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*Research Article*

# A New Class of Meromorphic Functions Associated with Spirallike Functions

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We introduce a new class of meromorphic functions associated with spirallike functions. Such results as subordination property, integral representation, convolution property, and coefficient inequalities are proved.

## 1. Introduction

Let $\Sigma$ denote the class of functions $f$ of the form

$$f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k z^k, \tag{1.1}$$

which are *analytic* in the *punctured* open unit disk

$$\mathbb{U}^* := \{z : z \in \mathbb{C},\ 0 < |z| < 1\} =: \mathbb{U} \setminus \{0\}. \tag{1.2}$$

Let $\mathcal{P}$ denote the class of functions $p$ given by

$$p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k \quad (z \in \mathbb{U}), \tag{1.3}$$

which are analytic in $\mathbb{U}$ and satisfy the condition

$$\operatorname{Re}\bigl(p(z)\bigr) > 0 \quad (z \in \mathbb{U}). \tag{1.4}$$

Let $f, g \in \Sigma$, where $f$ is given by (1.1) and $g$ is defined by

$$g(z) = \frac{1}{z} + \sum_{k=0}^{\infty} b_k z^k, \tag{1.5}$$

then the Hadamard product (or convolution) $f * g$ is defined by

$$(f * g)(z) := \frac{1}{z} + \sum_{k=0}^{\infty} a_k b_k z^k =: (g * f)(z). \tag{1.6}$$

For two functions $f$ and $g$, analytic in $\mathbb{U}$, we say that the function $f$ is subordinate to $g$ in $\mathbb{U}$ and write

$$f(z) \prec g(z) \quad (z \in \mathbb{U}), \tag{1.7}$$

if there exists a Schwarz function $\omega$, which is analytic in $\mathbb{U}$ with

$$\omega(0) = 0, \qquad |\omega(z)| < 1 \quad (z \in \mathbb{U}), \tag{1.8}$$

such that

$$f(z) = g(\omega(z)) \quad (z \in \mathbb{U}). \tag{1.9}$$

Indeed, it is known that

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Longrightarrow f(0) = g(0), \quad f(\mathbb{U}) \subset g(\mathbb{U}). \tag{1.10}$$

Furthermore, if the function $g$ is univalent in $\mathbb{U}$, then we have the following equivalence:

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Longleftrightarrow f(0) = g(0), \quad f(\mathbb{U}) \subset g(\mathbb{U}). \tag{1.11}$$

A function $f \in \Sigma$ is said to be in the class $\mathcal{MS}^*(\beta)$ of *meromorphic starlike functions of order* $\beta$ if it satisfies the inequality

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) < -\beta \quad (z \in \mathbb{U};\ 0 \leqq \beta < 1). \tag{1.12}$$

For the real number $\beta$ $(0 < \beta < 1)$, we know that

$$\left|\frac{f(z)}{zf'(z)} + \frac{1}{2\beta}\right| < \frac{1}{2\beta} \Longleftrightarrow \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) < -\beta. \tag{1.13}$$

If the complex number $\alpha$ satisfies the condition

$$\left|\alpha - \frac{1}{2}\right| < \frac{1}{2}, \tag{1.14}$$

it can be easily verified that

$$\left|\frac{f(z)}{zf'(z)} + \frac{1}{2\alpha}\right| < \frac{1}{2|\alpha|} \Longleftrightarrow \operatorname{Re}\left(-\frac{1}{\alpha}\frac{zf'(z)}{f(z)}\right) > 1. \tag{1.15}$$

We now introduce and investigate the following class of meromorphic functions.

*Definition 1.1.* A function $f \in \Sigma$ is said to be in the class $\mathcal{MS}_\alpha$ if it satisfies the inequality

$$\operatorname{Re}\left(-\frac{1}{\alpha}\frac{zf'(z)}{f(z)}\right) > 1 \quad \left(z \in \mathbb{U}; \left|\alpha - \frac{1}{2}\right| < \frac{1}{2}\right). \tag{1.16}$$

*Remark 1.2.* For $0 < \alpha < 1$, the class $\mathcal{MS}_\alpha$ is the familiar class of meromorphic starlike functions of order $\alpha$.

*Remark 1.3.* If $\alpha = |\alpha|e^{i\psi}$ $(-\pi/2 < \psi < \pi/2)$, then the condition (1.16) is equivalent to

$$\operatorname{Re}\left(e^{-i\psi}\frac{zf'(z)}{f(z)}\right) < -|\alpha| \quad (z \in \mathbb{U}), \tag{1.17}$$

which implies that $f$ belongs to the class of meromorphic spirallike functions. Thus, the class of meromorphic spirallike functions is a special case of the class $\mathcal{MS}_\alpha$.

