Research Article

Comparison Results on Preconditioned GAOR Methods for Weighted Linear Least Squares Problems

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We present preconditioned generalized accelerated overrelaxation methods for solving weighted linear least square problems. We compare the spectral radii of the iteration matrices of the preconditioned and the original methods. The comparison results show that the preconditioned GAOR methods converge faster than the GAOR method whenever the GAOR method is convergent. Finally, we give a numerical example to confirm our theoretical results.

1. Introduction

Consider the weighted linear least squares problem

$$\min_{x \in R^n}(Ax - b)^T W^{-1}(Ax - b), \tag{1.1}$$

where W is the variance-covariance matrix. The problem has many scientific applications. A typical source is parameter estimation in mathematical modeling.

This problem has been discussed in many books and articles. In order to solve it, one has to solve a nonsingular linear system as

$$Hy = f, \tag{1.2}$$

where

$$H = \begin{pmatrix} I - B_1 & U \\ C & I - B_2 \end{pmatrix}$$ (1.3)

is an invertible matrix with

$$B_1 = (b_{ij})_{p \times p}, \qquad B_2 = (b_{ij})_{(n-p) \times (n-p)}, \qquad C = (c_{ij})_{(n-p) \times p}, \qquad U = (u_{ij})_{p \times (n-p)}.$$ (1.4)

Yuan proposed a generalized SOR (GSOR) method to solve linear system (1) in [1]; afterwards, Yuan and Jin [2] established a generalized AOR (GAOR) method to solve linear system (1). In [3, 4], authors studied the convergence of the GAOR method for solving the linear system $Hy = f$. In [3], authors studied the convergence of the GAOR method when the coefficient matrices are consistently ordered matrices and gave the regions of convergence. In [4], authors studied the convergence of the GAOR method for diagonally dominant coefficient matrices and gave the regions of convergence.

In order to solve the linear system (1.2) using the GAOR method, we split H as

$$H = I - \begin{pmatrix} 0 & 0 \\ -C & 0 \end{pmatrix} - \begin{pmatrix} B_1 & -U \\ 0 & B_2 \end{pmatrix}.$$ (1.5)

Then, for $\omega \neq 0$, one GAOR method can be defined by

$$y^{(k+1)} = L_{r,\omega} y^{(k)} + \omega g, \quad k = 0, 1, 2, \ldots,$$ (1.6)

where

$$\begin{aligned} L_{r,\omega} &= \begin{pmatrix} I & 0 \\ rC & I \end{pmatrix}^{-1} \left[(1-\omega)I + (\omega-r)\begin{pmatrix} 0 & 0 \\ -C & 0 \end{pmatrix} + \omega\begin{pmatrix} B_1 & -U \\ 0 & B_2 \end{pmatrix} \right] \\ &= \begin{pmatrix} (1-\omega)I + \omega B_1 & -\omega U \\ \omega(r-1)C - \omega r C B_1 & (1-\omega)I + \omega B_2 + \omega r C U \end{pmatrix} \end{aligned}$$ (1.7)

is the iteration matrix and

$$g = \begin{pmatrix} I & 0 \\ -rC & I \end{pmatrix} f.$$ (1.8)

In order to decrease the spectral radius of $L_{r,\omega}$, an effective method is to precondition the linear system (1.2), namely,

$$PH = \begin{pmatrix} I - B_1^* & U^* \\ C^* & I - B_2^* \end{pmatrix},$$ (1.9)

then the preconditioned GAOR method can be defined by

$$y^{(k+1)} = L^*_{r,\omega} y^{(k)} + \omega g^*, \quad k = 0, 1, 2, \ldots, \tag{1.10}$$

where

$$L^*_{r,\omega} = \begin{pmatrix} (1-\omega)I + \omega B^*_1 & -\omega U^* \\ \omega(1-r)C^* - \omega r C^* B^*_1 & (1-\omega)I + \omega B^*_2 + \omega r C^* U^* \end{pmatrix},$$

$$g^* = \begin{pmatrix} I & 0 \\ -rC^* & I \end{pmatrix} Pf. \tag{1.11}$$

In [5], authors presented three kinds of preconditioners for preconditioned modified accelerated overrelaxation method to solve systems of linear equations. They showed that the convergence rate of the preconditioned modified accelerated overrelaxation method is better than that of the original method, whenever the original method is convergent.

