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Research Article

Stochastic PDEs and Infinite Horizon Backward Doubly Stochastic Differential Equations

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We give a sufficient condition on the coefficients of a class of infinite horizon BDSDEs, under which the infinite horizon BDSDEs have a unique solution for any given square integrable terminal values. We also show continuous dependence theorem and convergence theorem for this kind of equations. A probabilistic interpretation for solutions to a class of stochastic partial differential equations is given.

1. Introduction

Pardoux and Peng [1] brought forward a new kind of backward doubly stochastic differential equations (BDSDEs in short); these equations are with two different directions of stochastic integrals, that is, the equations involve both a standard (forward) stochastic integral dW_t and a backward stochastic integral dB_t. They have proved the existence and uniqueness of solutions to BDSDEs under uniformly Lipschitz conditions on coefficients on finite time interval $[0,T]$. That is, for a given terminal time $T > 0$, under the uniformly Lipschitz assumptions on coefficients f, g, for any square integrable terminal value ξ, the following BDSDE has a unique solution pair (y_t, z_t) in the interval $[0,T]$:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)dB_s - \int_t^T Z_s dW_s, \quad t \in [0,T]. \tag{1.1}$$

Pardoux and Peng also showed that BDSDEs can produce a probabilistic representation for certain quasilinear stochastic partial differential equations (SPDEs). Many researchers do their work in this area (refer to, e.g., [2–13] and the references therein). Infinite horizon BDSDEs are also very interesting to produce a probabilistic representation of certain quasilinear stochastic partial differential equations. Recently, Zhang and Zhao [14] got stationary solutions of SPDEs and infinite horizon BDSDEs, but their researches under the assumption that terminal value $\lim_{T\to\infty} e^{-KT} Y_T = 0$. Zhu and Han [15] also give a sufficient condition on the coefficients of a class of infinite horizon BDSDEs, but there the coefficient g is independent of z.

This paper studies the existence and uniqueness of BDSDE (1.1) when $T = \infty$. Our method is different from Zhang and Zhao. Due to sufficient utilization of the properties of martingales, this method is essential to the theory of BSDEs. In this paper we give a sufficient condition on coefficients f, g under which, for any square integrable random variable ξ, BDSDE (1.1) still has a unique solution pair when $T = \infty$. Our conditions are a special kind of Lipschitz conditions, which even include some cases of unbounded coefficients. This allows us to give a probabilistic interpretation for the solutions to a class of stochastic partial differential equations (SPDEs in short).

The paper is organized as follows: in Section 2 we introduce some preliminaries and notations; in Section 3 we prove the existence and uniqueness theorem of BDSDEs; in Section 4 we discuss continuous dependence theorem and convergence theorem; at the end, we give the connection of the solutions of SPDEs and BDSDEs in Section 5.

2. Setting of Infinite Horizon BDSDEs

Notation. The Euclidean norm of a vector $x \in R^k$ will be denoted by $|x|$, and, for a $d \times k$ matrix A, we define $\|A\| = \sqrt{TrAA^*}$, where A^* is the transpose of A.

Let $(\Omega, \mathcal{F}, P)$ be a completed probability space and let $\{W_t\}_{t\geq 0}$ and $\{B_t\}_{t\geq 0}$ be two mutually independent standard Brownian motions, with values, respectively, in R^d and R^l, defined on $(\Omega, \mathcal{F}, P)$. Let $\mathcal{N}$ denote the class of P-null sets of $\mathcal{F}$. For each $t \in [0, \infty)$, we define

$$\mathcal{F}_{0,t}^W \doteq \sigma\{W_r;\, 0 \leq r \leq t\} \bigvee \mathcal{N}, \qquad \mathcal{F}_{t,\infty}^B \doteq \sigma\{B_r - B_t;\, t \leq r < \infty\} \bigvee \mathcal{N},$$
$$\mathcal{F}_{0,\infty}^W \doteq \bigvee_{0\leq t<\infty} \mathcal{F}_{0,t}^W, \qquad \mathcal{F}_{\infty,\infty}^B \doteq \bigcap_{0\leq t<\infty} \mathcal{F}_{t,\infty}^B, \tag{2.1}$$
$$\mathcal{F}_t \doteq \mathcal{F}_{0,t}^W \bigvee \mathcal{F}_{t,\infty}^B, \quad t \in [0, \infty).$$

Note that $\{\mathcal{F}_{0,t}^W; t \in [0, \infty)\}$ is an increasing filtration and $\{\mathcal{F}_{t,\infty}^B; t \in [0, \infty)\}$ is a decreasing filtration, and the collection $\{\mathcal{F}_t, t \in [0, \infty)\}$ is neither increasing nor decreasing.

Suppose

$$\mathcal{F} = \mathcal{F}_\infty \doteq \mathcal{F}_{0,\infty}^W \bigvee \mathcal{F}_{\infty,\infty}^B. \tag{2.2}$$

For any $n \in N$, let $S^2(R^+; R^n)$ denote the space of all $\{\mathcal{F}_t\}$-measurable n-dimensional processes v with norm of $\|v\|_S \doteq [E(\sup_{s\geq 0}|v(s)|)^2]^{1/2} < \infty$.

We denote similarly by $M^2(R^+; R^n)$ the space of all (classes of $dP \otimes dt$ a.e. equal) $\{\mathcal{F}_t\}$-measurable n-dimensional processes v with norm of $\|v\|_M \doteq [E\int_0^\infty |v(s)|^2 ds]^{1/2} < \infty$.

For any $t \geq 0$, let $L^2(\Omega, \mathcal{F}_t, P; R^n)$ denote the space of all $\{\mathcal{F}_t\}$-measurable n-valued random variables ξ satisfying $E|\xi|^2 < \infty$.

We also denote

$$B^2 \doteq \left\{(X,Y); X \in S^2(R^+; R^n), Y \in M^2(R^+; R^n)\right\}. \tag{2.3}$$

For each $(X,Y) \in B^2$, we define the norm of (X,Y) by

$$\|(X,Y)\|_B \doteq \left(\|X\|_S^2 + \|Y\|_M^2\right)^{1/2}. \tag{2.4}$$

Obviously B^2 is a Banach space.

Consider the following infinite horizon backward doubly stochastic differential equation:

$$y_t = \xi + \int_t^\infty f(s, y_s, z_s)ds + \int_t^\infty g(s, y_s, z_s)dB_s - \int_t^\infty z_s dW_s, \quad t \geq 0, \tag{2.5}$$

where $\xi \in L^2(\Omega, \mathcal{F}, P; R^k)$ is given. We note that the integral with respect to $\{B_t\}$ is a backward Itô integral and the integral with respect to $\{W_t\}$ is a standard forward Itô integral. These two types of integrals are particular cases of the Itô-Skorohod integral; see Boufoussi et al. [12]. Our aim is to find some conditions under which BDSDE (2.5) has a unique solution. Now we give the definition of solution of BDSDE (2.5).

