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Research Article

Existence and Uniqueness of Homoclinic Solution for a Class of Nonlinear Second-Order Differential Equations

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The authors study the existence and uniqueness of a set with $2kT$-periodic solutions for a class of second-order differential equations by using Mawhin's continuation theorem and some analysis methods, and then a unique homoclinic orbit is obtained as a limit point of the above set of $2kT$-periodic solutions.

1. Introduction

In this paper, we study the existence and uniqueness of homoclinic solutions for the following nonlinear second-order differential equations:

$$u''(t) + g\left(u'(t)\right) + h(u(t)) = f(t), \tag{1.1}$$

where $u(t) \in R$, g, h and f are all in $C(R, R)$.

As usual we say that a nonzero solution $u(t)$ of (1.1) is homoclinic (to 0) if $u(t) \to 0$ and $u'(t) \to 0$ as $|t| \to +\infty$.

Equation (1.1) is important in the applied sciences such as nonlinear vibration of masses, see [1–3] and the references therein. But most of the authors in those papers are interested in the study of problems of periodic solutions. Recently, the existence of homoclinic solutions for some second-order ordinary differential equation (system) has been extensively studied by using critical point theory, see [4–13] and the references therein. For example, in [9], by using the Mountain Pass theorem, Lv et al. discussed the existence of homoclinic solutions for the following second-order Hamiltonian systems:

$$u''(t) - L(t)u(t) + \nabla w(t, u(t)) = 0, \tag{1.2}$$

and in [13], the authors by means of variational method studied the problem of homoclinic solutions for the forced pendulum equation without the first derivative term. But, as far as we know, there were few papers studying the existence of homoclinic solution for the equation such as (1.1). This is due to the fact that (1.1) contains the first derivative term $g(u'(t))$. This implies that the differential equation is not the Euler Lagrange equation associated with some functional $I : W_{2kT}^{1,p} \to R$. So the method of critical point theory (or variational method) in [4–13] cannot be applied directly. Although paper [13] discussed the existence of homoclinic solutions for the following equation containing the first derivative term:

$$x''(t) + f(t)x'(t) + \beta(t)x(t) + g(t, x(t)) = 0, \tag{1.3}$$

the term containing the first derivative is only linear with respect to $x'(t)$.

In order to investigate the homoclinic solutions to (1.1), firstly, we study the existence of $2kT$-periodic solutions to the following equation for each $k \in \mathbf{N}$:

$$u''(t) + g\big(u'(t)\big) + h(u(t)) = f_k(t), \tag{1.4}$$

where $f_k : R \to R$ is a $2kT$-periodic function such that

$$f_k(t) = \begin{cases} f(t), & t \in [-kT, kT - \varepsilon_0] \\ f(kT - \varepsilon_0) + \dfrac{f(-kT) - f(kT - \varepsilon_0)}{\varepsilon_0}(t - kT + \varepsilon_0), & t \in [kT - \varepsilon_0, kT], \end{cases} \tag{1.5}$$

$T > 0$ is a given constant, and $\varepsilon_0 \in (0, T)$ is a constant independent of k. Then a homoclinic solution to (1.1) is obtained as a limit point of the set $\{u_k(t)\}$, where $u_k(t)$ is an arbitrary $2kT$-periodic solution to (1.4) for each $k \in \mathbf{N}$.

The significance of present paper is that we not only investigate the existence of homoclinic solution to (1.1), but also study the uniqueness of the homoclinic solution and, the existence of $2kT$-periodic solutions to (1.4) is obtained by using Mawhin's continuation theorem [14], not by using the methods of critical point theory, which is quite different from the approaches of [4–13, 15]. Furthermore, the method to obtain the homoclinic solution to (1.1) is also different from the corresponding ones of [15].

