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Research Article

System of Nonlinear Set-Valued Variational Inclusions Involving a Finite Family of $H(\cdot,\cdot)$-Accretive Operators in Banach Spaces

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We study a new system of nonlinear set-valued variational inclusions involving a finite family of $H(\cdot,\cdot)$-accretive operators in Banach spaces. By using the resolvent operator technique associated with a finite family of $H(\cdot,\cdot)$-accretive operators, we prove the existence of the solution for the system of nonlinear set-valued variational inclusions. Moreover, we introduce a new iterative scheme and prove a strong convergence theorem for finding solutions for this system.

1. Introduction

Variational inequality theory has become a very effective and powerful tool for studying a wide range of problems arising in pure and applied sciences which include work on differential equations, control problems, mechanics, general equilibrium problems in transportation and economics. In 1994, Hassouni and Moudafi [1] introduced and studied a class of variational inclusions and developed a perturbed algorithm for finding approximate solutions of the variational inclusions. In 1996, Adly [2] obtained some important extensions and generalizations of the results in [1] for nonlinear variational inclusions. Recently, Ding [3] introduced and studied a class of generalized quasivariational inclusions and Kazmi [4] introduced and studied another class of quasivariational inclusions in the same year. In [5, 6], Ansari et al. introduced the system of vector equilibrium problems and they proved the existence of solutions for such problems (see also in [7–9]). In 2004, Verma [10] studied nonlinear variational inclusion problems based on the generalized resolvent operator technique involving A-monotone mapping. For existence result and approximating solution of the system of set-valued variational inclusions and the class of nonlinear relaxed cocoercive variational inclusions, we refer the reader to Yan et al. [11], Plubtieng and Sriprad [12], Verma [13] and Cho et al. [14].

Very recently, Verma [15] introduced and studied approximation solvability of a general class of nonlinear variational inclusion problems based on (A, η)-resolvent operator technique in a Hilbert space. On the other hand, Zou and Huang [16] studied the Lipschitz continuity of resolvent operator for the $H(\cdot,\cdot)$-accretive operator in Banach spaces. Moreover, they also applied these new concepts to solve a variational-like inclusion problem. One year later, Zou and Huang [17] introduced and studied a new class of system of variational inclusions involving $H(\cdot,\cdot)$-accretive operator in Banach spaces. By using the resolvent operator technique associated with $H(\cdot,\cdot)$-accretive operator, they proved the existence of the solution for the system of inclusions. Moreover, they also develop a step-controlled iterative algorithm to approach the unique solution.

In this paper, we introduce a new system of nonlinear set-valued variational inclusions involving a finite family of $H(\cdot,\cdot)$-accretive operators in Banach spaces. By using the resolvent operators technique associated with a finite family of $H(\cdot,\cdot)$-accretive operator, we prove the existence of the solution for the system of nonlinear set-valued variational inclusions. Moreover, we introduce a new iterative scheme and prove a strong convergence theorem for finding solutions of this system.

2. Preliminaries

Let X be a real Banach space with dual space X^*, $\langle\cdot,\cdot\rangle$ the dual pair between X and X^* and 2^X and $C(X)$ denote the family of all the nonempty subsets of X and the family of all closed subsets of X, respectively. The generalized duality mapping $J_q : X \to 2^{X^*}$ is defined by

$$J_q(X) = \left\{ f^* \in X^* : \langle x, f^* \rangle = \|x\|^q, \|f^*\| = \|x\|^{q-1} \right\}, \quad \forall x \in X, \tag{2.1}$$

where $q > 1$ is a constant. It is known that, in general, $J_q(x) = \|x\|^{q-1} J_2(x)$ for all $x \neq 0$ and J_q is single-valued if X^* is strictly convex. In the sequel, we always assume that X is a real Banach space such that J_q is single-valued.

The modulus of smoothness of X is the function $\rho_X : [0,\infty) \to [0,\infty)$ defined by

$$\rho_X(t) = \sup\left\{ \frac{\|x+y\| + \|x-y\|}{2} - 1 : \|x\| \leq 1, \|y\| \leq t \right\}. \tag{2.2}$$

A Banach space X is called uniformly smooth if

$$\lim_{t\to 0} \frac{\rho_X(t)}{t} = 0. \tag{2.3}$$

X is called q-uniformly smooth if there exists a constant $c > 0$ such that

$$\rho_X(t) \leq ct^q, \qquad q > 1. \tag{2.4}$$

Note that J_q is single valued if X is uniformly smooth. In the study of characteristic inequalities in q-uniformly smooth Banach spaces, Xu [18] proved the following result.

Definition 2.1. Let $H, \eta : X \times X \to X$ be two single-valued mappings and $A, B : X \to X$ two single-valued mappings.

(i) A is said to be accretive if

$$\langle Ax - Ay, J_q(x-y)\rangle \geq 0, \qquad \forall x, y \in X, \tag{2.5}$$

(ii) A is said to be strictly accretive if A is accretive and

$$\langle Ax - Ay, J_q(x-y)\rangle = 0, \qquad \forall x, y \in X, \tag{2.6}$$

if and only if $x = y$;

(iii) $H(A, \cdot)$ is said to be α-strongly accretive with respect to A if there exists a constant $\alpha > 0$ such that

$$\langle H(Ax,u) - H(Ay,u), J_q(x-y)\rangle \geq \alpha\|x-y\|^q, \qquad \forall x, y, u \in X; \tag{2.7}$$

(iv) $H(\cdot, B)$ is said to be β-relaxed accretive with respect to B if there exists a constant $\beta > 0$ such that

$$\langle H(u,Bx) - H(u,By), J_q(x-y)\rangle \geq -\beta\|x-y\|^q, \qquad \forall x, y, u \in X; \tag{2.8}$$