For some recent investigations on spirallike functions and related functions, see, for example, the earlier works [1–9] and the references cited in each of these earlier investigations.

*Remark 1.4.* The function

$$f(z) = z^{-1}(1-z)^{2\alpha[\operatorname{Re}(1/\alpha)-1]} \quad \left(z \in \mathbb{U}^*; \left|\alpha - \frac{1}{2}\right| < \frac{1}{2}\right) \tag{1.18}$$

belongs to the class $\mathcal{MS}_\alpha$.

It is clear that

$$\operatorname{Re}\left(\frac{1}{\alpha}\right) > 1 \quad \left(\left|\alpha - \frac{1}{2}\right| < \frac{1}{2}\right). \tag{1.19}$$

Then, for the function $f$ given by (1.18), we know that

$$\begin{aligned}\operatorname{Re}\left(-\frac{1}{\alpha}\frac{zf'(z)}{f(z)}\right) &= \operatorname{Re}\left(\frac{1}{\alpha}+2\left[\operatorname{Re}\left(\frac{1}{\alpha}\right)-1\right]\frac{z}{1-z}\right) \\ &> \operatorname{Re}\left(\frac{1}{\alpha}\right)-\operatorname{Re}\left(\frac{1}{\alpha}\right)+1=1,\end{aligned} \tag{1.20}$$

which implies that $f \in \mathcal{MS}_\alpha$.

In this paper, we aim at deriving the subordination property, integral representation, convolution property, and coefficient inequalities of the function class $\mathcal{MS}_\alpha$.

## 2. Preliminary Results

In order to derive our main results, we need the following lemmas.

**Lemma 2.1.** *Let $\lambda$ be a complex number. Suppose also that the sequence $\{A_k\}_{k=0}^{\infty}$ is defined by*

$$A_0 = 2\lambda, \qquad A_{k+1} = \frac{2\lambda}{k+2}\left(1+\sum_{l=0}^{k}A_l\right) \quad (k \in \mathbb{N}_0 := \mathbb{N}\cup\{0\}). \tag{2.1}$$

*Then*

$$A_k = \frac{1}{(k+1)!}\prod_{j=0}^{k}(2\lambda+j) \quad (k \in \mathbb{N}_0). \tag{2.2}$$

*Proof.* From (2.1), we know that

$$\begin{aligned}(k+2)A_{k+1} &= 2\lambda\left(1+\sum_{l=0}^{k}A_l\right), \\ (k+1)A_k &= 2\lambda\left(1+\sum_{l=0}^{k-1}A_l\right).\end{aligned} \tag{2.3}$$

By virtue of (2.3), we find that

$$\frac{A_{k+1}}{A_k} = \frac{k+1+2\lambda}{k+2} \quad (k \in \mathbb{N}_0). \tag{2.4}$$

Thus, for $k \geqq 1$, we deduce from (2.4) that

$$A_k = \frac{A_k}{A_{k-1}}\cdots\frac{A_3}{A_2}\cdot\frac{A_2}{A_1}\cdot\frac{A_1}{A_0}\cdot A_0 = \frac{1}{(k+1)!}\prod_{j=0}^{k}(2\lambda+j). \tag{2.5}$$

By virtue of (2.1) and (2.5), we get the desired assertion (2.2) of Lemma 2.1. □

**Lemma 2.2** (Jack's Lemma [10]). *Let $\phi$ be a nonconstant regular function in $\mathbb{U}$. If $|\phi|$ attains its maximum value on the circle $|z| = r < 1$ at $z_0$, then*

$$z_0\phi'(z_0) = t\phi(z_0), \tag{2.6}$$

*for some real number $t$ $(t \geqq 1)$.*

## 3. Main Results

We begin by deriving the following subordination property of functions belonging to the class $\mathcal{MS}_\alpha$.