Definition 2.1. A pair of processes $(y,z) : \Omega \times R^+ \to R^k \times R^{k\times d}$ is called a solution of BDSDE (2.5), if $(y,z) \in B^2$ and satisfies BDSDE (2.5).

Let

$$\begin{aligned} f &: \Omega \times R^+ \times R^k \times R^{k\times d} \longrightarrow R^k, \\ g &: \Omega \times R^+ \times R^k \times R^{k\times d} \longrightarrow R^{k\times l} \end{aligned} \tag{2.6}$$

satisfy the following assumptions:

(H1) for any $(y,z) \in R^k \times R^{k\times d}$, $f(\cdot, y, z)$ and $g(\cdot, y, z)$ are $\{\mathcal{F}_t\}$-progressively measurable processes, such that

$$E\left(\int_0^\infty f(t,0,0)dt\right)^2 < \infty; \qquad g(\cdot,0,0) \in M^2\left(R^+; R^{k\times l}\right), \tag{2.7}$$

(H2) f and g satisfy Lipschitz condition with Lipschitz coefficient $v := \{v(t)\}$; that is, there exists a positive nonrandom function $\{v(t)\}$ such that

$$\begin{aligned} \left|f(t,y_1,z_1) - f(t,y_2,z_2)\right| &\leq v(t)\left(\left|y_1 - y_2\right| + \|z_1 - z_2\|\right), \\ \left\|g(t,y_1,z_1) - g(t,y_2,z_2)\right\| &\leq v(t)\left(\left|y_1 - y_2\right| + \|z_1 - z_2\|\right), \end{aligned} \tag{2.8}$$

for all $(t, y_i, z_i) \in R^+ \times R^k \times R^{k\times d}$, $i = 1,2$,

(H3) $\int_0^\infty v^2(t)dt < \infty$.

3. Existence and Uniqueness Theorem

The following existence and uniqueness theorem is our main result.

Theorem 3.1. *Under the above conditions, in particular (H1), (H2), and (H3),* (2.5) *has unique solution* $(y, z) \in B^2$.

In order to prove the existence and uniqueness theorem, one first gives an a priori estimate.

Lemma 3.2. *Suppose (H1), (H2), and (H3) hold for f and g. For any $T \in [0,\infty]$, let $Y_T^i \in L^2(\Omega, \mathcal{F}_T, P; R^k)$, (Y^i, Z^i) and $(y^i, z^i) \in B^2$ $(i = 1,2)$ satisfy the following equation:*

$$Y_t^i = Y_T^i + \int_t^T f\left(s, y_s^i, z_s^i\right) ds + \int_t^T g\left(s, y_s^i, z_s^i\right) dB_s - \int_t^T Z_s^i dW_s, \quad 0 \le t \le T \le \infty. \tag{3.1}$$

Then there is a constant $C > 0$, such that, for any $\tau \in [0, T]$,

$$\begin{aligned} &\left\| \left(\left(Y^1 - Y^2\right) I_{[\tau,T]}, \left(Z^1 - Z^2\right) I_{[\tau,T]} \right) \right\|_B^2 \\ &\quad \le C \left[E\left| Y_T^1 - Y_T^2 \right|^2 + l_{[\tau,T]} \left\| \left(\left(y^1 - y^2\right) I_{[\tau,T]}, \left(z^1 - z^2\right) I_{[\tau,T]} \right) \right\|_B^2 \right], \end{aligned} \tag{3.2}$$

where $l_{[\tau,T]} = \int_\tau^T v^2(s) ds$ and $I_{[\tau,T]}(\cdot)$ is an indicator function.

Proof. Firstly, we assume that $\tau = 0$, $T = \infty$.

Set

$$\begin{aligned} &\widehat{Y}_t = Y_t^1 - Y_t^2, \qquad \widehat{Z}_t = Z_t^1 - Z_t^2, \qquad \widehat{y}_t = y_t^1 - y_t^2, \qquad \widehat{z}_t = z_t^1 - z_t^2, \\ &\widehat{f}_t = f\left(t, y_t^1, z_t^1\right) - f\left(t, y_t^2, z_t^2\right), \qquad \widehat{g}_t = g\left(t, y_t^1, z_t^1\right) - g\left(t, y_t^2, z_t^2\right). \end{aligned} \tag{3.3}$$

Then

$$\widehat{Y}_t = \widehat{Y}_\infty + \int_t^\infty \widehat{f}_s ds + \int_t^\infty \widehat{g}_s dB_s - \int_t^\infty \widehat{Z}_s dW_s. \tag{3.4}$$

We define the filtration $\{\mathcal{G}_t\}_{0 \le t \le T}$ by

$$\mathcal{G}_t \doteq \mathcal{F}_{0,t}^W \bigvee \mathcal{F}_{0,\infty}^B. \tag{3.5}$$

Obviously $\mathcal{G}_t$ is an increasing filtration. Since $(\widehat{Y}, \widehat{Z}) \in B^2$, $\{\int_0^t \widehat{Z}_s dW_s\}$ is a $\mathcal{G}_t$-martingale, thus from (3.4) it follows that

$$\widehat{Y}_t = E^{\mathcal{G}_t}\left[\widehat{Y}_\infty + \int_t^\infty \widehat{f}_s ds + \int_t^\infty \widehat{g}_s dB_s\right]. \tag{3.6}$$

Note that

$$\begin{aligned}
E\left(\int_0^\infty |\widehat{f}_s| ds\right)^2 &\le E\left(\int_0^\infty \left(v(s)\left|\widehat{y}_s\right| + v(s)\|\widehat{z}_s\|\right) ds\right)^2 \\
&\le 2E\left(\sup_{t\ge 0}\left|\widehat{y}_t\right|^2\right)\cdot\int_0^\infty v^2(s)ds + 2E\left(\int_0^\infty v^2(s)ds\cdot\int_0^\infty \|\widehat{z}_s\|^2 ds\right) \\
&= 2\int_0^\infty v^2(s)ds\cdot\left\|(\widehat{y},\widehat{z})\right\|_B^2 < \infty, \\
E\int_0^\infty \|\widehat{g}_s\|^2 ds &\le E\int_0^\infty \left(v(s)\left|\widehat{y}_s\right| + v(s)\|\widehat{z}_s\|\right)^2 ds \\
&\le 2\int_0^\infty v^2(s)ds\cdot\left\|(\widehat{y},\widehat{z})\right\|_B^2 < \infty.
\end{aligned} \tag{3.7}$$

