated by $X^1(f, 0)$, $X^1(f, k_1)$, and $X^2(g)$, respectively, where f, g, and k_1 are viewed as parameters. Let

$$L_{S,0} = L_0 \dotplus L_2 \tag{3.12}$$

be the subspace of the Lie algebra $L_S = L_1 \dotplus L_2$ of G_S. We note that $L_{S,0}$ is obtained from L_S simply by setting $k_1 = 0$ in the generator $X^1(f, k_1)$ of G_S, which according to (3.8b) amounts to ignoring the constant factor $\lambda = e^{k_1 t}$ in the transformation of the dependent variable under G. Moreover, we have $\dim L_{S,0} = \dim L_S - 1$, while L_S itself is infinite dimensional, in general.

Theorem 3.1. (a) *The vector spaces L_0, L_1, and L_2 are all Lie subalgebras of L_S.*

(b) *L_0 and L_2 are the components of the Levi decomposition of the Lie algebra $L_{S,0}$, that is,*

$$[L_0, L_2] \subset L_2, \tag{3.13}$$

and L_2 is a solvable ideal while L_0 is semisimple.

Proof. A computation of the commutation relations of the vector fields shows that

$$\left[X^1(f_1, k_1), X^1(f_2, k_2)\right] = X^1(-f_2 f_1' + f_1 f_2', 0), \tag{3.14a}$$

$$\left[X^2(g_1), X^2(g_2)\right] = 0, \tag{3.14b}$$

$$\left[X^1(f_1, k_1), X^2(g_1)\right] = X^2(f_1 g_1' - g_1(k_1 + f_1')), \tag{3.14c}$$

where the f_j and g_j are arbitrary functions, while the k_j are arbitrary constants. Consequently, it readily follows from (3.14a) and (3.14b) that L_1 and L_2 are Lie subalgebras of L_S, while setting $k_1 = k_2 = 0$ in (3.14a) shows that L_0 is also a Lie subalgebra, and this proves the first part of the theorem. Moreover, it follows from the commutation relations (3.14a), (3.14b), and (3.14c) that $[L_S, L_S] \subset L_{S,0}$, and hence that $L_{S,0}$ is an ideal of L_S, while (3.14b) and (3.14c) show that L_2 is an abelian ideal in L_S, and in particular in $L_{S,0}$. Thus, we are only left with showing that L_0 is a semisimple subalgebra of $L_{S,0}$. Clearly, $[L_0, L_0] \neq 0$, and if L_0 had a proper ideal A, then for a given nonzero operator $X^1(H, 0)$ in A, all operators $X^1(-fH' + f'H, 0)$ would be in A for all possible functions f. However, since for every function h of x the equation

$$-fH' + f'H = h \tag{3.15}$$

admits a solution in f, it follows that A would be equal to L_0. This contradiction shows that L_0 has no proper ideal and is therefore a simple subalgebra of $L_{S,0}$. □

Note that part (b) of Theorem 3.1 can also be interpreted as stating that up to a constant factor, X^1 and X^2 generate the components of the Levi decomposition of L_S. The theorem therefore shows that the decomposition $X = X^1 + X^2$ is not fortuitous, as it is intimately associated with the the Levi decomposition of L_S, and this decomposition is unique up to isomorphism for any given Lie algebra.

Although we have stated the results of this theorem only for the general linear third order equation (3.1) in its normal reduced form, these results can certainly be extended to the general linear ODE

$$y^{(n)} + a^{n-1} y^{(n-1)} + a^{n-2} y^{(n-2)} + \cdots + a^0 y = 0 \tag{3.16}$$

of an arbitrary order $n \geq 3$. We first note that if we write the infinitesimal generator X of the symmetry group G_S of this equation in the form

$$X = \{\xi, \eta, \phi\} \equiv \xi \partial_x + \eta \partial_y + \phi \partial_A, \tag{3.17}$$