(v) $H(\cdot, \cdot)$ is said to be γ-Lipschitz continuous with respect to A if there exists a constant $\gamma > 0$ such that

$$\|H(Ax,u) - H(Ay,u)\| \leq \gamma\|x-y\|^q, \qquad \forall x, y, u \in X; \tag{2.9}$$

(vi) A is said to be θ-Lipschitz continuous if there exists a constant $\theta > 0$ such that

$$\|Ax - Ay\| \leq \theta\|x-y\|^q, \qquad \forall x, y \in X; \tag{2.10}$$

(vii) $\eta(\cdot,\cdot)$ is said to be strongly accretive with respect to $H(A,B)$ if there exists a constant $\rho > 0$ such that

$$\langle \eta(x,u) - \eta(y,u), J_q(H(Ax,Bx) - H(Ay,By)) \rangle \geq \rho \|x - y\|^q, \quad \forall x, y, u \in X. \tag{2.11}$$

Definition 2.2. Let $\eta : X \times X \to X$ be single-valued mapping. Let $M : X \to 2^X$ be a set-valued mapping.

(i) η is said to be τ-Lipschitz continuous if there exists a constant $\tau > 0$ such that

$$\|\eta(x,y)\| \leq \tau \|x - y\|, \quad \forall x, y \in X; \tag{2.12}$$

(ii) M is said to be accretive if

$$\langle u - v, J_q(x - y) \rangle \geq 0, \quad \forall x, y \in X,\ u \in M(x),\ v \in M(y); \tag{2.13}$$

(iii) M is said to be η-accretive if

$$\langle u - v, J_q(\eta(x,y)) \rangle \geq 0, \quad \forall x, y \in X,\ u \in M(x),\ v \in M(y); \tag{2.14}$$

(iv) M is said to be strictly η-accretive if M is η-accretive and equality holds if and only if $x = y$;

(v) M is said to be γ-strongly η-accretive if there exists a positive constant $\gamma > 0$ such that

$$\langle u - v, J_q(\eta(x,y)) \rangle \geq \gamma \|x - y\|^q, \quad \forall x, y \in X,\ u \in M(x),\ v \in M(y); \tag{2.15}$$

(vi) M is said to be α-relaxed η-accretive if there exists a positive constant $\alpha > 0$ such that

$$\langle u - v, J_q(\eta(x,y)) \rangle \geq -\alpha \|x - y\|^q, \quad \forall x, y \in X,\ u \in M(x),\ v \in M(y). \tag{2.16}$$

Definition 2.3. Let $A, B : X \rightarrow X$, $H : X \times X \rightarrow X$ be three single-valued mappings. Let $M : X \rightarrow 2^X$ be a set-valued mapping. M is said to be $H(\cdot,\cdot)$-accretive with respect to A and B (or simply $H(\cdot,\cdot)$-accretive in the sequel), if M is accretive and $(H(A,B) + \lambda M)(X) = X$ for every $\lambda > 0$.

Lemma 2.4. *Let X be a real uniformly smooth Banach space. Then X is q-uniformly smooth if and only if there exists a constant $c_q > 0$ such that for all $x, y \in X$*

$$\|x + y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + c_q\|y\|^q. \tag{2.17}$$

Lemma 2.5 (see[16]). *Let $H(A,B)$ be α-strongly accretive with respect to A, β-relaxed accretive with respect to B, and $\alpha > \beta$. Let M be an $H(\cdot,\cdot)$-accretive operator with respect to A and B. Then, the operator $H((A,B) + \lambda M)^{-1}$ is single valued. Based on Lemma 2.4, one can define the resolvent operator $R_{M,\lambda}^{H(\cdot,\cdot)}$ as follows.*

Definition 2.6. Let H, A, B, M be defined as in Definition 2.3. Let $H(A,B)$ be α-strongly accretive with respect to A, β-relaxed accretive with respect to B, and $\alpha > \beta$. Let M be an $H(\cdot,\cdot)$-accretive operator with respect to A and B. The resolvent operator $R_{M,\lambda}^{H(\cdot,\cdot)} : X \rightarrow X$ is defined by

$$R_{M,\lambda}^{H(\cdot,\cdot)}(z) = (H(A,B) + \lambda M)^{-1}(z), \quad \forall z \in X, \tag{2.18}$$

where $\lambda > 0$ is a constant.

Lemma 2.7 (see [16]). *Let H, A, B, M be defined as in Definition 2.3. Let $H(A,B)$ be α-strongly accretive with respect to A, β-relaxed accretive with respect to B, and $\alpha > \beta$. Suppose that $M : X \rightarrow 2^X$ is an $H(\cdot,\cdot)$-accretive operator. Then resolvent operator $R_{M,\lambda}^{H(\cdot,\cdot)}$ defined by* (2.18) *is* $1/(\alpha - \beta)$ *Lipschitz continuous. That is,*

$$\left\|R_{M,\lambda}^{H(\cdot,\cdot)}(x) - R_{M,\lambda}^{H(\cdot,\cdot)}(y)\right\| \leq \frac{1}{\alpha - \beta}\|x - y\|, \quad \forall x, y \in X. \tag{2.19}$$

We define a Hausdorff pseudometric $D : 2^X \times 2^X \rightarrow [0, +\infty]$ by

$$D(U,V) = \max\left\{\sup_{u\in U}\inf_{v\in V}\|u - v\|, \sup_{u\in V}\inf_{v\in U}\|u - v\|\right\} \tag{2.20}$$

for any given $U, V \in 2^X$. Note that if the domain of D is restricted to closed bounded subsets, then D is the Hausdorff metric.

Lemma 2.8 (see [19]). *Let $\{c_n\}$ and $\{k_n\}$ be two real sequences of nonnegative numbers that satisfy the following conditions:*

(i) $0 < k_n < 1$ *for* $n = 0, 1, 2, \ldots$, *and* $\limsup_n k_n < 1$;

(ii) $c_{n+1} \leq k_n c_n$ *for* $n = 0, 1, 2, \ldots$.