Applying Doob inequality and B-D-G inequality, we can deduce

$$\begin{aligned}
\left\|\widehat{Y}\right\|_S^2 &= E\left(\sup_{t\ge 0}\left|\widehat{Y}_t\right|\right)^2 \\
&= E\left(\sup_{t\ge 0}\left|E^{\mathcal{G}_t}\left[\widehat{Y}_\infty + \int_t^\infty \widehat{f}_s ds + \int_t^\infty \widehat{g}_s dB_s\right]\right|\right)^2 \\
&\le 2E\left(\sup_{t\ge 0} E^{\mathcal{G}_t}\left[\left|\widehat{Y}_\infty\right| + \int_t^\infty \left|\widehat{f}_s\right| ds\right]\right)^2 + 2E\left(\sup_{t\ge 0} E^{\mathcal{G}_t}\left[\left|\int_t^\infty \widehat{g}_s dB_s\right|\right]\right)^2 \\
&\le 8E\left(\left|\widehat{Y}_\infty\right| + \int_0^\infty \left|\widehat{f}_s\right| ds\right)^2 + 2c_0 E\int_0^\infty \|\widehat{g}_s\|^2 ds \\
&\le 16\left(E\left|\widehat{Y}_\infty\right|^2 + E\left(\int_0^\infty \left|\widehat{f}_s\right| ds\right)^2\right) + 2c_0 E\int_0^\infty \|\widehat{g}_s\|^2 ds \\
&\le (16 + 2c_0)\left(E\left|\widehat{Y}_\infty\right|^2 + E\left(\int_0^\infty \left|\widehat{f}_s\right| ds\right)^2 + E\int_0^\infty \|\widehat{g}_s\|^2 ds\right),
\end{aligned} \tag{3.8}$$

where $c_0 > 0$ is a constant.

On the other hand, from (3.4) it follows that

$$\begin{aligned}
\left\|\widehat{Z}\right\|_M^2 &= E\left\langle \int_0^{\cdot} \widehat{Z}_s dW_s \right\rangle_\infty \\
&= E\left(\widehat{Y}_\infty + \int_0^\infty \widehat{f}_s ds + \int_0^\infty \widehat{g}_s dB_s\right)^2 - \left[E\left(\widehat{Y}_\infty + \int_0^\infty \widehat{f}_s ds + \int_0^\infty \widehat{g}_s dB_s\right)\right]^2 \\
&\le E\left(\widehat{Y}_\infty + \int_0^\infty \widehat{f}_s ds + \int_0^\infty \widehat{g}_s dB_s\right)^2 \\
&\le 3E\left(\left|\widehat{Y}_\infty\right|^2 + \left(\int_0^\infty \left|\widehat{f}_s\right| ds\right)^2 + \int_0^\infty \|\widehat{g}_s\|^2 ds\right),
\end{aligned} \tag{3.9}$$

where $\langle M \rangle$ is the variation process generated by the martingale M.

Consequently, (3.8) and (3.9) imply that

$$\begin{aligned}
\left\|\left(\widehat{Y}, \widehat{Z}\right)\right\|_B^2 &= \left\|\widehat{Y}\right\|_S^2 + \left\|\widehat{Z}\right\|_M^2 \\
&\le (19 + 2c_0)\left(E\left|\widehat{Y}_\infty\right|^2 + E\left(\int_0^\infty \left|\widehat{f}_s\right| ds\right)^2 + E\int_0^\infty \|\widehat{g}_s\|^2 ds\right) \\
&\le (57 + 6c_0)\left(E\left|\widehat{Y}_\infty\right|^2 + l_{[0,\infty]} \|(\widehat{y}, \widehat{z})\|_B^2\right) \\
&= C\left(E\left|\widehat{Y}_\infty\right|^2 + l_{[0,\infty]} \|(\widehat{y}, \widehat{z})\|_B^2\right),
\end{aligned} \tag{3.10}$$

where $C = (57 + 6c_0)$ is a constant, and $l_{[0,\infty]} = \int_0^\infty v^2(s) ds$.

For any $\tau, T \in [0, \infty]$, we set $f_1(t, y_t, z_t) = f(t, y_t, z_t) I_{[\tau,T]}$, and $g_1(t, y_t, z_t) = g(t, y_t, z_t) I_{[\tau,T]}$. Then f_1, and g_1 satisfy the assumptions (H1), (H2), and (H3), and their Lipschitz constants are $v I_{[\tau,T]}$.

Obviously,

$$\widehat{Y}_t I_{[\tau,T]} = \widehat{Y}_T + \int_t^T \widehat{f}_s I_{[\tau,T]} ds + \int_t^T \widehat{g}_s I_{[\tau,T]} dB_s - \int_t^T \widehat{Z}_s I_{[\tau,T]} dW_s. \tag{3.11}$$

Since $(\widehat{Y} I_{[\tau,T]}, \widehat{Z} I_{[\tau,T]}) \in B^2$, $\{\int_0^t \widehat{Z}_s I_{[\tau,T]} dW_s\}$ is a $\mathcal{G}_t$-martingale, thus from (3.11) it follows that

$$\widehat{Y}_t I_{[\tau,T]} = E^{\mathcal{G}_t}\left[\widehat{Y}_T + \int_t^T \widehat{f}_s I_{[\tau,T]} ds + \int_t^T \widehat{g}_s I_{[\tau,T]} dB_s\right]. \tag{3.12}$$

Note that

$$
\begin{aligned}
E\left(\int_0^T \left|\widehat{f}_s I_{[\tau,T]}\right| ds\right)^2 &\le E\left(\int_0^T \left(v(s) I_{[\tau,T]} \left|\widehat{y}_s\right| + v(s) I_{[\tau,T]} \|\widehat{z}_s\|\right) ds\right)^2 \\
&\le 2E\left(\sup_{t\ge 0}\left|\widehat{y}_t I_{[\tau,T]}\right|^2\right) \cdot \int_0^T v^2(s) I_{[\tau,T]} ds \\
&\quad + 2E\left(\int_0^T v^2(s) I_{[\tau,T]} ds \cdot \int_0^T \left\|\widehat{z}_s I_{[\tau,T]}\right\|^2 ds\right) \\
&= 2\int_\tau^T v^2(s) ds \cdot \left\|\left(\widehat{y} I_{[\tau,T]}, \widehat{z} I_{[\tau,T]}\right)\right\|_B^2 < \infty, \\
E\int_0^T \left\|\widehat{g}_s I_{[\tau,T]}\right\|^2 ds &\le E\int_0^T \left(v(s) I_{[\tau,T]} \left|\widehat{y}_s\right| + v(s) I_{[\tau,T]} \|\widehat{z}_s\|\right)^2 ds \\
&\le 2\int_\tau^T v^2(s) ds \cdot \left\|\left(\widehat{y} I_{[\tau,T]}, \widehat{z} I_{[\tau,T]}\right)\right\|_B^2 < \infty.
\end{aligned}
\tag{3.13}
$$