2. Main Lemmas

For each $k \in \mathbf{N}$, let $C_{2kT} = \{x \mid x \in C(R, R), x(t + 2kT) \equiv x(t)\}$, $C_{2kT}^1 = \{x \mid x \in C^1(R, R), x(t + 2kT) \equiv x(t)\}$, the norms of C_{2kT} and C_{2kT}^1 are defined by $\|\cdot\|_\infty = \max_{t\in[-kT,kT]}|x(t)|$ and $\|x\|_{C_{2kT}^1} = \max\{\|x\|_\infty, \|x'\|_\infty\}$, respectively, then C_{2kT} and C_{2kT}^1 are all Banach spaces. Furthermore for $x \in C_{2kT}$, $\|x\|_r = \left(\int_{-kT}^{kT} |x(t)|^r dt\right)^{1/r}$, where $r \in (1, +\infty)$.

Lemma 2.1 (see [12]). *Let $a > 0$ and $q \in W^{1,p}(R,R)$, then for every $t \in R$, the following inequality holds:*

$$|q(t)| \leq (2a)^{-1/\mu}\left(\int_{t-a}^{t+a}|q(s)|^{\mu}ds\right)^{1/\mu} + a(2a)^{-1/p}\left(\int_{t-a}^{t+a}|q'(s)|^{p}ds\right)^{1/p}, \tag{2.1}$$

where $\mu, p \in (1,+\infty)$ are constants.

Lemma 2.2 (see [12]). *Let $q \in W_{2kT}^{1,p}(R,R^n)$, then the following inequality holds:*

$$\|q\|_{\infty} \leq T^{-1/\nu}\left(\int_{-kT}^{kT}|q(s)|^{\nu}ds\right)^{1/\nu} + T^{(p-1)/p}\left(\int_{-kT}^{kT}|q'(s)|^{p}ds\right)^{1/p}, \tag{2.2}$$

where ν and p are constants with $\nu > 1$ and $p > 1$.

In order to use Mawhin's continuation theorem for investigating the existence of $2kT$-periodic solutions to (1.4), we give some definitions associated with Mawhin's continuation theorem.

Definition 2.3 (see [14]). Let X and Y be two Banach spaces with norms $\|x\|_X$ and $\|x\|_Y$, respectively. A linear operator

$$L : D(L) \subset X \longrightarrow Y \tag{2.3}$$

is said to be a Fredholm opeartor with index zero provided that

(1) Im L is a closed subset of Y;

(2) $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < \infty$.

If $L : D(L) \subset X \to Y$ is a Fredholm operator with index zero, then $X = \ker L \oplus X_1$ and $Y = \operatorname{Im} L \oplus Y_1$. Let $P : X \to \ker L$ and $Q : Y \to Y_1$ be the continuous projectors. Clearly, $\ker L \cap (D(L) \cap X_1) = \{0\}$, thus the restriction $L_P := L|_{D(L)\cap X_1}$ is invertible. Denote by K_P the inverse of L_P.

Definition 2.4 (see [14]). Let X and Y be two Banach spaces with norms $\|x\|_X$ and $\|x\|_Y$, respectively, and the operator

$$L : D(L) \subset X \longrightarrow Y \tag{2.4}$$

is a Fredholm operator with index zero, $\Omega \subset X$ is an open bounded set with $D(L) \cap \Omega \neq \phi$. A continuous operator $N : \overline{\Omega} \subset X \to Y$ is said to be L-compact in $\overline{\Omega}$, provided that

(1) $K_P(I-Q)N(\overline{\Omega})$ is a relative compact set of X;

(2) $QN(\overline{\Omega})$ is a bounded set of Y.

Lemma 2.5 (see [14]). *Suppose that X and Y are two Banach spaces, and $L : D(L) \subset X \to Y$ is a Fredholm operator with index zero. Furthermore, $\Omega \subset X$ is an open bounded subset and $N : \overline{\Omega} \to Y$ is L-compact on $\overline{\Omega}$. If all the following conditions hold:*

(1) $Lx \neq \lambda Nx$, *for all* $x \in \partial\Omega \cap D(L), \lambda \in (0,1)$;

(2) $Nx \notin \operatorname{Im} L$, *for all* $x \in \partial\Omega \cap \operatorname{Ker} L$;

(3) $\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$,

where $J : \operatorname{Im} Q \to \operatorname{Ker} L$ is an isomorphism. Then equation $Lx = Nx$ has a solution on $\overline{\Omega} \cap D(L)$.