Then, c_n converges to 0 as $n \rightarrow \infty$.

3. Main Result

Let X be q-uniformly smooth real Banach space and $C(X)$ a nonempty closed convex set. Let $S_i, H_i : X \times X \to X$, $A_i, B_i : X \to X$ be single-valued operators, for all $i = 1, 2, \ldots, N$. For any fix $i \in \{1, 2, \ldots, N\}$, we let $M_i : X \to 2^X$, $H_i(A_i, B_i)$-accretive set-valued operator and $U_i : X \to 2^X$ a set-valued mapping which nonempty values. The system of nonlinear set-valued variational inclusions is to find $a_1, \ldots, a_N \in X$, $u_1 \in U_1(a_N), \ldots, u_N \in U_N(a_1)$ such that

$$0 \in S_i(a_i, u_i) + M_i(a_i), \quad \forall i = 1, 2, \ldots, N. \tag{3.1}$$

If $N = 2$, then system of nonlinear set-valued variational inclusions (3.1) becomes following system of variational inclusions: finding $a_1, a_2 \in X$, $u_1 \in U_1(a_2)$ and $u_2 \in U_2(a_1)$ such that

$$\begin{gathered} 0 \in S_1(a_1, u_1) + M_1(a_1), \\ 0 \in S_2(a_2, u_2) + M_2(a_2). \end{gathered} \tag{3.2}$$

If $N = 1$, then system of nonlinear set-valued variational inclusions (3.1) becomes the following class of nonlinear set-valued variational inclusions see [15]: finding $a \in X$, $u \in U(a)$ such that

$$0 \in S(a, u) + M(a). \tag{3.3}$$

For solving the system of nonlinear set-valued variational inclusions involving a finite family of $H(\cdot,\cdot)$-accretive operators in Banach spaces, let us give the following assumptions.

For any $i \in \{1, 2, \ldots, N\}$, we suppose that

(A1) $H(A_i, B_i)$ is α_i-strongly accretive with respect to A_i, β_i-relaxed accretive with respect to B_i and $\alpha_i > \beta_i$,

(A2) $M_i : X \to 2^X$ is an $H_i(\cdot,\cdot)$-accretive single-valued mapping,

(A3) $U_i : X \to C(X)$ is a contraction set-valued mapping with $0 \le L_i < 1$ and nonempty values,

(A4) $H_i(A_i, B_i)$ is r_i-Lipschitz continuous with respect to A_i and t_i-Lipschitz continuous with respect to B_i,

(A5) $S_i : X \times X \to X$ is l_i-Lipschitz continuous with respect to its first argument and m_i-Lipschitz continuous with respect to its second argument,

(A6) $S_i(\cdot, u)$ is s_i-strongly accretive with respect to $H_i(A_i, B_i)$.

Theorem 3.1. *For given $a_1, \ldots, a_N \in X$, $u_1 \in U_1(a_N), \ldots, u_N \in U_N(a_1)$, it is a solution of problem* (3.1) *if and only if*

$$a_i = R_{M_i,\lambda_i}^{H_i(\cdot,\cdot)}\left[H_i(A_i(a_i), B_i(a_i)) - \lambda_i S_i(a_i, u_i)\right], \tag{3.4}$$

where $\lambda_i > 0$ are constants.

Proof. We note from the Definition 2.6 that $a_1, \ldots, a_N \in X$, $u_1 \in U_1(a_N), \ldots, u_N \in U_N(a_1)$ is a solution of (3.1) if and only if, for each $i \in \{1, 2, \ldots, N\}$, we have

$$\begin{aligned} a_i &= R^{H_i(\cdot,\cdot)}_{M_i,\lambda_i}[H_i(A_i(a_i), B_i(a_i)) - \lambda_i S_i(a_i, u_i)] \\ &\Longleftrightarrow a_i = [H_i(A_i, B_i) + \lambda_i M_i]^{-1}[H_i(A_i(a_i), B_i(a_i)) - \lambda_i S_i(a_i, u_i)] \\ &\Longleftrightarrow [H_i(A_i(a_i), B_i(a_i)) - \lambda_i S_i(a_i, u_i)] \in [H_i(A_i, B_i) + \lambda_i M_i](a_i) \\ &\Longleftrightarrow \quad -\lambda_i S_i(a_i, u_i) \in \lambda_i M_i(a_i) \\ &\Longleftrightarrow 0 \in S_i(a_i, u_i) + M_i(a_i). \end{aligned} \tag{3.5}$$

□

Algorithm 3.2. For given $a_0^1, \ldots, a_0^N \in X$, $u_0^1 \in U_1(a_0^N), \ldots, u_0^N \in U_N(a_0^1)$, we let

$$a_1^i = \sigma_0 a_0^i + (1 - \sigma_0) R^{H_i(\cdot,\cdot)}_{M_i,\lambda_i}\left[H_i\left(A_i\left(a_0^i\right), B_i\left(a_0^i\right)\right) - \lambda_i S_i\left(a_0^i, u_0^i\right)\right], \tag{3.6}$$

for all $i = 1, 2, \ldots, N$, where $0 < \sigma_0 \leq 1$. By Nadler theorem [20], there exists $u_1^1 \in U_1$ $(a_1^N), \ldots,$ $u_1^N \in U_N(a_1^1)$ such that

$$\left\|u_1^i - u_0^i\right\| \leq (1+1) D\left(U_i\left(a_1^{N-(i-1)}\right), U_i\left(a_0^{N-(i-1)}\right)\right), \quad \forall i = 1, 2, \ldots, N, \tag{3.7}$$