Applying Doob inequality and B-D-G inequality, we can deduce

$$
\begin{aligned}
\left\|\widehat{Y} I_{[\tau,T]}\right\|_S^2 &= E\left(\sup_{t\ge 0}\left|\widehat{Y}_t I_{[\tau,T]}\right|\right)^2 \\
&\le 2E\left(\sup_{t\ge 0} E^{\mathcal{G}_t}\left[\left|\widehat{Y}_T\right| + \int_t^T \left|\widehat{f}_s I_{[\tau,T]}\right| ds\right]\right)^2 + 2E\left(\sup_{t\ge 0} E^{\mathcal{G}_t}\left[\left|\int_t^T \widehat{g}_s I_{[\tau,T]} dB_s\right|\right]\right)^2 \\
&\le 8E\left(\left|\widehat{Y}_T\right| + \int_0^T \left|\widehat{f}_s I_{[\tau,T]}\right| ds\right)^2 + 2c_0 E\int_0^T \left\|\widehat{g}_s I_{[\tau,T]}\right\|^2 ds \\
&\le 16\left(E\left|\widehat{Y}_T\right|^2 + E\left(\int_0^T \left|\widehat{f}_s I_{[\tau,T]}\right| ds\right)^2\right) + 2c_0 E\int_0^T \left\|\widehat{g}_s I_{[\tau,T]}\right\|^2 ds \\
&\le (16 + 2c_0)\left(E\left|\widehat{Y}_T\right|^2 + E\left(\int_0^T \left|\widehat{f}_s I_{[\tau,T]}\right| ds\right)^2 + E\int_0^T \left\|\widehat{g}_s I_{[\tau,T]}\right\|^2 ds\right),
\end{aligned}
\tag{3.14}
$$

where $c_0 > 0$ is a constant.

On the other hand, from (3.11) it follows that

$$\begin{aligned}
\left\|\widehat{Z}I_{[\tau,T]}\right\|_M^2 &= E\left\langle\int_0^{\cdot}\widehat{Z}_sI_{[\tau,T]}dW_s\right\rangle_\infty \\
&= E\left(\widehat{Y}_T+\int_0^T\widehat{f}_sI_{[\tau,T]}ds+\int_0^T\widehat{g}_sI_{[\tau,T]}dB_s\right)^2 \\
&\quad-\left[E\left(\widehat{Y}_T+\int_0^T\widehat{f}_sI_{[\tau,T]}ds+\int_0^T\widehat{g}_sI_{[\tau,T]}dB_s\right)\right]^2 \\
&\le E\left(\widehat{Y}_T+\int_0^T\widehat{f}_sI_{[\tau,T]}ds+\int_0^T\widehat{g}_sdI_{[\tau,T]}B_s\right)^2 \\
&\le 3E\left(\left|\widehat{Y}_T\right|^2+\left(\int_0^T\left|\widehat{f}_sI_{[\tau,T]}\right|ds\right)^2+\int_0^T\left\|\widehat{g}_sI_{[\tau,T]}\right\|^2ds\right).
\end{aligned} \tag{3.15}$$

Consequently, (3.14) and (3.15) imply that

$$\begin{aligned}
\left\|(\widehat{Y}I_{[\tau,T]},\widehat{Z}I_{[\tau,T]})\right\|_B^2 &= \left\|\widehat{Y}I_{[\tau,T]}\right\|_S^2+\left\|\widehat{Z}I_{[\tau,T]}\right\|_M^2 \\
&\le (19+2c_0)\left(E\left|\widehat{Y}_T\right|^2+E\left(\int_0^T\left|\widehat{f}_sI_{[\tau,T]}\right|ds\right)^2+E\int_0^T\left\|\widehat{g}_sI_{[\tau,T]}\right\|^2ds\right) \\
&\le (57+6c_0)\left(E\left|\widehat{Y}_T\right|^2+l_{[\tau,T]}\left\|(\widehat{y}I_{[\tau,T]},\widehat{z}I_{[\tau,T]})\right\|_B^2\right) \\
&= C\left(E\left|\widehat{Y}_T\right|^2+l_{[\tau,T]}\left\|(\widehat{y}I_{[\tau,T]},\widehat{z}I_{[\tau,T]})\right\|_B^2\right),
\end{aligned} \tag{3.16}$$

where $C=(57+6c_0)$ is a constant, and $l_{[\tau,T]}=\int_\tau^T v^2(s)ds$. □

Martingale Representation Theorem [4]

Suppose Y is a random variable, such that $E|Y^2|<\infty$. Note that $M_t=E[Y\mid\mathcal{G}_t]$ is a square integrable martingale with respect to $\mathcal{G}_t$ and can be represented using martingale representation theorem as $M_t=M_0+\int_0^t Z_sdW_s$, where $E\int_0^\infty\|Z_t\|^2dt<\infty$.

Now we give the proof of the Theorem 3.1.

Proof. The proof of Theorem 3.1 is divided into two steps.

Step 1. We assume $l_{[0,\infty]}=\int_0^\infty v^2(s)ds\le 1/2C$. For any $(y,z)\in B^2$, let

$$M_t\doteq E^{\mathcal{G}_t}\left[\xi+\int_0^\infty f(s,y_s,z_s)ds+\int_0^\infty g(s,y_s,z_s)dB_s\right],\quad t\ge 0. \tag{3.17}$$

We will prove $\{M_t\}$ is a square integrable $\mathcal{G}_t$-martingale. From (H1)–(H3), it follows that

$$\begin{aligned}
&E\left(\left|\xi+\int_0^\infty f(s,y_s,z_s)ds+\int_0^\infty g(s,y_s,z_s)dB_s\right|^2\right)\\
&\le E\left(|\xi|+\int_0^\infty |f(s,y_s,z_s)|ds+\left|\int_0^\infty g(s,y_s,z_s)dB_s\right|\right)^2\\
&\le 3E|\xi|^2+9E\left(\int_0^\infty |f(s,0,0)|ds\right)^2+9E\left(\int_0^\infty v(s)|y_s|ds\right)^2\\
&\quad+9E\left(\int_0^\infty v(s)\|z_s\|ds\right)^2+9E\int_0^\infty\left(\|g(s,0,0)\|^2+v^2(s)|y_s|^2+v^2(s)\|z_s\|^2\right)ds\\
&\le 3E|\xi|^2+9E\left(\int_0^\infty |f(s,0,0)|ds\right)^2+9\int_0^\infty v^2(s)ds\cdot\|y\|_S^2\\
&\quad+9\int_0^\infty v^2(s)ds\cdot\|z\|_M^2+9E\int_0^\infty\|g(s,0,0)\|^2ds\\
&\quad+9\int_0^\infty v^2(s)ds\cdot\|y\|_S^2+9\int_0^\infty v^2(s)ds\cdot\|z\|_M^2\\
&<\infty,
\end{aligned}\tag{3.18}$$

which means $\{M_t\}$ is a square integrable $\mathcal{G}_t$-martingale. According to the martingale representation theorem, there exists a unique $\mathcal{G}_t$-progressively measurable process Z_t with value in $R^{k\times d}$ such that

$$\begin{gathered}
E\int_0^\infty\|Z_t\|^2dt<\infty,\\
M_t=E^{\mathcal{G}_0}\left[\xi+\int_0^\infty f(s,y_s,z_s)ds+\int_0^\infty g(s,y_s,z_s)dB_s\right]+\int_0^t Z_s dW_s,\quad 0\le t\le\infty.
\end{gathered}\tag{3.19}$$