where $A = \{a^{n-1}, a^{n-2}, \dots, a^0\}$ is the set of all arbitrary functions, then on account of the linearity of the equation, we must have

$$\eta = hy + g \tag{3.18}$$

for some arbitrary functions h and g. Now, let again $X^1 = \{\xi^0, \eta^0, \phi^0\}$ and X^2 be given by

$$\mathrm{X}^1 = \mathrm{X}_{|_{g=0}}, \qquad \mathrm{X}^2 = \mathrm{X} - \mathrm{X}^1, \tag{3.19}$$

and set $\mathrm{X}^0 = \{\xi^0, \phi^0\}$. We have shown in another recent paper [16] that X^0 thus obtained using $g = 0$ as the minimum set of conditions is the infinitesimal generator of the group G_c for $n = 3, 4, 5$. This should certainly also hold for the linear equation (3.16) of a general order, and we thus propose the following.

Conjecture 3.2. *For the general linear* ODE (3.16), $\mathrm{X}^0 = \{\xi, \phi\}_{|_{g=0}}$ *is the infinitesimal generator of* G_c, *where* $\mathrm{X} = \{\xi, \eta, \phi\}$ *is the generator of* G_S.

As already noted, it has been proved [7] that for any family $\mathcal{F}$ of (linear or nonlinear) differential equations of any order in which the arbitrary functions depend on the independent variables alone, if $X^1 = \{\xi^0, \eta^0, \phi^0\}$ is obtained by setting $X^1 = \mathrm{X}_{|\Omega}$ for some set Ω of minimum conditions that reduce $V = \{\xi, \eta\}$ into a generator of G, then $X^0 = \{\xi^0, \phi^0\}$ is the generator of G_c. However, the difficulty lies in finding the set Ω of minimum conditions, and we have proved that for (3.1), Ω is given by $\{g = 0\}$ and extended this as a conjecture for a general linear homogeneous ODE.

Moreover, calculations done for equations of low order up to five suggest that all subalgebras appearing in Theorem 3.1 can also be defined in a similar way for the general linear equation (3.16) and that all the results of the theorem also hold for this general equation.

We now wish to pay some attention to the converse of part (a) of Theorem 2.1 which states that for any given family $\mathcal{F}$ of differential equations, the group G can be identified with a subgroup of G_S. From the proof of that theorem, it appears that the symmetry group G_S is much larger in general, because there are symmetry transformations that do not arise from type I equivalence transformations. A simple example of such a symmetry is given by the term X^2 appearing in (3.10) of the symmetry generator of (3.1). Indeed, by construction, its projection $X^{2,0} = \{0, g\}$ in the (x, y)-space does not match any particular form of the generic infinitesimal generator $V^0 = \{f, (k_1 + f')y\}$ of G, where f is an arbitrary function and k_1 an arbitrary constant.

Nevertheless, although (3.1) gives an example in which the inclusion $G \subset G_S$ is strict, there are equations for which the two groups are isomorphic. Such an equation is given by the nonhomogeneous version of (3.1) which may be put in the form

$$y^{(3)} + a^1(x)y' + a^0(x)y + r(x) = 0, \tag{3.20}$$

where r is also an arbitrary function, in addition to a^1 and a^0. The linearity of this equation forces its equivalence transformations to be of the form

$$x = f(z), \qquad y = h(z)w + g(z), \tag{3.21}$$

and the latter change of variables transforms (3.20) into an equation of the form

$$w''' + B_2 w'' + B_1 w' + B_0 w + B_{-1} = 0, \tag{3.22}$$

where the B_j, for $j = -1, \ldots, 2$ are functions of z and

$$B_2 = 3\left(\frac{h'}{h} - \frac{f''}{f'}\right). \tag{3.23}$$

The required vanishing of B_2 shows that the necessary and sufficient condition for (3.21) to represent an equivalence transformation of (3.20) is to have $h = \lambda f'$ for some arbitrary constant λ. The equivalence transformations of (3.20) are therefore given by