where $D(\cdot, \cdot)$ is the Hausdorff pseudo metric on 2^X. Continuing the above process inductively, we can obtain the sequences $\{a_n^i\}$ and $\{u_n^i\}$ such that

$$a_{n+1}^i = \sigma_n a_n^i + (1 - \sigma_n) R^{H_i(\cdot,\cdot)}_{M_i,\lambda_i}\left[H_i\left(A_i\left(a_n^i\right), B_i\left(a_n^i\right)\right) - \lambda_i S_i\left(a_n^i, u_n^i\right)\right], \tag{3.8}$$

for all $n = 1, 2, 3, \ldots,$ $i = 1, 2, \ldots, N$, where $0 < \sigma_n \leq 1$ with $\limsup_{n\to\infty} \sigma_n < 1$. Therefore, by Nadler theorem [20], there exists $u_{n+1}^1 \in U_1(a_{n+1}^N), \ldots, u_{n+1}^N \in U_N(a_{n+1}^1)$ such that

$$\begin{aligned} &\left\|u_{n+1}^i - u_n^i\right\| \leq \left(1 + (1+n)^{-1}\right) D\left(U_i\left(a_{n+1}^{N-(i-1)}\right),\right. \\ &\left. U_i\left(a_n^{N-(i-1)}\right)\right), \quad \forall n = 1, 2, 3, \ldots, i = 1, 2, \ldots, N. \end{aligned} \tag{3.9}$$

The idea of the proof of the next theorem is contained in the paper of Verma [15] and Zou and Huang [17].

Theorem 3.3. *Let X be q-uniformly smooth real Banach space. Let $A_i, B_i : X \to X$ be single-valued operators, $H_i : X \times X \to X$ a single-valued operator satisfy (A1) and M_i, U_i, $H_i(A_i, B_i)$, S_i, $S_i(\cdot, u)$ satisfy conditions (A2)–(A6), respectively. If there exists a constant $c_{q,i}$ such that*

$$\frac{\sqrt[q]{(r_i+t_i)^q - q\lambda_i s_i + c_{q,i}\lambda_i^q l_i^q}}{\alpha_i - \beta_i} + \frac{\lambda_i m_i}{\alpha_i - \beta_i} < 1 \tag{3.10}$$

for all $i = 1, 2, \ldots, N$, then problem (3.1) *has a solution $a_1, \ldots, a_N$, $u_1 \in U_1(a_N), \ldots, u_N \in U_N(a_1)$.*

Proof. For any $i \in \{1, 2, \ldots, N\}$ and $\lambda_i > 0$, we define $F_i : X \times X \to X$ by

$$F_i(u, v) = R_{M_i,\lambda_i}^{H_i(\cdot,\cdot)}\left[H_i(A_i(u), B_i(u)) - \lambda_i S_i(u, v)\right], \tag{3.11}$$

for all $u, v \in X$. Let $J_i(x, y) = H_i(A_i(x), B_i(y))$. For any $(u_1, v_1), (u_2, v_2) \in X \times X$, we note by (3.11) and Lemma 2.7 that

$$\begin{aligned}
\|F_i(u_1, v_1) - F_i(u_2, v_2)\| &= \Big\| R_{M_i,\lambda_i}^{H_i(\cdot,\cdot)}\left[H_i(A_i(u_1), B_i(u_1)) - \lambda_i S_i(u_1, v_1)\right] \\
&\qquad - R_{M_i,\lambda_i}^{H_i(\cdot,\cdot)}\left[H_i(A_i(u_2), B_i(u_2)) - \lambda_i S_i(u_2, v_2)\right]\Big\| \\
&= \Big\| R_{M_i,\lambda_i}^{H_i(\cdot,\cdot)}\left[J_i(u_1, u_1) - \lambda_i S_i(u_1, v_1)\right] - R_{M_i,\lambda_i}^{H_i(\cdot,\cdot)}\left[J_i(u_2, u_2) - \lambda_i S_i(u_2, v_2)\right]\Big\| \\
&\le \frac{1}{\alpha_i - \beta_i}\left\|\left[J_i(u_1, u_1) - \lambda_i S_i(u_1, v_1)\right] - \left[J_i(u_2, u_2) - \lambda_i S_i(u_2, v_2)\right]\right\| \\
&= \frac{1}{\alpha_i - \beta_i}\left\|\left[J_i(u_1, u_1) - J_i(u_2, u_2)\right] - \lambda_i\left[S_i(u_1, v_1) - S_i(u_2, v_2)\right]\right\| \\
&\le \frac{1}{\alpha_i - \beta_i}\left\|\left[J_i(u_1, u_1) - J_i(u_2, u_2)\right] - \lambda_i\left[S_i(u_1, v_1) - S_i(u_2, v_1)\right]\right\| \\
&\quad + \frac{\lambda_i}{\alpha_i - \beta_i}\left\|\left[S_i(u_2, v_1) - S_i(u_2, v_2)\right]\right\|.
\end{aligned} \tag{3.12}$$

By Lemma 2.4, we have

$$\begin{aligned}
&\left\|J_i(u_1, u_1) - J_i(u_2, u_2) - \lambda_i\left[S_i(u_1, v_1) - S_i(u_2, v_1)\right]\right\|^q \\
&\quad \le \|J_i(u_1, u_1) - J_i(u_2, u_2)\|^q \\
&\qquad - q\lambda_i\left\langle S_i(u_1, v_1) - S_i(u_2, v_1), J_q(J_i(u_1, u_1) - J_i(u_2, u_2))\right\rangle \\
&\qquad + c_{q,i}\lambda_i^q\|S_i(u_1, v_1) - S_i(u_2, v_1)\|^q.
\end{aligned} \tag{3.13}$$