**Theorem 3.1.** *A function $f \in \mathcal{MS}_\alpha$ if and only if*

$$-\frac{zf'(z)}{f(z)} \prec 1 + 2\alpha\left[\operatorname{Re}\left(\frac{1}{\alpha}\right) - 1\right]\frac{z}{1-z} \qquad \left(z \in \mathbb{U}^*;\ \left|\alpha - \frac{1}{2}\right| < \frac{1}{2}\right). \tag{3.1}$$

*Proof.* Suppose that

$$h(z) := \frac{-(1/\alpha)(zf'(z)/f(z)) - 1 - i\operatorname{Im}(1/\alpha)}{\operatorname{Re}(1/\alpha) - 1} \qquad (z \in \mathbb{U};\ f \in \mathcal{MS}_\alpha). \tag{3.2}$$

We easily know that $h \in \mathcal{P}$, which implies that

$$\frac{-(1/\alpha)(zf'(z)/f(z)) - 1 - i\operatorname{Im}(1/\alpha)}{\operatorname{Re}(1/\alpha) - 1} = \frac{1+\omega(z)}{1-\omega(z)} \qquad (z \in \mathbb{U};\ f \in \mathcal{MS}_\alpha), \tag{3.3}$$

where $\omega$ is analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ $(z \in \mathbb{U})$.

It follows from (3.3) that

$$-\frac{zf'(z)}{f(z)} = 1 + 2\alpha\left[\operatorname{Re}\left(\frac{1}{\alpha}\right) - 1\right]\frac{\omega(z)}{1-\omega(z)} \qquad (z \in \mathbb{U}), \tag{3.4}$$

which is equivalent to the subordination relationship (3.1).

On the other hand, the above deductive process can be converse. The proof of Theorem 3.1 is thus completed. □

**Theorem 3.2.** *Let $f \in \mathcal{MS}_\alpha$. Then*

$$f(z) = \frac{1}{z}\cdot\exp\left(-2\alpha\left[\operatorname{Re}\left(\frac{1}{\alpha}\right) - 1\right]\int_0^z \frac{\omega(t)}{t(1-\omega(t))}dt\right) \qquad (z \in\ \mathbb{U}^*), \tag{3.5}$$

*where $\omega$ is analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ $(z \in \mathbb{U})$.*

*Proof.* For $f \in \mathcal{MS}_\alpha$, by Theorem 3.1, we know that (3.1) holds true. It follows that

$$-\frac{zf'(z)}{f(z)} = 1 + 2\alpha\left[\operatorname{Re}\left(\frac{1}{\alpha}\right) - 1\right]\frac{\omega(z)}{1-\omega(z)} \quad (z \in \mathbb{U}), \tag{3.6}$$

where $\omega$ is analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ $(z \in \mathbb{U})$.

We now find from (3.6) that

$$\frac{f'(z)}{f(z)} + \frac{1}{z} = -2\alpha\left[\operatorname{Re}\left(\frac{1}{\alpha}\right) - 1\right]\frac{\omega(z)}{z(1-\omega(z))} \quad (z \in \mathbb{U}^*), \tag{3.7}$$

which, upon integration, yields

$$\log(zf(z)) = -2\alpha\left[\operatorname{Re}\left(\frac{1}{\alpha}\right) - 1\right]\int_0^z \frac{\omega(t)}{t(1-\omega(t))}dt \quad (z \in \mathbb{U}). \tag{3.8}$$

The assertion (3.5) of Theorem 3.2 can be easily derived from (3.8). □

**Theorem 3.3.** *Let $f \in \mathcal{MS}_\alpha$. Then*

$$f(z) * \frac{(1-e^{i\theta})z + 2\alpha[\operatorname{Re}(1/\alpha) - 1]e^{i\theta}(1-z)}{z(1-z)^2} \neq 0 \quad (z \in \mathbb{U}^*;\ 0 < \theta < 2\pi). \tag{3.9}$$

*Proof.* Assume that $f \in \mathcal{MS}_\alpha$. By Theorem 3.1, we know that (3.1) holds, which implies that

$$-\frac{zf'(z)}{f(z)} \neq 1 + 2\alpha\left[\operatorname{Re}\left(\frac{1}{\alpha}\right) - 1\right]\frac{e^{i\theta}}{1-e^{i\theta}} \quad (z \in \mathbb{U}^*;\ 0 < \theta < 2\pi). \tag{3.10}$$

It is easy to see that the condition (3.10) can be written as follows:

$$\left(1-e^{i\theta}\right)zf'(z) + \left(1 - e^{i\theta} + 2\alpha\left[\operatorname{Re}\left(\frac{1}{\alpha}\right) - 1\right]e^{i\theta}\right)f(z) \neq 0. \tag{3.11}$$