This paper is organized as follows. In Section 2, we give some important definition and the known results as the preliminaries of the paper. In Section 3, we propose three preconditioners and give the comparison theorems between the preconditioned and original methods. These results show that the preconditioned GAOR methods converge faster than the GAOR method whenever the GAOR method is convergent. In Section 4, we give an example to confirm our theoretical results.

2. Preliminaries

We need the following definition and results.

Definition 2.1. Let $A = (a_{ij})_{n \times n}$ and $B = (b_{ij})_{n \times n}$. We say $A \geq B$ if $a_{ij} \geq b_{ij}$ for all $i, j = 1, 2, \ldots, n$.
This definition can be carried over to vectors by identifying them with $n \times 1$ matrices.
In this paper, $\rho(\cdot)$ denotes the spectral radius of a matrix.

Lemma 2.2 (see [6]). *Let $A \in R^{n \times n}$ be nonnegative and irreducible. Then*

(i) *A has a positive real eigenvalue equal to its spectral radius $\rho(A)$;*

(ii) *for $\rho(A)$, there corresponds an eigenvector $x > 0$.*

Lemma 2.3 (see [7]). *Let $A \in R^{n \times n}$ be nonnegative and irreducible. If*

$$0 \neq \alpha x \leq Ax \leq \beta x, \quad \alpha x \neq Ax, \quad Ax \neq \beta x, \tag{2.1}$$

for some nonnegative vector x, then $\alpha < \rho(A) < \beta$ and x is a positive vector.

3. Comparison Results

We consider the preconditioned linear system

$$\widetilde{H}y = \widetilde{f}, \tag{3.1}$$

where $\widetilde{H} = (I + \widetilde{S})H$ and $\widetilde{f} = (I + \widetilde{S})f$ with

$$\widetilde{S} = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix}, \tag{3.2}$$

S is a $p \times p$ matrix with $1 < p < n$.

We take S as follows:

$$S_1 = \begin{pmatrix} 0 & \alpha_2 b_{12} & \cdots & 0 & 0 \\ \beta_2 b_{21} & 0 & \ddots & 0 & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \ddots & 0 & \alpha_p b_{p-1,p} \\ 0 & 0 & \cdots & \beta_p b_{p,p-1} & 0 \end{pmatrix}, \qquad S_2 = \begin{pmatrix} 0 & \alpha_2 b_{12} & \cdots & \alpha_p b_{1p} \\ \beta_2 b_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \beta_p b_{p1} & 0 & \cdots & 0 \end{pmatrix}. \tag{3.3}$$

Now, we obtain two preconditioned linear systems with coefficient matrices

$$\widetilde{H}_i = \begin{pmatrix} I - [B_1 - S_i(I - B_1)] & (I + S_i)U \\ C & I - B_2 \end{pmatrix}, \quad \text{for } i = 1, 2, \tag{3.4}$$

where

$B_1 - S_1(I - B_1)$

$$= \begin{pmatrix} b_{11} + \alpha_2 b_{21} b_{12} & \cdots & b_{1p} + \alpha_2 b_{2p} b_{12} \\ b_{21} + \alpha_3 b_{23} b_{31} - \beta_2 b_{21}(1 - b_{11}) & \cdots & b_{2p} + \beta_2 b_{1p} b_{21} + \alpha_3 b_{3p} b_{23} \\ \vdots & \vdots & \vdots \\ b_{p-1,1} + \beta_{p-1} b_{p-1,p-2} b_{p-2,1} + \alpha_p b_{p-1,p} b_{p1} & \cdots & b_{p-1,p} + \beta_{p-1} b_{p-1,p-2} b_{p-2,p} + \alpha_p b_{p-2,p} b_{p-1,p-2} \\ b_{p1} + \beta_p b_{p-1,1} b_{p,p-1} & \cdots & b_{pp} + \beta_p b_{p,p-1} b_{p-1,p} \end{pmatrix},$$

$B_1 - S_2(I - B_1)$

$$= \begin{pmatrix} b_{11} + \alpha_2 b_{12} b_{21} + \cdots + \alpha_p b_{1p} b_{p1} & \cdots & b_{1p} + \alpha_2 b_{12} b_{2p} + \cdots + \alpha_p b_{1p}(1 - b_{pp}) \\ b_{21} - \beta_2 b_{21}(1 - b_{11}) & \cdots & b_{2p} + \beta_2 b_{21} b_{1p} \\ \vdots & \ddots & \vdots \\ b_{p1} - \beta_p b_{p1}(1 - b_{11}) & \cdots & b_{pp} + \beta_p b_{p1} b_{1p} \end{pmatrix}. \tag{3.5}$$