$$x = f(z), \qquad y = \lambda f'(z) w + g(z). \tag{3.24}$$

On the other hand, the generator X of the symmetry group G_S of the nonhomogeneous equation (3.20) in the coordinates system (x, y, a^1, a^0, r) is found to be of the form

$$\mathrm{X} = \left\{J, (k_1 + J')y + P, -2\left(a^1 J' + J'''\right), C_3, \phi^4\right\}, \tag{3.25a}$$

where

$$C_3 = \frac{-1}{y}\left(a^0 P + \phi^4 + 2rJ' - rk_1 + a^1 P' + P^{(3)}\right) - \left(3a^0 J' + a^1 J'' + J^{(4)}\right), \tag{3.25b}$$

and where J and P are arbitrary functions of x and k_1 is an arbitrary constant, while ϕ^4 is an arbitrary function of x, y, a^1, a^0, and r. Thus, X has projection $V = \{J, (k_1 + J')y + P\}$ on (x, y)-space and this is exactly the infinitesimal transformation of (3.24). Consequently, the minimum set Ω of conditions to be imposed on V to reduce it into the infinitesimal generator $V^0 = \{\xi^0, \eta^0\}$ of G is void in this case, and hence

$$\mathrm{X} = \mathrm{X}^1 = \left\{\xi^0, \eta^0, \phi^0\right\}. \tag{3.26}$$

It thus follows from Lemma 2.2 that the finite transformations associated with X are given precisely by (3.24), together with the corresponding induced transformations of the arbitrary functions a^1, a^0, and r. Consequently, to each symmetry transformation X in G_S, there corresponds a unique equivalence transformation in G and vice versa. We have thus proved the following results.

Proposition 3.3. *For the nonhomogeneous equation* (3.20), *the groups G and G_S are isomorphic.*

This proposition should certainly also hold for the nonhomogeneous version of the general linear equation (3.16) of an arbitrary order n. In such cases, invariants of the differential equation are determined simply by searching the symmetry generator X of G_S, which must satisfy (3.26) and then solving the resulting system of linear first-order partial differential equations (PDEs) resulting from the determining equation of the form

$$X^{0,m} \cdot F = 0, \tag{3.27}$$

where $X^{0,m}$ is the generator $X^0 = \{\xi^0, \phi^0\}$ of G_c prolonged to the desired order m of the unknown invariants F.

4. Concluding Remarks

Because type I equivalence group G can be identified with a subgroup of type II equivalence group G_S, every function invariant under G_S must be invariant under G, and hence G has much more invariant functions than G_S, and functions invariant under G are naturally much easier to find than those invariant under G_S. If we consider for instance the third order linear equation (3.1), it is well known [6] that its first nontrivial invariant function is given by the third order differential invariant

$$\Psi = -\frac{4\left(9a^1\mu^2 + 7\mu'^2 - 6\mu\mu''\right)^3}{\mu^8}, \tag{4.1}$$

where $\mu(x) = -2a^0 + a^{1'}$, while at order four [16] it has two differential invariants,

$$\begin{aligned}
\Psi_1 &= \Psi, \\
\Psi_2 &= \frac{-1}{18\mu^4}\left(216a^{0^4} - 324a^{0^3}a^{1'} + 18a^{0^2}\left(9a^{1'2} + 2a^1\mu'\right) + 9\mu^2\mu^{(3)}\right) \\
&\quad + \frac{-1}{18\mu^4}\left(\mu'\left(28\mu'^2 + 9a^{1'}\left(a^1\ a^{1'} - 4\mu''\right)\right) - 9a^0\left(3a^{1'3} + 4a^1a^{1'}\mu' - 8\mu'\mu''\right)\right).
\end{aligned} \tag{4.2}$$

It can be verified on the other hand that G_S has no nontrivial differential invariants up to the order four.