Let

$$Y_t\doteq E^{\mathcal{G}_t}\left[\xi+\int_t^\infty f(s,y_s,z_s)ds+\int_t^\infty g(s,y_s,z_s)dB_s\right],\quad 0\le t\le\infty.\tag{3.20}$$

So

$$\begin{aligned}
M_t&=E^{\mathcal{G}_t}\left[\xi+\int_0^\infty f(s,y_s,z_s)ds+\int_0^\infty g(s,y_s,z_s)dB_s\right]\\
&=E^{\mathcal{G}_t}\left[\xi+\int_t^\infty f(s,y_s,z_s)ds+\int_t^\infty g(s,y_s,z_s)dB_s\right]\\
&\quad+\int_0^t f(s,y_s,z_s)ds+\int_0^t g(s,y_s,z_s)dB_s
\end{aligned}$$

$$
\begin{aligned}
&= Y_t + \int_0^t f(s, y_s, z_s)ds + \int_0^t g(s, y_s, z_s)dB_s \\
&= E^{\mathcal{G}_0}\left[\xi + \int_0^\infty f(s, y_s, z_s)ds + \int_0^\infty g(s, y_s, z_s)dB_s\right] + \int_0^t Z_s dW_s.
\end{aligned} \tag{3.21}
$$

Then

$$
Y_t = \xi + \int_t^\infty f(s, y_s, z_s)ds + \int_t^\infty g(s, y_s, z_s)dB_s - \int_t^\infty Z_s dW_s, \quad t \geq 0. \tag{3.22}
$$

We show that $\{Y_t\}$ and $\{Z_t\}$ are in fact $\mathcal{F}_t$-measurable. For Y_t, this is obvious since, for each t,

$$
Y_t = E^{\mathcal{G}_t}\left[\xi + \int_t^\infty f(s, y_s, z_s)ds + \int_t^\infty g(s, y_s, z_s)dB_s\right] = E\left(\frac{\Theta}{\mathcal{F}_t \bigvee \mathcal{F}_{0,t}^B}\right), \tag{3.23}
$$

where $\Theta = \xi + \int_t^\infty f(s, y_s, z_s)ds + \int_t^\infty g(s, y_s, z_s)dB_s$ is indeed $\mathcal{F}_{0,\infty}^W \bigvee \mathcal{F}_{t,\infty}^B$-measurable. Hence $\mathcal{F}_{0,t}^B$ is independent of $\mathcal{F}_t \bigvee \sigma(\Theta)$, and

$$
Y_t = E\left(\frac{\Theta}{\mathcal{F}_t}\right). \tag{3.24}
$$

Now

$$
\int_t^\infty Z_s dW_s = \xi + \int_t^\infty f(s, y_s, z_s)ds + \int_t^\infty g(s, y_s, z_s)dB_s - Y_t \tag{3.25}
$$

and the right side is $\mathcal{F}_{0,\infty}^W \bigvee \mathcal{F}_{t,\infty}^B$-measurable. Hence, from Itô's martingale representation theorem, $\{Z_s, s > t\}$ is $\mathcal{F}_{0,s}^W \bigvee \mathcal{F}_{t,\infty}^B$-adapted. Consequently Z_s is $\mathcal{F}_{0,s}^W \bigvee \mathcal{F}_{t,\infty}^B$-measurable, for any $t < s$, and, thus, Z_t is $\mathcal{F}_t$-measurable. So $(Y, Z) \in B^2$. Therefore (3.22) has constructed a mapping from B^2 to B^2, and we denote it by ϕ, that is,

$$
\phi : (y, z) \longrightarrow (Y, Z). \tag{3.26}
$$

If ϕ is a contractive mapping with respect to the norm $\|\cdot\|_B$, by the fixed point theorem, there exists a unique $(y, z) \in B^2$, satisfying (3.22), which is just the unique solution to BDSDE (2.5).

Now we are in the position to prove that ϕ is a contractive mapping. Supposing that $(y^i, z^i) \in B^2$, let (Y^i, Z^i) be the map ϕ of (y^i, z^i), $(i = 1, 2)$, that is

$$
\phi\left(y^i, z^i\right) = \left(Y^i, Z^i\right), \quad i = 1, 2. \tag{3.27}
$$

We denote

$$\widehat{Y} = Y^1 - Y^2, \qquad \widehat{Z} = Z^1 - Z^2, \qquad \widehat{y} = y^1 - y^2, \qquad \widehat{z} = z^1 - z^2,$$
$$\widehat{f}_t = f\left(t, y^1, z^1\right) - f\left(t, y^2, z^2\right), \qquad \widehat{g}_t = g\left(t, y^1, z^1\right) - g\left(t, y^2, z^2\right). \tag{3.28}$$

By Lemma 3.2, we have

$$\left\|\phi(y^1, z^1) - \phi(y^2, z^2)\right\|_B^2 = \left\|(\widehat{Y}, \widehat{Z})\right\|_B^2 \le C l_{[0,\infty]} \|(\widehat{y}, \widehat{z})\|_B^2. \tag{3.29}$$

Due to $l_{[0,\infty]} \le 1/2C$, it follows that ϕ is a contractive mapping from B^2 to B^2.

Step 2. Since $\int_0^\infty v^2(t)dt < \infty$, then there exists a sufficiently large constant T such that

$$\int_T^\infty v^2(s)ds \le \frac{1}{2C}. \tag{3.30}$$

Let

$$f_1(t, y, z) \doteq I_{[T,\infty]}(t) f(t, y, z), \qquad g_1(t, y, z) \doteq I_{[T,\infty]}(t) g(t, y, z), \tag{3.31}$$

then (H1)–(H3) hold for f_1 and g_1, whose Lipschitz coefficients are $\overline{v}(t) = I_{[T,\infty]} v(t)$. Obviously,

$$\int_0^\infty \overline{v}^2(s)ds \le \frac{1}{2C}. \tag{3.32}$$

By Step 1, there exists a unique $(\widetilde{y}, \widetilde{z}) \in B^2$ satisfying

$$\widetilde{y}_t = \xi + \int_t^\infty f_1(s, \widetilde{y}_s, \widetilde{z}_s)ds + \int_t^\infty g_1(s, \widetilde{y}_s, \widetilde{z}_s)dB_s - \int_t^\infty \widetilde{z}_s dW_s, \quad 0 \le t \le \infty. \tag{3.33}$$

For $(\widetilde{y}_t, \widetilde{z}_t)$ given as above, let us consider the following infinite BDSDE:

$$\overline{y}_t = \int_t^T f(s, \overline{y}_s + \widetilde{y}_s, \overline{z}_s + \widetilde{z}_s)ds + \int_t^T g(s, \overline{y}_s + \widetilde{y}_s, \overline{z}_s + \widetilde{z}_s)dB_s - \int_t^T \overline{z}_s dW_s, \quad 0 \le t \le T,$$
$$\overline{y}_t \equiv 0, \quad \overline{z}_t \equiv 0, \quad t > T. \tag{3.34}$$

According to the results of Pardoux and Peng [1], the above BDSDE has a unique solution $(\overline{y}, \overline{z})$ in $[0, T]$, thus the above BDSDE has a unique solution such that $(\overline{y}, \overline{z}) \equiv (0, 0)$ for every $t > T$. Let

$$y \doteq \overline{y} + \widetilde{y}, \qquad z \doteq \overline{z} + \widetilde{z}. \tag{3.35}$$

It is easy to check that (y_t, z_t) is the unique solution of (2.5). □

Remark 3.3. Suppose v is a constant, if we choose $v(t) = vI_{[0,T]}(t)$, then Theorem 3.1 is the main theorem in the paper by Pardoux and Peng [1].