Moreover, by (A4), we obtain

$$\begin{aligned}\|J_i(u_1,u_1)-J_i(u_2,u_2)\| &\le \|J_i(u_1,u_1)-J_i(u_2,u_1)\|+\|J_i(u_2,u_1)-J_i(u_2,u_2)\| \\ &\le r_i\|u_1-u_2\|+t_i\|u_1-u_2\| \\ &\le (r_i+t_i)\|u_1-u_2\|.\end{aligned} \tag{3.14}$$

From (A6), we have

$$-q\lambda_i\langle S_i(u_1,v_1)-S_i(u_2,v_1), J_q(J_i(u_1,u_1)-J_i(u_2,u_2))\rangle \le -q\lambda_i s_i\|u_1-u_2\|^q. \tag{3.15}$$

Moreover, from (A5), we obtain

$$\|S_i(u_1,v_1)-S_i(u_2,v_1)\| \le l_i\|u_1-u_2\|, \tag{3.16}$$

$$\|S_i(u_2,v_1)-S_i(u_2,v_2)\| \le m_i\|v_1-v_2\|. \tag{3.17}$$

From (3.13)–(3.16), we have

$$\|J_i(u_1,u_1)-J_i(u_2,u_2)-\lambda_i[S_i(u_1,v_1)-S_i(u_2,v_1)]\|^q \le \sqrt[q]{(r_i+t_i)^q-q\lambda_i s_i+c_{q,i}\lambda_i^q l_i^q}\|u_1-u_2\|. \tag{3.18}$$

It follows from (3.12), (3.17), and (3.18) that

$$\|F_i(u_1,v_1)-F_i(u_2,v_2)\| \le \frac{\sqrt[q]{(r_i+t_i)^q-q\lambda_i s_i+c_{q,i}\lambda_i^q l_i^q}}{\alpha_i-\beta_i}\|u_1-u_2\|+\frac{\lambda_i m_i}{\alpha_i-\beta_i}\|v_1-v_2\|. \tag{3.19}$$

Put

$$\theta_1^i=\frac{\sqrt[q]{(r_i+t_i)^q-q\lambda_i s_i+c_{q,i}\lambda_i^q l_i^q}}{\alpha_i-\beta_i}, \qquad \theta_2^i=\frac{\lambda_i m_i}{\alpha_i-\beta_i}. \tag{3.20}$$

Define $\|\cdot\|$ on $\underbrace{X\times\cdots\times X}_{N\text{-times}}$ by $\|(x_1,\dots,x_N)\|=\|x_1\|+\cdots\|x_N\|$ for all $(x_1,\dots,x_N)\in\underbrace{X\times\cdots\times X}_{N\text{-times}}$. It is easy to see that $(\underbrace{X\times\cdots\times X}_{N\text{-times}},\|\cdot\|)$ is a Banach space. For any given $x_1,\dots,x_N\in X$, we choose a finite sequence $w_1\in U_1(x_N),\dots,w_N\in U_N(x_1)$. Define $Q:\underbrace{X\times\cdots\times X}_{N\text{-times}}\to\underbrace{X\times\cdots\times X}_{N\text{-times}}$ by $Q(x_1,\dots,x_N)=(F_1(x_1,w_1),\dots,F_N(x_N,w_N))$. Set $k=\max\{(\theta_1^1+\theta_2^N L_N),\dots,(\theta_2^1 L_1+\theta_1^N)\}$, where $L_1,\dots,L_N$ are contraction constants of $U_1,\dots,U_N$, respectively. We note that $\theta_1^i+\theta_2^i L_i<\theta_1^i+\theta_2^i<1$, for all $i=1,2,\dots,N$, and so $k<1$. Let $x_1,\dots,x_N\in X$,

$w_1 \in U_1(x_N), \ldots, w_N \in U_N(x_1)$ and $y_1, \ldots, y_N \in X$, $z_1 \in U_1(y_N), \ldots, z_N \in U_N(y_1)$. By (A3), we get

$$
\begin{aligned}
\left\|Q(x_1,\ldots,x_N)-Q(y_1,\ldots,y_N)\right\| &= \|(F_1(x_1,w_1),\ldots,F_N(x_N,w_N)) \\
&\qquad -(F_1(y_1,z_1),\ldots,F_N(y_N,z_N))\| \\
&= \left\|F_1(x_1,w_1)-F_1(y_1,z_1)\right\| \\
&\quad +\cdots+\left\|F_N(x_N,w_N)-F_N(y_N,z_N)\right\| \\
&\leq \left(\theta_1^1\left\|x_1-y_1\right\|+\theta_2^1\|w_1-z_1\|\right) \\
&\quad +\cdots+\left(\theta_1^N\left\|x_N-y_N\right\|+\theta_2^N\|w_N-z_N\|\right) \\
&\leq \left(\theta_1^1\left\|x_1-y_1\right\|+\theta_2^1 L_1\left\|x_N-y_N\right\|\right) \\
&\quad +\cdots+\left(\theta_1^N\left\|x_N-y_N\right\|+\theta_2^N L_N\left\|x_1-y_1\right\|\right) \\
&= \left(\theta_1^1+\theta_2^N L_N\right)\left\|x_1-y_1\right\| \\
&\quad +\cdots+\left(\theta_1^N+\theta_2^1 L_1\right)\left\|x_N-y_N\right\| \\
&\leq k\left\|x_1-y_1\right\|+\cdots+k\left\|x_N-y_N\right\| \\
&= k\left(\left\|x_1-y_1\right\|+\cdots+\left\|x_N-y_N\right\|\right) \\
&= k\left\|(x_1,\ldots,x_N)-(y_1,\ldots,y_N)\right\|,
\end{aligned}
\tag{3.21}
$$

and so Q is a contraction on $\underbrace{X\times\cdots\times X}_{N\text{-times}}$. Hence there exists $a_1, \ldots, a_N \in X$, $u_1 \in U_1(a_N), \ldots, u_N \in U_N(a_1)$ such that $a_1 = F_1(a_1, u_1), \ldots, a_N = F_N(a_N, u_N)$. From Theorem 3.1, $a_1, \ldots, a_N \in X$, $u_1 \in U_1(a_N), \ldots, u_N \in U_N(a_1)$ is the solution of the problem (3.1). □