We note that

$$\begin{aligned} f(z) &= f(z) * \left(\frac{1}{z} + 1 + \frac{z}{1-z}\right) = f(z) * \frac{1}{z(1-z)}, \\ -zf'(z) &= f(z) * \left(\frac{1}{z} - \frac{z}{(1-z)^2}\right) = f(z) * \frac{1-2z}{z(1-z)^2}. \end{aligned} \tag{3.12}$$

Thus, by substituting (3.12) into (3.11), we get the desired assertion (3.9) of Theorem 3.3. □

**Theorem 3.4.** *Let* $\lambda = [\mathrm{Re}(1/\alpha) - 1]|\alpha|$. *If* $f \in \mathcal{MS}_\alpha$, *then*

$$|a_k| \leqq \frac{1}{(k+1)!}\prod_{j=0}^{k}(2\lambda + j) \quad (k \in \mathbb{N}_0). \tag{3.13}$$

*The inequality* (3.13) *is sharp for the function given by*

$$f(z) = \frac{1}{z(1-z)^{2-2\alpha}} \quad (0 < \alpha < 1). \tag{3.14}$$

*Proof.* Suppose that

$$h(z) := \frac{-(1/\alpha)\big(zf'(z)/f(z)\big) - 1 - i\,\mathrm{Im}(1/\alpha)}{\mathrm{Re}(1/\alpha) - 1}. \tag{3.15}$$

We easily know that $h \in \mathcal{P}$.

If we put

$$h(z) = 1 + h_1 z + h_2 z^2 + \cdots, \tag{3.16}$$

it is known that

$$|h_k| \leqq 2 \quad (k \in \mathbb{N}). \tag{3.17}$$

From (3.15), we have

$$-\frac{1}{\alpha}\frac{zf'(z)}{f(z)} - 1 - i\,\mathrm{Im}\left(\frac{1}{\alpha}\right) = \left[\mathrm{Re}\left(\frac{1}{\alpha}\right) - 1\right]h(z). \tag{3.18}$$

We now set

$$\begin{aligned} A &:= 1 + i\,\mathrm{Im}\left(\frac{1}{\alpha}\right), \\ B &:= \mathrm{Re}\left(\frac{1}{\alpha}\right) - 1. \end{aligned} \tag{3.19}$$

It follows from (3.18) that

$$-zf'(z) = [\alpha A + \alpha B h(z)]f(z). \tag{3.20}$$

Combining (1.1), (3.16), and (3.20), we obtain

$$\begin{aligned} & -z\left(-\frac{1}{z^2}+a_1+2a_2z+\cdots+ka_kz^{k-1}+\cdots\right) \\ & \quad =\left(1+\alpha Bh_1z+\cdots+\alpha Bh_kz^k+\cdots\right)\left(\frac{1}{z}+a_0+a_1z+\cdots+a_kz^k+\cdots\right). \end{aligned} \tag{3.21}$$

In view of (3.21), we get

$$a_0+\alpha Bh_1=0, \tag{3.22}$$

$$-ka_k=a_k+\alpha B(a_{k-1}h_1+a_{k-2}h_2+\cdots+a_0h_k+h_{k+1}) \quad (k\in\mathbb{N}). \tag{3.23}$$

From (3.17) and (3.22), we obtain

$$|a_0|\leqq 2|\alpha|B=2\lambda. \tag{3.24}$$

Moreover, we deduce from (3.17) and (3.23) that

$$|a_k|\leqq\frac{2|\alpha|B}{k+1}\left(1+\sum_{l=0}^{k-1}|a_l|\right)=\frac{2\lambda}{k+1}\left(1+\sum_{l=0}^{k-1}|a_l|\right) \quad (k\in\mathbb{N}). \tag{3.25}$$

Next, we define the sequence $\{A_k\}_{k=0}^{\infty}$ as follows:

$$A_0=2\lambda, \qquad A_{k+1}=\frac{2\lambda}{k+2}\left(1+\sum_{l=0}^{k}A_l\right) \quad (k\in\mathbb{N}_0). \tag{3.26}$$

In order to prove that

$$|a_k|\leqq A_k, \tag{3.27}$$

we make use of the principle of mathematical induction. By noting that

$$|a_0|\leqq 2\lambda=A_0. \tag{3.28}$$

Therefore, assuming that

$$|a_l|\leqq A_l \quad (l=0,1,2,\ldots,k;\ k\in\mathbb{N}_0). \tag{3.29}$$

Combining (3.25) and (3.26), we get

$$|a_{k+1}|\leqq\frac{2\lambda}{k+2}\left(1+\sum_{l=0}^{k}|a_l|\right)\leqq\frac{2\lambda}{k+2}\left(1+\sum_{l=0}^{k}A_l\right)=A_{k+1}. \tag{3.30}$$

Hence, by the principle of mathematical induction, we have

$$|a_k| \leqq A_k \quad (k \in \mathbb{N}_0) \tag{3.31}$$

as desired.