Remark 3.4. The condition (H3) is usually necessary. That is, if for any $\xi \in L^2(\Omega, \mathcal{F}, P; R^k)$ and f, g hold in (H1) and (H2), BDSDE (2.5) has a unique solution in B^2, then the (H3) is necessary.

In fact, let us choose $f(s, y_s, z_s) = (1/(1+s))z_s$, $g(s, y_s, z_s) = 0$ and any $\xi \in L^2(\Omega, \mathcal{F}, P; R^k)$, then the solution of BDSDE

$$y_t = \xi + \int_t^\infty \frac{1}{1+s} z_s ds - \int_t^\infty z_s dW_s \tag{3.36}$$

should be

$$\begin{gathered} y_t = E\left[\xi \exp \frac{\left(-(1/2)\int_t^\infty (1/(1+s))^2 ds + \int_t^\infty (1/(1+s))dW_t\right)}{\mathcal{F}_t}\right], \\ z_t = \frac{d\langle y_t, W_t\rangle}{dt}, \end{gathered} \tag{3.37}$$

where $\langle y_t, W_t\rangle$ is the variation process generated by the semimartingale y_t and Brownian motion W_t.

Thus the assumption (H3) is necessary.

Remark 3.5. The following example shows that if the coefficients f and g of BDSDE (2.5) satisfy the uniformly Lipschitz, the BDSDE (2.5) has no solution.

For all $T > 0$, let $\xi_T = \int_0^T (1/(1+s))dW_s$, then the BDSDE $y_t = \xi + \int_t^T v|z_s|ds - \int_t^T z_s dW_s$ has a unique solution pair (y_t^T, z_t^T),

$$\begin{gathered} y_t^T = \begin{cases} \xi, & t > T, \\ \displaystyle\int_0^t \frac{1}{1+s} dW_s + \int_t^T \frac{v}{1+s} ds, & t \le T, \end{cases} \\ z_t^T = \begin{cases} 0, & t > T, \\ \dfrac{1}{1+t}, & t \le T. \end{cases} \end{gathered} \tag{3.38}$$

When $T \to \infty$, $\xi_T \to \int_0^\infty (1/(1+s))dW_s$ and $z_t^T \to 1/(1+t)$ in $L^2(\Omega, \mathcal{F}, P)$, but $z_t = 1/(1+t)$ is not the solution of the following infinite horizon BDSDE:

$$y_t = \int_0^\infty \frac{1}{1+s} dW_s + \int_t^\infty v|z_s|ds - \int_t^\infty z_s\, dW_s \tag{3.39}$$

because $\int_0^\infty (v/(1+s))ds = \infty$.

4. Continuous Dependence Theorem

In this section we will discuss the convergence of solutions of infinite horizon BDSDEs. First we give the following continuous dependence theorem.

Theorem 4.1. *Suppose $\xi_i \in L^2(\Omega, \mathcal{F}, P; R^k)$, $(i = 1, 2)$, and consider (H1)–(H3). Let (y^i, z^i) be the solutions of BDSDE* (2.5) *corresponding to the terminal data $\xi = \xi_1$, $\xi = \xi_2$, respectively. Then there exists a constant $\overline{C} > 0$ such that*

$$\left\| \left(y^1 - y^2, z^1 - z^2 \right) \right\|_B^2 \le \overline{C} E |\xi_1 - \xi_2|^2. \tag{4.1}$$

Proof. Set $\widehat{y} := y^1 - y^2$, $\widehat{z} := z^1 - z^2$. Since $\int_0^\infty v^2(s)ds < \infty$, we can choose a strictly increasing sequence $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = \infty$ such that

$$l_{[t_i, t_{i+1}]} = \int_{t_i}^{t_{i+1}} v^2(s)ds \le \frac{1}{2C}, \quad i = 0, 1, \dots, n. \tag{4.2}$$

Applying Lemma 3.2, we have

$$\begin{aligned} \left\| (\widehat{y}, \widehat{z}) I_{[t_i, t_{i+1}]} \right\|_B^2 &\le C E \left| \widehat{y}_{t_{i+1}} \right|^2 + C l_{[t_i, t_{i+1}]} \left\| (\widehat{y}, \widehat{z}) I_{[t_i, t_{i+1}]} \right\|_B^2 \\ &\le C E \left| \widehat{y}_{t_{i+1}} \right|^2 + \frac{1}{2} \left\| (\widehat{y}, \widehat{z}) I_{[t_i, t_{i+1}]} \right\|_B^2. \end{aligned} \tag{4.3}$$

Thus

$$\begin{aligned} \left\| (\widehat{y}, \widehat{z}) I_{[t_i, t_{i+1}]} \right\|_B^2 &\le 2CE \left| \widehat{y}_{t_{i+1}} \right|^2 \\ &\le 2CE \left(\left(\sup_{t_{i+1} \le s \le t_{i+2}} \left| \widehat{y}_s \right| \right)^2 + \int_{t_{i+1}}^{t_{i+2}} \|\widehat{z}_s\|^2 ds \right) \\ &= 2C \left\| (\widehat{y}, \widehat{z}) I_{[t_{i+1}, t_{i+2}]} \right\|_B^2, \quad i = 0, 1, \dots, n-1. \end{aligned} \tag{4.4}$$

In particular, we have

$$\left\| (\widehat{y}, \widehat{z}) I_{[t_n, \infty]} \right\|_B^2 \le 2CE |\xi_1 - \xi_2|^2. \tag{4.5}$$

From (4.4) and (4.5), it follows that

$$\begin{aligned} \left\| (\widehat{y}, \widehat{z}) \right\|_B^2 &\le \sum_{i=0}^n \left\| (\widehat{y}, \widehat{z}) I_{[t_i, t_{i+1}]} \right\|_B^2 \\ &\le \left(2C + (2C)^2 + \cdots + (2C)^{n+1} \right) E |\xi_1 - \xi_2|^2 \\ &= \frac{2C \left((2C)^{n+1} - 1 \right)}{2C - 1} E |\xi_1 - \xi_2|^2 = \overline{C} E |\xi_1 - \xi_2|^2. \end{aligned} \tag{4.6}$$

Thus the desired result is obtained. □

Now we can assert the following convergence theorem for infinite horizon BDSDEs.