Theorem 3.4. *Let X be q-uniformly smooth real Banach space. For $i = 1, 2, \ldots, N$. Let $A_i, B_i : X \to X$ be single-valued operators, $H_i : X \times X \to X$ single-valued operator satisfy (A1) and suppose that M_i, U_i, $H_i(A_i, B_i)$, S_i, $S_i(\cdot, u)$ satisfy conditions (A2)–(A6), respectively. Then, for any $i \in \{1, 2, \ldots, N\}$, the sequences $\{a_n^i\}_{n=1}^{\infty}$ and $\{u_n^i\}_{n=1}^{\infty}$ generated by Algorithm 3.2 converge strongly to a_i, $u_i \in U_i(a_{N-(i-1)})$, respectively.*

Proof. By Theorem 3.3, the problem (3.1) has a solution $a_1, \ldots, a_N \in X$, $u_1 \in U_1(a_N), \ldots, u_N \in U_N(a_1)$. From Theorem 3.1, we note that

$$
a_i = \sigma_n a_i + (1-\sigma_n) R_{M_i,\lambda_i}^{H_i(\cdot,\cdot)}\left[H_i(A_i(a_i), B_i(a_i)) - \lambda_i S_i(a_i, u_i)\right], \tag{3.22}
$$

for all $i = 1, 2, \ldots, N$. Hence, by (3.8) and (3.22), we have

$$\begin{aligned}
\left\| a^i_{n+1} - a^i_n \right\| = \Big\| & \sigma_n a^i_n + (1-\sigma_n) R^{H_i(\cdot,\cdot)}_{M_i,\lambda_i}\left[H_i\left(A_i\left(a^i_n\right), B_i\left(a^i_n\right)\right) - \lambda_i S_i\left(a^i_n, u^i_n\right)\right] \\
& - \left[\sigma_n a^i_{n-1} + (1-\sigma_n) R^{H_i(\cdot,\cdot)}_{M_i,\lambda_i}\left[H_i\left(A_i\left(a^i_{n-1}\right), B_i\left(a^i_{n-1}\right)\right) - \lambda_i S_i\left(a^i_{n-1}, u^i_{n-1}\right)\right]\right] \Big\| \\
\leq{}& \sigma_n \left\| a^i_n - a^i_{n-1} \right\| + (1-\sigma_n) \Big\| R^{H_i(\cdot,\cdot)}_{M_i,\lambda_i}\left[H_i\left(A_i\left(a^i_n\right), B_i\left(a^i_n\right)\right) - \lambda_i S_i\left(a^i_n, u^i_n\right)\right] \\
& - R^{H_i(\cdot,\cdot)}_{M_i,\lambda_i}\Big[H_i\left(A_i\left(a^i_{n-1}\right), B_i\left(a^i_{n-1}\right)\right) \\
& - \lambda_i S_i\left(a^i_{n-1}, u^i_{n-1}\right)\Big] \Big\| \\
={}& \sigma_n \left\| a^i_n - a^i_{n-1} \right\| + (1-\sigma_n) \Big\| R^{H_i(\cdot,\cdot)}_{M_i,\lambda_i}\left[J_i\left(a^i_n, a^i_n\right) - \lambda_i S_i\left(a^i_n, u^i_n\right)\right] \\
& - R^{H_i(\cdot,\cdot)}_{M_i,\lambda_i}\left[J_i\left(a^i_{n-1}, a^i_{n-1}\right) - \lambda_i S_i\left(a^i_{n-1}, u^i_{n-1}\right)\right] \Big\| \\
\leq{}& \sigma_n \left\| a^i_n - a^i_{n-1} \right\| + (1-\sigma_n) \frac{1}{\alpha_i - \beta_i} \Big\| \left[J_i\left(a^i_n, a^i_n\right) - \lambda_i S_i\left(a^i_n, u^i_n\right)\right] \\
& - \left[J_i\left(a^i_{n-1}, a^i_{n-1}\right) - \lambda_i S_i\left(a^i_{n-1}, u^i_{n-1}\right)\right] \Big\| \\
={}& \sigma_n \left\| a^i_n - a^i_{n-1} \right\| + (1-\sigma_n) \frac{1}{\alpha_i - \beta_i} \Big\| \left[J_i\left(a^i_n, a^i_n\right) - J_i\left(a^i_{n-1}, a^i_{n-1}\right)\right] \\
& - \lambda_i \left[S_i\left(a^i_n, u^i_n\right) - S_i\left(a^i_{n-1}, u^i_{n-1}\right)\right] \Big\| \\
\leq{}& \sigma_n \left\| a^i_n - a^i_{n-1} \right\| + (1-\sigma_n) \frac{1}{\alpha_i - \beta_i} \Big\| \left[J_i\left(a^i_n, a^i_n\right) - J_i\left(a^i_{n-1}, a^i_{n-1}\right)\right] \\
& - \lambda_i \left[S_i\left(a^i_n, u^i_n\right) - S_i\left(a^i_{n-1}, u^i_n\right)\right] \Big\| \\
& + (1-\sigma_n) \frac{1}{\alpha_i - \beta_i} \left\| S_i\left(a^i_{n-1}, u^i_n\right) - S_i\left(a^i_{n-1}, u^i_{n-1}\right)\right\|.
\end{aligned} \tag{3.23}$$