By means of Lemma 2.1 and (3.26), we know that

$$A_k = \frac{1}{(k+1)!}\prod_{j=0}^{k}(2\lambda + j) \quad (k \in \mathbb{N}_0). \tag{3.32}$$

Combining (3.31) and (3.32), we readily get the coefficient estimates asserted by Theorem 3.4.

For the sharpness, we consider the function $f$ given by (3.14). A simple calculation shows that

$$\operatorname{Re}\left(-\frac{zf'(z)}{f(z)}\right) = \operatorname{Re}\left(\frac{1+(2\alpha-3)z}{1-z}\right) \geqq 2-\alpha > \alpha. \tag{3.33}$$

Thus, the function $f$ belongs to the class $\mathcal{MS}_\alpha$. Since $0 < \alpha < 1$, we have

$$\lambda = 1 - \alpha. \tag{3.34}$$

Then $f$ becomes

$$f(z) = z^{-1}(1-z)^{-2\lambda} = z^{-1}\left(\sum_{n=0}^{\infty}\binom{-2\lambda}{n}(-z)^n\right) = \frac{1}{z} + \sum_{n=0}^{\infty}\frac{2\lambda(2\lambda+1)\cdots(2\lambda+n)}{(n+1)!}z^n. \tag{3.35}$$

This completes the proof of Theorem 3.4. □

**Theorem 3.5.** *If $f \in \Sigma$ satisfies the inequality*

$$\sum_{k=0}^{\infty}(k + |k+2\alpha|)|a_k| \leqq 1 - |2\alpha - 1| \quad \left(\left|\alpha - \frac{1}{2}\right| < \frac{1}{2}\right), \tag{3.36}$$

*then $f \in \mathcal{MS}_\alpha$.*

*Proof.* To prove $f \in \mathcal{MS}_\alpha$, it suffices to show that

$$\left|\frac{f(z)}{zf'(z)} + \frac{1}{2\alpha}\right| < \frac{1}{2|\alpha|} \quad (z \in \mathbb{U}), \tag{3.37}$$

which is equivalent to

$$\left|\frac{zf'(z) + 2\alpha f(z)}{zf'(z)}\right| < 1 \quad (z \in \mathbb{U}^*). \tag{3.38}$$

From (3.36), we know that

$$1-\sum_{k=0}^{\infty}k|a_k| \geqq |2\alpha-1|+\sum_{k=0}^{\infty}|k+2\alpha||a_k|>0. \tag{3.39}$$

Now, by the maximum modulus principle, we deduce from (1.1) and (3.39) that

$$\begin{aligned}\left|\frac{zf'(z)+2\alpha f(z)}{zf'(z)}\right| &= \left|\frac{(2\alpha-1)+\sum_{k=0}^{\infty}(k+2\alpha)a_k z^{k+1}}{-1+\sum_{k=0}^{\infty}ka_k z^{k+1}}\right| \\ &\leqq \frac{|2\alpha-1|+\sum_{k=0}^{\infty}|k+2\alpha||a_k||z|^{k+1}}{1-\sum_{k=0}^{\infty}k|a_k||z|^{k+1}} \\ &< \frac{|2\alpha-1|+\sum_{k=0}^{\infty}|k+2\alpha||a_k|}{1-\sum_{k=0}^{\infty}k|a_k|} \\ &\leqq 1,\end{aligned} \tag{3.40}$$

which implies that the assertion of Theorem 3.5 holds. □

**Theorem 3.6.** *If $f \in \Sigma$ satisfies the condition*

$$\left|1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right| < \frac{1-\alpha}{2\alpha} \quad \left(\frac{1}{2}<\alpha<1\right), \tag{3.41}$$

*then $f \in \mathcal{MS}_{\alpha}$.*

*Proof.* Define the function $\varphi$ by

$$\varphi(z) := \frac{\left(zf'(z)/f(z)\right)+1}{\left(zf'(z)/f(z)\right)+2\alpha-1} \quad (z \in \mathbb{U}). \tag{3.42}$$

Then we see that $\varphi$ is analytic in $\mathbb{U}$ with $\varphi(0)=0$.