Theorem 4.2. *Suppose* $\xi, \xi_i \in L^2(\Omega, \mathcal{F}, P; R^k)$, $(i = 1, 2, \ldots)$, *(H1)–(H3) hold for* f *and* g. *Let* (y^i, z^i) *be the solutions of the following BDSDE:*

$$y_t^i = \xi_i + \int_t^\infty f\left(s, y_s^i, z_s^i\right) ds + \int_t^\infty g\left(s, y_s^i, z_s^i\right) dB_s - \int_t^\infty z_s^i dW_s, \quad t \geq 0. \tag{4.7}$$

If $E|\xi_i - \xi|^2 \to 0$ *as* $i \to \infty$, *then there exists a pair* $(y, z) \in B^2$ *such that* $\|(y^i - y, z^i - z)\|_B \to 0$ *as* $i \to \infty$. *Furthermore,* (y, z) *is the solution of the following BDSDE:*

$$y_t = \xi + \int_t^\infty f(s, y_s, z_s) ds + \int_t^\infty g(s, y_s, z_s) dB_s - \int_t^\infty z_s dW_s, \quad 0 \leq t < \infty. \tag{4.8}$$

Proof. For any $n, m \geq 1$, let (y^n, z^n) and (y^m, z^m) be the solutions of (4.7) corresponding to ξ_n and ξ_m, respectively. Due to Theorem 4.1, there exists a constant $\overline{C} > 0$ such that

$$\begin{aligned} \|(y^n - y^m, z^n - z^m)\|_B^2 &\leq \overline{C} E|\xi_n - \xi_m|^2 \\ &\leq 2\overline{C}\left(E|\xi_n - \xi|^2 + E(\xi_m - \xi)^2\right) \longrightarrow 0, \quad \text{as } n, m \longrightarrow \infty, \end{aligned} \tag{4.9}$$

which means that $\{(y^i, z^i),\ i = 1, 2, \ldots\}$ is a Cauchy sequence in B^2. Thus there exists a pair $(y, z) \in B^2$ such that $\|(y^i - y, z^i - z)\|_B \to 0$ as $i \to \infty$. Since

$$\begin{aligned} &E\left|\int_t^\infty \left(f\left(s, y_s^i, z_s^i\right) - f(s, y_s, z_s)\right) ds\right|^2 \\ &\quad \leq E\left(\int_0^\infty \left(v(s)\left|y_s^i - y_s\right| + v(s)\left\|z_s^i - z_s\right\|\right) ds\right)^2 \\ &\quad \leq 2\int_0^\infty v^2(s) ds \cdot \left\|\left(y^i - y, z^i - z\right)\right\|_B^2 \longrightarrow 0, \quad \text{as } i \longrightarrow \infty, \\ &E\left|\int_t^\infty \left(g\left(s, y_s^i, z_s^i\right) - g(s, y_s, z_s)\right) ds\right|^2 \\ &\quad \leq 2\int_0^\infty v^2(s) ds \cdot \left\|\left(y^i - y, z^i - z\right)\right\|_B^2 \longrightarrow 0, \quad \text{as } i \longrightarrow \infty. \end{aligned} \tag{4.10}$$

Thus for any $t \in R^+$, $\int_t^\infty f(s, y_s^i, z_s^i) ds \to \int_t^\infty f(s, y_s, z_s) ds$ and $\int_t^\infty g(s, y_s^i, z_s^i) dB_s \to \int_t^\infty g(s, y_s, z_s) dB_s$ in $L^2(\Omega, \mathcal{F}, P)$. Taking the limit on both sides of (4.7), we deduce that (y, z) is the solution to BDSDE (4.8). The desired result is obtained. □

The following corollary shows the relation between the solution of infinite horizon BDSDE (2.5) and the following finite time BDSDE:

$$y_t = E^{\mathcal{F}_T}[\xi] + \int_t^T f(s, y_s, z_s)ds + \int_t^T g(s, y_s, z_s)dB_s - \int_t^T z_s dW_s, \quad 0 \le t \le T < \infty. \tag{4.11}$$

Corollary 4.3. *Assume $\xi \in L^2(\Omega, \mathcal{F}, P; R^k)$, (H1)–(H3) hold for f and g. Let (y, z) be the solution of BDSDE* (2.5)*. For any $T > 0$, let (y^T, z^T) be the solutions of the finite time interval BDSDE* (4.11)*, then$(y^T, z^T) \to (y, z)$ in B^2 as $T \to \infty$.*

Proof. Note that $E^{\mathcal{F}_T}[\xi] \to \xi$ in $L^2(\Omega, \mathcal{F}, P; R^k)$ as $T \to \infty$. The proof is straightforward from Theorem 4.2. □

5. BDSDEs and Systems of Quasilinear SPDEs

In this section, we study the link between BDSDEs and the solution of a class of SPDEs.

Let us first give some notations. $C^k(R^p; R^q)$, $C^k_{l,b}(R^p; R^q)$, $C^k_p(R^p; R^q)$ will denote, respectively, the set of functions of classes from R^p into R^q, the set of those functions of class C^k whose partial derivatives of order less than or equal to k are bounded (and hence the function itself grows at most linearly at infinity), and the set of those functions of class C^k which, together with all their partial derivatives of order less than or equal to k, grow at most like a polynomial function of the variable x at infinity.

For $s \ge t$, let $X_s^{t,x}$ be a diffusion process given by the solution of

$$X_s^{t,x} = x + \int_t^s b\left(X_\mu^{t,x}\right)d\mu + \int_t^s \sigma\left(X_\mu^{t,x}\right)dW_\mu, \tag{5.1}$$

where $b \in C^3_{l,b}(R^d; R^d)$, $\sigma \in C^3_{l,b}(R^d; R^d \times R^d)$, and, for $0 \le s < t$, we regulate $X_s^{t,x} = x$.

It is well known that the solution defines a stochastic flow of diffeomorphism $X_s^{t,\cdot} : R^d \to R^d$ and denotes by $\widehat{X}_s^{t,\cdot}$ the inverse flow (see e.g., [15]). The random field $X_s^{t,x}$; $s \ge t$, $x \in R^d$ has a version which is a.s. of class C^2 in x, the function and its derivatives being a.s. continuous with respect to (t, s, x).

Now the coefficients of the BDSDE will be of the form (with an obvious abuse of notations):

$$f(s, y, z) = f\left(s, X_s^{t,x}, y, z\right); \qquad g(s, y, z) = g\left(s, X_s^{t,x}, y, z\right), \tag{5.2}$$

where $f : [0, T] \times R^d \times R^k \times R^{k\times d} \to R^k$; $g : [0, T] \times R^d \times R^k \times R^{k\times d} \to R^{k\times l}$.

We assume that for any $s \ge t$, $(x, y, z) \to (f(s, x, y, z), g(s, x, y, z))$ is of class C^3, and all derivatives are bounded on $[0, \infty) \times R^d \times R^k \times R^{k\times d}$.