By Lemma 2.4, we obtain

$$\begin{aligned}
&\left\| J_i\left(a^i_n, a^i_n\right) - J_i\left(a^i_{n-1}, a^i_{n-1}\right) - \lambda_i \left[S_i\left(a^i_n, u^i_n\right) - S_i\left(a^i_{n-1}, u^i_n\right)\right] \right\|^q \\
&\quad \leq \left\| J_i\left(a^i_n, a^i_n\right) - J_i\left(a^i_{n-1}, a^i_{n-1}\right) \right\|^q \\
&\quad - q\lambda_i \left\langle S_i\left(a^i_n, u^i_n\right) - S_i\left(a^i_{n-1}, u^i_n\right), J_{q,i}\left(J_i\left(a^i_n, a^i_n\right) - J_i\left(a^i_{n-1}, a^i_{n-1}\right)\right)\right\rangle \\
&\quad + c_{q,i} \lambda_i^q \left\| S_i\left(a^i_n, u^i_n\right) - S_i\left(a^i_{n-1}, u^i_n\right)\right\|^q.
\end{aligned} \tag{3.24}$$

From (A4), we note that

$$\begin{aligned}\left\|J_i\left(a_n^i, a_n^i\right) - J_i\left(a_{n-1}^i, a_{n-1}^i\right)\right\| &= \left\|H_i\left(A_i\left(a_n^i\right), B_i\left(a_n^i\right)\right) - H_i\left(A_i\left(a_{n-1}^i\right), B_i\left(a_{n-1}^i\right)\right)\right\| \\ &\leq \left\|H_i\left(A_i\left(a_n^i\right), B_i\left(a_n^i\right)\right) - H_i\left(A_i\left(a_{n-1}^i\right), B_i\left(a_n^i\right)\right)\right\| \\ &\quad + \left\|H_i\left(A_i\left(a_{n-1}^i\right), B_i\left(a_n^i\right)\right) - H_i\left(A_i\left(a_{n-1}^i\right), B_i\left(a_{n-1}^i\right)\right)\right\| \\ &\leq (r_i + t_i)\left\|a_n^i - a_{n-1}^i\right\|.\end{aligned} \tag{3.25}$$

From (3.24) and (A6), it follows that

$$-q\lambda_i\left\langle S_i\left(a_n^i, u_n^i\right) - S_i\left(a_{n-1}^i, u_n^i\right), J_{q,1}\left(J_i\left(a_n^i, a_n^i\right) - J_i\left(a_{n-1}^i, a_{n-1}^i\right)\right)\right\rangle \leq -q\lambda_i s_i\left\|a_n^i - a_{n-1}^i\right\|^q. \tag{3.26}$$

By (3.23), (3.24), and (A5), we have

$$\begin{aligned}\left\|S_i\left(a_{n-1}^i, u_n^i\right) - S_i\left(a_{n-1}^i, u_{n-1}^i\right)\right\| &\leq m_i\left\|u_n^i - u_{n-1}^i\right\| \\ &\leq m_i d_i\left(1 + n^{-1}\right)\left\|a_n^i - a_{n-1}^i\right\|,\end{aligned} \tag{3.27}$$

$$\left\|S_i\left(a_n^i, u_n^i\right) - S_i\left(a_{n-1}^i, u_n^i\right)\right\| \leq l_i\left\|a_n^i - a_{n-1}^i\right\|. \tag{3.28}$$

From (3.23)–(3.28), we obtain

$$\begin{aligned}&\left\|J_i\left(a_n^i, a_n^i\right) - J_i\left(a_{n-1}^i, a_{n-1}^i\right) - \lambda_i\left[S_i\left(a_n^i, u_n^i\right) - S_i\left(a_{n-1}^i, u_n^i\right)\right]\right\|^q \\ &\quad\leq \frac{\sqrt[q]{(r_i + t_i)^q - q\lambda_i s_i + c_{q,i}\lambda_i^q l_i^q}}{\alpha_i - \beta_i}\left\|a_n^i - a_{n-1}^i\right\| \\ &\qquad + \frac{\lambda_i m_i}{\alpha_i - \beta_i} d_i\left(1 + n^{-1}\right)\left\|a_n^i - a_{n-1}^i\right\|.\end{aligned} \tag{3.29}$$

Hence, by (3.23), (3.28) and (3.29), we have

$$\begin{aligned}\left\|a_{n+1}^i - a_n^i\right\| \leq \sigma_n\left\|a_n^i - a_{n-1}^i\right\| &+ (1 - \sigma_n)\frac{\sqrt[q]{(r_i + t_i)^q - q\lambda_i s_i + c_{q,i}\lambda_i^q l_i^q}}{\alpha_i - \beta_i}\left\|a_n^i - a_{n-1}^i\right\| \\ &+ (1 - \sigma_n)\frac{\lambda_i m_i}{\alpha_i - \beta_i} d_i\left(1 + n^{-1}\right)\left\|a_n^i - a_{n-1}^i\right\|.\end{aligned} \tag{3.30}$$

Put $k = \max\{\pi_1 \ldots, \pi_N\}$, where

$$\pi_i = \frac{\sqrt[q]{(r_i + t_i)^q - q\lambda_i s_i + c_{q,i}\lambda_i^q l_i^q}}{\alpha_i - \beta_i} + \frac{\lambda_i m_i d_i(1 + n^{-1})}{\alpha_i - \beta_i}. \tag{3.31}$$

It follows from (3.30) that

$$\begin{aligned}\left\|a_{n+1}^1 - a_n^1\right\| + \cdots + \left\|a_{n+1}^N - a_n^N\right\| \le \sigma_n\left\|a_n^1 - a_{n-1}^1\right\| + (1-\sigma_n)k\left\|a_n^1 - a_{n-1}^1\right\| \\ + \cdots + \sigma_n\left\|a_n^N - a_{n-1}^N\right\| + (1-\sigma_n)k\left\|a_n^N - a_{n-1}^N\right\|.\end{aligned} \tag{3.32}$$