We split $\widetilde{H}_i$ ($i = 1, 2$) as

$$\widetilde{H}_i = I - \begin{pmatrix} 0 & 0 \\ -C & 0 \end{pmatrix} - \begin{pmatrix} [B_1 - S_i(I - B_1)] & -(I + S_i)U \\ 0 & B_2 \end{pmatrix}, \quad \text{for } i = 1, 2, \tag{3.6}$$

then the preconditioned GAOR methods for solving (3.1) are defined as follows

$$y^{(k+1)} = L_{r,\omega}^{(i)} y^{(k)} + \omega \widetilde{g}, \quad k = 0, 1, 2, \ldots, \tag{3.7}$$

where

$$L_{r,\omega}^{(i)} = \begin{pmatrix} (1-\omega)I + \omega[B_1 - S_i(I - B_1)] & -\omega(I + S_i)U \\ \omega(r-1)C - \omega rC[B_1 - S_i(I - B_1)] & (1-\omega)I + \omega B_2 + \omega rC(I + S_i)U \end{pmatrix} \tag{3.8}$$

are iteration matrices and

$$\widetilde{g} = \begin{pmatrix} I & 0 \\ -rC & I \end{pmatrix} \widetilde{f}. \tag{3.9}$$

Now, we give comparison results between the preconditioned GAOR methods defined by (3.7) and the corresponding GAOR method defined by (1.6).

Theorem 3.1. *Let $L_{r,\omega}, L_{r,\omega}^{(1)}$ be the iteration matrices associated with the GAOR and preconditioned GAOR methods, respectively. If the matrix H in (1.2) is irreducible with $C \leq 0, U \leq 0, B_1 \geq 0, B_2 \geq 0, 0 < \omega \leq 1, 0 \leq r < 1, b_{i,i+1} > 0, b_{i+1,i} > 0$ for some $i \in \{2, \ldots, p\}$, when $0 \leq b_{ii} < 1$ $(i \in \{2, \ldots, p\})$,*

$$0 < \alpha_i < \frac{b_{i-1,i-2}b_{i-2,i} + b_{i-1,i}(1 - b_{i-2,i-2})}{b_{i-1,i-2}[(1 - b_{ii})(1 - b_{i-2,i-2}) - b_{i,i-2}b_{i-2,i}]} \quad \textit{for } i \in \{3, \ldots, p\}, \qquad \alpha_2 < \frac{1}{1 - b_{22}},$$

$$0 < \beta_i < \frac{b_{i,i-1}(1 - b_{i+1,i+1}) + b_{i-1,i}b_{i+1,i-1}}{b_{i,i-1}[(1 - b_{i-1,i-1})(1 - b_{i+1,i+1}) - b_{i-1,i+1}b_{i+1,i-1}]} \quad \textit{for } i \in \{2, \ldots, p-1\}, \qquad \beta_p < \frac{1}{1 - b_{pp}}, \tag{3.10}$$

or when $b_{ii} \geq 1, \alpha_i > 0, \beta_i > 0$ $(i \in \{2, \ldots, p\})$, then either

$$\rho\left(L_{r,\omega}^{(1)}\right) < \rho(L_{r,\omega}) < 1 \tag{3.11}$$

or

$$\rho\left(L_{r,\omega}^{(1)}\right) > \rho(L_{r,\omega}) > 1. \tag{3.12}$$

Proof. By direct operation, we have

$$L_{r,\omega} = \begin{pmatrix} (1-\omega)I + \omega B_1 & -\omega U \\ -\omega(1-r)C & (1-\omega)I + \omega B_2 \end{pmatrix} + \omega r \begin{pmatrix} 0 & 0 \\ -CB_1 & CU \end{pmatrix}. \tag{3.13}$$

Since $0 < \omega \le 1, 0 \le r < 1, C \le 0, U \le 0, B_1 \ge 0, B_2 \ge 0$, then

$$
\begin{pmatrix} 0 & 0 \\ -CB_1 & CU \end{pmatrix} \ge 0 \tag{3.14}
$$

and $L_{r,\omega}$ is nonnegative. Since H is irreducible, from (3.13), it is easy to see that the matrix $L_{r,\omega}$ is nonnegative and irreducible.

Similarly, we can prove that the matrix $L_{r,\omega}^{(1)}$ is a nonnegative and irreducible matrix.