We assume again that (H1), (H2), and (H3) hold, then the following BDSDE has a unique solution:

$$Y_s^{t,x} = \xi + \int_s^{\infty} f\left(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}\right) dr + \int_s^{\infty} g\left(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}\right) dB_r - \int_s^{\infty} Z_r^{t,x} dW_r, \quad t \geq 0. \tag{5.3}$$

Let $(Y_s^{t,x}, Z_s^{t,x})$ denote the unique solution of (4.11). We shall define $X_s^{t,x}, Y_s^{t,x}$, and $Z_s^{t,x}$ for all $(s,t) \in [0,\infty) \times [0,\infty)$ by letting $X_s^{t,x} = X_{s\vee t}^{t,x}$, $Y_s^{t,x} = Y_{s\vee t}^{t,x}$, and $Z_s^{t,x} = 0$ for $s < t$.

We now relate our BDSDE to the following system of quasilinear backward stochastic partial differential equations:

$$\begin{aligned} d\kappa(t,x) = & -\left[\mathcal{L}\kappa(t,x) + f(x, \kappa(t,x), \sigma^*(x) D\kappa(t,x))\right] dt \\ & - g(x, \kappa(t,x), \sigma^*(x) D\kappa(t,x)) dB_t, \quad t \geq 0. \end{aligned} \tag{5.4}$$

$\mathcal{L}$ is the infinitesimal generator of a diffusion process $X_s^{t,x}$ (solution of (5.1)) given by

$$\mathcal{L} = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x) \frac{\partial}{\partial x_i}, \tag{5.5}$$

where $(a_{i,j}(x)) = \sigma\sigma^*(x)$.

Theorem 5.1. *Let $\kappa(t,x)$; $t \geq 0$, $x \in R^d$ be a random field such that $\kappa(t,x)$ is $\mathcal{F}_{t,\infty}^B$-measurable for each (t,x), $\kappa \in C^{0,2}([0,\infty) \times R^d; R^k)$ a.s., and κ satisfies* (5.4). *Then $\kappa(t,x) = Y_t^{t,x}$, where $(Y_s^{t,x}, Z_s^{t,x}; s \geq t)_{t \geq 0}$ solves the BDSDE* (5.3).

Proof. We can apply the extension of the Itô formula [5] to the solution κ of (5.4):

$$\begin{aligned} d\kappa\left(s, X_s^{t,x}\right) = {} & \mathcal{L}\kappa\left(s, X_s^{t,x}\right) ds + \sigma^* D\kappa\left(s, X_s^{t,x}\right) dW_s \\ & - \left[\mathcal{L}\left(\kappa\left(s, X_s^{t,x}\right)\right) + f\left(s, X_s^{t,x}, \kappa\left(s, X_s^{t,x}\right), \sigma^* D\kappa\left(s, X_s^{t,x}\right)\right)\right] ds \\ & - g\left(s, X_s^{t,x}, \kappa\left(s, X_s^{t,x}\right), \sigma^* D\kappa\left(s, X_s^{t,x}\right)\right) dB_s. \end{aligned} \tag{5.6}$$

We can see that $(\kappa(s, X_s^{t,x}), \sigma^* D\kappa(s, X_s^{t,x}))$ coincides with the unique solution of (5.3). It follows that $\kappa(t,x) = Y_t^{t,x}$. □

We have also a converse to Theorem 5.1.

Theorem 5.2. *Let f and g satisfy (H1), (H2), and (H3). Then $\{\kappa(t,x) = Y_t^{t,x}; t \geq 0, x \in R^d\}$ is the unique classical solution of the system of backward SPDEs* (5.3).

We can finish the proof exactly as in Theorem 3.2 of Hu and Ren [13].

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References

[1] E. Pardoux and S. Peng, "Backward doubly stochastic differential equations and systems of quasilinear SPDEs," *Probability Theory and Related Fields*, vol. 98, no. 2, pp. 209–227, 1994.
[2] V. Bally and A. Matoussi, "Weak solutions for SPDEs and backward doubly stochastic differential equations," *Journal of Theoretical Probability*, vol. 14, no. 1, pp. 125–164, 2001.
[3] R. Buckdahn and J. Ma, "Stochastic viscosity solutions for nonlinear stochastic partial differential equations. I," *Stochastic Processes and their Applications*, vol. 93, no. 2, pp. 181–204, 2001.
[4] R. Buckdahn and J. Ma, "Stochastic viscosity solutions for nonlinear stochastic partial differential equations. II," *Stochastic Processes and their Applications*, vol. 93, no. 2, pp. 205–228, 2001.
[5] E. Pardoux, *Stochastic Partial Differential Equations*, Fudan lecture notes, 2007.
[6] S. Peng and Y. Shi, "A type of time-symmetric forward-backward stochastic differential equations," *Comptes Rendus de l'Académie des Sciences—Series I*, vol. 336, no. 9, pp. 773–778, 2003.
[7] H. Kunita, *Stochastic Flows and Stochastic Differential Equations*, vol. 24 of *Cambridge Studies in Advanced Mathematics*, Cambridge University Press, Cambridge, UK, 1990.
[8] Y. Ren, A. Lin, and L. Hu, "Stochastic PDIEs and backward doubly stochastic differential equations driven by Lévy processes," *Journal of Computational and Applied Mathematics*, vol. 223, no. 2, pp. 901–907, 2009.
[9] Q. Zhu, Y. Shi, and X. Gong, "Solutions to general forward-backward doubly stochastic differential equations," *Applied Mathematics and Mechanics*, vol. 30, no. 4, pp. 517–526, 2009.
[10] B. Zhu and B. Y. Han, "Backward doubly stochastic differential equations with non-Lipschitz coefficients," *Acta Mathematica Scientia Series A*, vol. 28, no. 5, pp. 977–984, 2008.
[11] B. Zhu and B. Han, "Comparison theorems for the multidimensional BDSDEs and applications," *Journal of Applied Mathematics*, vol. 2012, Article ID 304781, 14 pages, 2012.
[12] B. Boufoussi, J. van Casteren, and N. Mrhardy, "Generalized backward doubly stochastic differential equations and SPDEs with nonlinear Neumann boundary conditions," *Bernoulli*, vol. 13, no. 2, pp. 423–446, 2007.
[13] L. Hu and Y. Ren, "Stochastic PDIEs with nonlinear Neumann boundary conditions and generalized backward doubly stochastic differential equations driven by Lévy processes," *Journal of Computational and Applied Mathematics*, vol. 229, no. 1, pp. 230–239, 2009.
[14] Q. Zhang and H. Zhao, "Stationary solutions of SPDEs and infinite horizon BDSDEs," *Journal of Functional Analysis*, vol. 252, no. 1, pp. 171–219, 2007.
[15] B. Zhu and B. Han, "Backward doubly stochastic differential equations with infinite time horizon," *Applications of Mathematics*, vol. 57, no. 6, pp. 641–653, 2012.