Set $c_n = \|a_n^1 - a_{n-1}^1\| + \cdots + \|a_n^N - a_{n-1}^N\|$ and $k_n = k + (1-k)\sigma_n$. From (3.32), we obtain

$$c_{n+1} \le k_n c_n, \qquad \forall n = 0, 1, 2, \ldots. \tag{3.33}$$

Since $\limsup_{n\to\infty}\sigma_n < 1$, we have $\limsup_{n\to\infty} k_n < 1$. Thus, it follows from Lemma 2.8 that $c_{n+1} \to 0$ and hence $\lim_{n\to\infty}\|a_{n+1}^i - a_n^i\| = 0$. Therefore, $\{a_n^i\}$ is a Cauchy sequence and hence there exists $a_i \in X$ such that $a_n^i \to a_i$ as $n \to \infty$, for all $i = 1, 2, \ldots, N$. Next, we will show that $u_n^1 \to u_1 \in U_1(a_N)$ as $n \to \infty$. Hence, it follows from (3.9) that $\{u_n^1\}$ is also a Cauchy sequence. Thus there exists $u_1 \in X$ such that $u_n^1 \to u_1$ as $n \to \infty$. Consider

$$\begin{aligned} d(u_1, U_1(a_N)) &= \inf\{\|u_1 - q\| : q \in U_1(a_N)\} \\ &\le \left\|u_1 - u_n^1\right\| + d\left(u_n^1, U_1(a_N)\right) \\ &\le \left\|u_1 - u_n^1\right\| + D\left(U_1\left(a_n^N\right), U_1(a_N)\right) \\ &\le \left\|u_1 - u_n^1\right\| + d_1\left\|a_n^N - a_N\right\| \longrightarrow 0 \end{aligned} \tag{3.34}$$

as $n \to \infty$. Since $U_1(a_N)$ is a closed set and $d(u_1, U_1(a_N)) = 0$, we have $u_1 \in U_1(a_N)$. By continuing the above process, there exist $u_2 \in U_2(a_{N-1}), \ldots, u_N \in U_N(a_1)$ such that $u_n^2 \to u_2, \ldots, u_n^N \to u_N$ as $n \to \infty$. Hence, by (3.8), we obtain

$$a_i = R_{M_i,\lambda_i}^{H_i(\cdot,\cdot)}[H_i(A_i(a_i), B_i(a_i)) - \lambda_i S_i(a_i, u_i)]. \tag{3.35}$$

Therefore, it follows from Theorem 3.1 that $a_1, \ldots, a_N$ is a solution of problem (3.1).

Setting $N = 2$ in Theorem 3.3, we have the following result. □

Corollary 3.5. *Let X be q-uniformly smooth real Banach spaces. Let $A_i, B_i : X \to X$ be singled valued operators, $H_i : X \times X \to X$ a single-valued operator such that $H(A_i, B_i)$ is α_i-strongly accretive with respect to A_i, β_i-relaxed accretive with respect to B_i and $\alpha_i > \beta_i$ and suppose that $M_i : X \to 2^X$ is an $H_i(\cdot,\cdot)$-accretive set-valued mapping and $U_i : X \to C(X)$ contraction set-valued mapping with $0 \le L_i < 1$ and nonempty values, for all $i = 1, 2$. Assume that $H_i(A_i, B_i)$ is r_i-Lipschitz continuous with respect to A_i and t_i-Lipschitz continuous with respect to B_i, $S_i : X \times X \to X$ is l_i-Lipschitz continuous with respect to its first argument and m_i-Lipschitz continuous with respect* *to its second argument,* $S_1(\cdot, y)$ *is* s_1*-strongly accretive with respect to* $H_1(A_1, B_1)$*, and* $S_2(x, \cdot)$ *is* s_2*-strongly accretive with respect to* $H_2(A_2, B_2)$*, for all* $i = 1, 2$*. If*

$$\frac{\sqrt[q]{(r_i + t_i)^q - q\lambda_i s_i + c_{q,i}\lambda_i^q l_i^q}}{\alpha_i - \beta_i} + \frac{\lambda_i m_i}{\alpha_i - \beta_i} < 1, \tag{3.36}$$

for all $i \in \{1, 2\}$*, then problem* (3.2) *has a solution* $a_1, a_2 \in X$*,* $u_1 \in U_1(a_2), u_2 \in U_2(a_1)$*.*

Setting $N = 1$ in Theorem 3.3, we have the following result.

Corollary 3.6. *Let* X *be* q*-uniformly smooth real Banach spaces. Let* $A, B : X \to X$ *be two singled valued operators,* $H : X \times X \to X$ *a single-valued operator such that* $H(A, B)$ *is* α*-strongly accretive with respect to* A*,* β*-relaxed accretive with respect to* B*, and* $\alpha > \beta$ *and suppose that* $M : X \to 2^X$ *is an* $H(\cdot, \cdot)$*-accretive set-valued mapping,* $U : X \to C(X)$ *is contraction set-valued mapping with* $0 \leq L < 1$ *and nonempty values. Assume that* $H(A, B)$ *is* r*-Lipschitz continuous with respect to* A *and* t*-Lipschitz continuous with respect to* B*,* $S : X \times X \to X$ *is* l*-Lipschitz continuous with respect to its first argument and* m*-Lipschitz continuous with respect to its second argument,* $S(\cdot, y)$ *is* s*-strongly accretive with respect to* $H(A, B)$*. If*

$$\frac{\sqrt[q]{(r + t)^q - q\lambda s + c_q, \lambda^q l^q}}{\alpha - \beta} + \frac{\lambda m}{\alpha - \beta} < 1, \tag{3.37}$$

then problem (3.3) *has a solution* $a \in X$ *and* $u \in U(a)$*.*

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