By Lemma 2.2, there is a positive vector x such that

$$
L_{r,\omega} x = \lambda x, \tag{3.15}
$$

where $\lambda = \rho(L_{r,\omega})$. Since the matrix H is nonsingular, $\lambda \ne 1$. Hence, we get either $\lambda > 1$ or $\lambda < 1$.

Now, from (3.15) and by the definitions of $L_{r,\omega}$ and $L_{r,\omega}^{(1)}$, we have

$$
\begin{aligned}
L_{r,\omega}^{(1)} x - \lambda x &= \left(L_{r,\omega}^{(1)} - L_{r,\omega} \right) x \\
&= \begin{pmatrix} -\omega S_1(I - B_1) & -\omega S_1 U \\ \omega r C S_1(I - B_1) & \omega r C S_1 U \end{pmatrix} x \\
&= \begin{pmatrix} S_1 & 0 \\ -r C S_1 & 0 \end{pmatrix} \begin{pmatrix} -\omega(I - B_1) & -\omega U \\ 0 & 0 \end{pmatrix} x \\
&= \begin{pmatrix} S_1 & 0 \\ -r C S_1 & 0 \end{pmatrix} \begin{pmatrix} -\omega(I - B_1) & -\omega U \\ \omega(r - 1)C - \omega r C B_1 & -\omega I + \omega B_2 + \omega r C U \end{pmatrix} x \\
&= \begin{pmatrix} S_1 & 0 \\ -r C S_1 & 0 \end{pmatrix} (L_{r,\omega} - I) x = (\lambda - 1) \begin{pmatrix} S_1 & 0 \\ -r C S_1 & 0 \end{pmatrix} x.
\end{aligned} \tag{3.16}
$$

Since $b_{i,i+1} > 0, b_{i+1,i} > 0$ for some $i \in \{2, \ldots, p\}$, when $0 \le b_{ii} < 1$ $(i \in \{2, \ldots, p\})$,

$$
0 < \alpha_i < \frac{b_{i-1,i-2} b_{i-2,i} + b_{i-1,i}(1 - b_{i-2,i-2})}{b_{i-1,i-2}[(1 - b_{ii})(1 - b_{i-2,i-2}) - b_{i,i-2} b_{i-2,i}]} \quad \text{for } i \in \{3, \ldots, p\}, \qquad \alpha_2 < \frac{1}{1 - b_{22}},
$$

$$
0 < \beta_i < \frac{b_{i,i-1}(1 - b_{i+1,i+1}) + b_{i-1,i} b_{i+1,i-1}}{b_{i,i-1}[(1 - b_{i-1,i-1})(1 - b_{i+1,i+1}) - b_{i-1,i+1} b_{i+1,i-1}]} \quad \text{for } i \in \{2, \ldots, p-1\}, \qquad \beta_p < \frac{1}{1 - b_{pp}},
$$

$$
\tag{3.17}
$$

or when $b_{ii} \geq 1$, $\alpha_i > 0$, $\beta_i > 0$ ($i \in \{2, \ldots, p\}$), then $S_1 \geq 0$ and $S_1 \neq 0$. So we have $\begin{pmatrix} S_1 & 0 \\ -rCS_1 & 0 \end{pmatrix} x \geq 0$, $\begin{pmatrix} S_1 & 0 \\ -rCS_1 & 0 \end{pmatrix} x \neq 0$.

If $\lambda < 1$, then $L_{r,\omega}^{(1)} x - \lambda x \leq 0$, $L_{r,\omega}^{(1)} x - \lambda x \neq 0$.
By Lemma 2.3, the inequality (3.11) is proved.
If $\lambda > 1$, then $L_{r,\omega}^{(1)} x - \lambda x \geq 0$, $L_{r,\omega}^{(1)} x - \lambda x \neq 0$.
By Lemma 2.3, the inequality (3.12) is proved. □