It follows from (3.42) that

$$-\frac{zf'(z)}{f(z)} = \frac{1+(1-2\alpha)\varphi(z)}{1-\varphi(z)}. \tag{3.43}$$

By differentiating both sides of (3.43) logarithmically, we obtain

$$1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)} = \frac{2(1-\alpha)z\varphi'(z)}{\left[1+(1-2\alpha)\varphi(z)\right]\left(1-\varphi(z)\right)}. \tag{3.44}$$

From (3.41) and (3.44), we find that

$$\left|1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right|=\left|\frac{2(1-\alpha)z\varphi'(z)}{[1+(1-2\alpha)\varphi(z)](1-\varphi(z))}\right|<\frac{1-\alpha}{2\alpha}. \tag{3.45}$$

Next, we claim that $|\varphi(z)|<1$. Indeed, if not, there exists a point $z_0\in\mathbb{U}$ such that

$$\max_{|z|\leqq|z_0|}|\varphi(z)|=|\varphi(z_0)|=1. \tag{3.46}$$

By Lemma 2.2, we have

$$\varphi(z_0)=e^{i\theta},\qquad z_0\varphi'(z_0)=te^{i\theta}\quad(t\geqq 1). \tag{3.47}$$

Moreover, for $z=z_0$, we find from (3.44) and (3.47) that

$$\begin{aligned}
&\left|1+\frac{z_0f''(z_0)}{f'(z_0)}-\frac{z_0f'(z_0)}{f(z_0)}\right|\\
&\quad=\left|\frac{2(1-\alpha)te^{i\theta}}{(1+(1-2\alpha)e^{i\theta})(1-e^{i\theta})}\right|\\
&\quad=\frac{2(1-\alpha)t}{\sqrt{1+2(1-2\alpha)\cos\theta+(1-2\alpha)^2}\cdot\sqrt{2-2\cos\theta}}\\
&\quad\geqq\frac{1-\alpha}{2\alpha}\quad\left(\frac{1}{2}<\alpha<1\right).
\end{aligned} \tag{3.48}$$

But (3.48) contradicts to (3.45). Therefore, we conclude that $|\varphi(z)|<1$, that is

$$\left|\frac{(zf'(z)/f(z))+1}{(zf'(z)/f(z))+2\alpha-1}\right|<1, \tag{3.49}$$

which shows that $f\in\mathcal{MS}_\alpha$. □

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## References


[1] M. Elin, "Covering and distortion theorems for spirallike functions with respect to a boundary point," *International Journal of Pure and Applied Mathematics*, vol. 28, no. 3, pp. 387–400, 2006.

[2] M. Elin and D. Shoikhet, "Angle distortion theorems for starlike and spirallike functions with respect to a boundary point," *International Journal of Mathematics and Mathematical Sciences*, vol. 2006, Article ID 81615, 13 pages, 2006.

[3] K. Hamai, T. Hayami, K. Kuroki, and S. Owa, "Coefficient estimates of functions in the class concerning with spirallike functions," *Applied Mathematics E-Notes*, vol. 11, pp. 189–196, 2011.

[4] Y. C. Kim and T. Sugawa, "The Alexander transform of a spirallike function," *Journal of Mathematical Analysis and Applications*, vol. 325, no. 1, pp. 608–611, 2007.

[5] Y. C. Kim and H. M. Srivastava, "Some subordination properties for spirallike functions," *Applied Mathematics and Computation*, vol. 203, no. 2, pp. 838–842, 2008.

[6] A. Lecko, "The class of functions spirallike with respect to a boundary point," *International Journal of Mathematics and Mathematical Sciences*, no. 37–40, pp. 2133–2143, 2004.

[7] H. M. Srivastava, Q.-H. Xu, and G.-P. Wu, "Coefficient estimates for certain subclasses of spiral-like functions of complex order," *Applied Mathematics Letters*, vol. 23, no. 7, pp. 763–768, 2010.

[8] Y. Sun, W.-P. Kuang, and Z.-G. Wang, "On meromorphic starlike functions of reciprocal order $\alpha$," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 35, no. 2, pp. 469–477, 2012.

[9] N. Uyanik, H. Shiraishi, S. Owa, and Y. Polatoglu, "Reciprocal classes of $p$-valently spirallike and $p$-valently Robertson functions," *Journal of Inequalities and Applications*, vol. 2011, article 61, 2011.

[10] I. S. Jack, "Functions starlike and convex of order $\alpha$," *Journal of the London Mathematical Society*, vol. 3, pp. 469–474, 1971.