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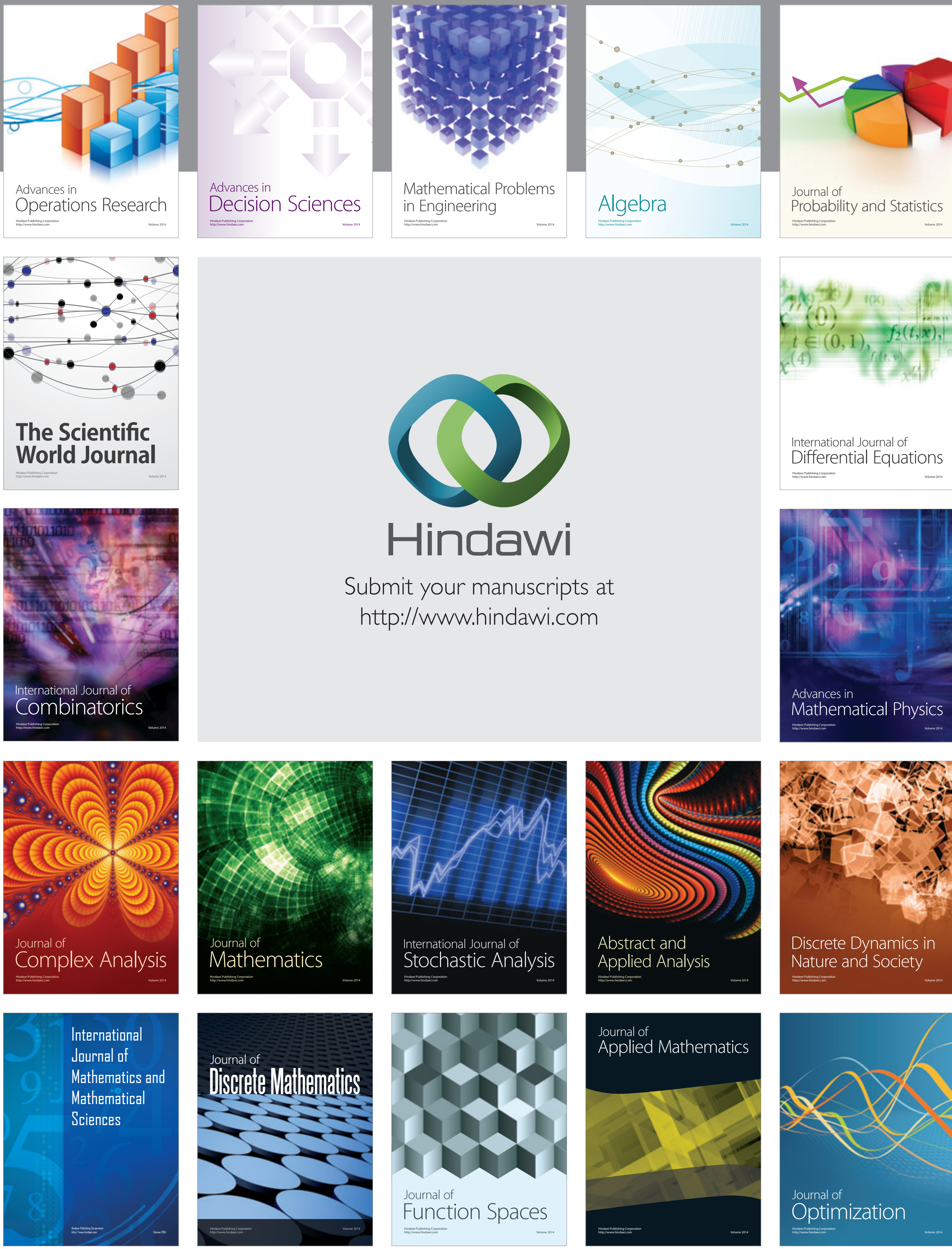
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