Theorem 3.2. *Let $L_{r,\omega}, L_{r,\omega}^{(2)}$ be the iteration matrices associated with the GAOR and preconditioned GAOR methods, respectively. If the matrix H in (1.2) is irreducible with $C \leq 0$, $U \leq 0$, $B_1 \geq 0$, $B_2 \geq 0$, $0 < \omega \leq 1$, $0 \leq r < 1$, $b_{i,1} > 0$, $b_{1,i} > 0$ for some $i \in \{2, 3, \ldots, p\}$, when $0 \leq b_{11} < 1$, $0 < \beta_i < 1/(1 - b_{11})$,*

$$0 < \alpha_i < \frac{b_{1i} + \alpha_2 b_{12} b_{2i} + \cdots \alpha_{i-1} b_{1,i-1} b_{i-1,i} + \alpha_{i+1} b_{1,i+1} b_{i+1,i} + \cdots + \alpha_p b_{1p} b_{pi}}{b_{1i}(1 - b_{ii})}, \quad i \in \{2, 3, \ldots, p\} \tag{3.18}$$

or when $b_{11} \geq 1$, $\alpha_i > 0$, $\beta_i > 0$, $i \in \{2, 3, \ldots, p\}$, then either

$$\rho\left(L_{r,\omega}^{(2)}\right) < \rho(L_{r,\omega}) < 1 \tag{3.19}$$

or

$$\rho\left(L_{r,\omega}^{(2)}\right) > \rho(L_{r,\omega}) > 1. \tag{3.20}$$

By the analogous proof of Theorem 3.1, we can prove Theorem 3.2.

4. Numerical Example

Now, we present an example to illustrate our theoretical results.

Example 4.1. The coefficient matrix H in (1.2) is given by

$$H = \begin{pmatrix} I - B_1 & U \\ C & I - B_2 \end{pmatrix}, \tag{4.1}$$

Table 1: The spectral radii of the GAOR and preconditioned GAOR iteration matrices.

n	ω	r	p	ρ	ρ_1	ρ_2
5	0.95	0.7	3	0.1450	0.1384	0.1348
10	0.9	0.85	5	0.2782	0.2726	0.2695
15	0.95	0.8	5	0.3834	0.3808	0.3796
20	0.75	0.65	10	0.6350	0.6317	0.6297
25	0.7	0.55	8	0.7872	0.7861	0.7855
30	0.65	0.55	16	0.9145	0.9136	0.9130
40	0.6	0.5	10	1.1426	1.1433	1.1436
50	0.6	0.5	10	1.3668	1.3683	1.3691

Where $\rho = \rho(L_{r,\omega})$, $\rho_1 = \rho(L_{r,\omega}^{(1)})$, $\rho_2 = \rho(L_{r,\omega}^{(2)})$.

where $B_1 = (b_{ij}^{(1)})_{p \times p}$, $B_2 = (b_{ij}^{(2)})_{(n-p) \times (n-p)}$, $C = (c_{ij})_{(n-p) \times p}$, and $U = (u_{ij})_{p \times (n-p)}$ with

$$
\begin{aligned}
b_{ii}^{(1)} &= \frac{1}{10 \times (i+1)}, \quad i = 1, 2, \ldots, p, \\[2mm]
b_{ij}^{(1)} &= \frac{1}{30} - \frac{1}{30 \times j + i}, \quad i < j, \; i = 1, 2, \ldots, p-1, \; j = 2, \ldots, p, \\[2mm]
b_{ij}^{(1)} &= \frac{1}{30} - \frac{1}{30 \times (i-j+1) + i}, \quad i > j, \; i = 2, \ldots, p, \; j = 1, 2, \ldots, p-1, \\[2mm]
b_{ii}^{(2)} &= \frac{1}{10 \times (p+i+1)}, \quad i = 1, 2, \ldots, n-p, \\[2mm]
b_{ij}^{(2)} &= \frac{1}{30} - \frac{1}{30 \times (p+j) + p + i}, \quad i < j, \; i = 1, 2, \ldots, n-p+1, \; j = 2, \ldots, n-p, \\[2mm]
b_{ij}^{(2)} &= \frac{1}{30} - \frac{1}{30 \times (i-j+1) + p + i}, \quad i > j, \; i =, 2, \ldots, n-p, \; j = 1, 2, \ldots, n-p-1, \\[2mm]
c_{ij} &= \frac{1}{30 \times (p+i-j+1) + p + i} - \frac{1}{30}, \quad i = 1, 2, \ldots, n-p, \; j = 1, 2, \ldots, p, \\[2mm]
u_{ij} &= \frac{1}{30 \times (p+j) + i} - \frac{1}{30}, \quad i = 1, 2, \ldots, p, \; j = 1, 2, \ldots, n-p.
\end{aligned}
\tag{4.2}
$$

Table 1 displays the spectral radii of the corresponding iteration matrices with some randomly chosen parameters r, ω, p. The randomly chosen parameters α_i and β_i satisfy the conditions of two theorems.

From Table 1, we see that these results accord with Theorems 3.1–3.2.

Acknowledgments

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