Lemma 2.6. *Assume that there are positive constants m, m_1, n, l_0, and l_1 with $l_0 \geq l_1$, such that the following conditions hold.*

(A1) $\sup_{t\in R}|f(t)| < +\infty$, $\int_R |f(t)|^{(l_0+1)/l_0} dt < +\infty$, $\int_R |f(t)|^{(l_1+1)/l_1} dt < +\infty$ *and* $\int_R |f(t)|^2 dt < +\infty$.

(A2)

$$\begin{gathered}
-m_1|x|^{l_0+1} \leq xg(x) \leq -m|x|^{l_0+1}, \quad \forall x \in R, \\
xh(x) \leq -n|x|^{l_1+1}, \quad \forall x \in R.
\end{gathered} \tag{2.5}$$

(A3) $h \in C^1(R,R)$ *with* $h'(x) \leq 0$ *for all* $x \in R$.

Then for every $k \in \mathbf{N}$, (1.4) possesses a $2kT$-periodic solution.

Remark 2.7. From (1.5), we see

$$\begin{aligned}
\|f_k\|_\infty &\leq \sup_{t\in R}|f(t)| < +\infty, \quad \forall k \in \mathbf{N}, \\
\|f_k\|_{(l_0+1)/l_0} &= \left(\int_{-kT}^{kT} |f_k(s)|^{(l_0+1)/l_0} ds\right)^{l_0/(l_0+1)} \\
&\leq \left(\int_{-kT}^{kT-\varepsilon_0} |f_k(s)|^{(l_0+1)/l_0} ds\right)^{l_0/(l_0+1)} + \left(\int_{kT-\varepsilon_0}^{kT} |f_k(s)|^{(l_0+1)/l_0} ds\right)^{l_0/(l_0+1)} \\
&\leq \left(\int_R |f(s)|^{(l_0+1)/l_0} ds\right)^{l_0/(l_0+1)} + \varepsilon_0^{l_0/(l_0+1)} \sup_{t\in R}|f(t)|, \quad \forall k \in \mathbf{N},
\end{aligned} \tag{2.6}$$

which together with assumption (A1) yields that $\|f_k\|_\infty$ and $\|f_k\|_{(l_0+1)/l_0}$ are two constants independent of $k \in \mathbf{N}$.

Similarly, we have that $\|f_k\|_{(l_1+1)/l_1} < +\infty$ and $\|f_k\|_2 < +\infty$ are two constants independent of $k \in \mathbf{N}$.

Proof. Set $X = C^1_{2kT}$, $Y = C_{2kT}$, $L : D(L) \subset X \to Y$, $Lu = u''$, where $D(L) = \{u \mid u \in C^2_{2kT}\}$, and

$$N : C^1_{2kT} \longrightarrow C_{2kT}, \qquad [Nu](t) = -g(u'(t)) - h(u(t)) + f_k(t). \tag{2.7}$$

Clearly, Ker $L = R$, Im $L = \{y \in Y : \int_{-kT}^{kT} y(s)ds = 0\}$, which implies that Im L is a closed subset of X, and dim Ker L = codim Im $L = 1 < +\infty$. So L is a Fredholm operator with index zero. Let

$$P : X \longrightarrow \text{Ker } L, \quad Q : Y \to Y/ \text{ Im } L \tag{2.8}$$

be defined respectively by $Px = (1/2kT)\int_0^{2kT} x(s)ds, Qx = (1/2kT)\int_0^{2kT} x(s)ds$ and let

$$L_P = L|_{X\cap \text{Ker } P} : X \cap \text{ Ker } P \longrightarrow \text{Im } L. \tag{2.9}$$

Then L_P has a unique continuous pseudo-inverse L_P^{-1} on Im L defined by $(L_P^{-1}y)(t) = [Fy](t)$, where

$$\begin{gathered}
[Fy](t) = \int_0^{2kT} G(t,s)y(s)ds, \\
G(t,s) = \begin{cases} \dfrac{s(2kT-t)}{2kT}, & 0 \le s < t, \\ \dfrac{t(2kT-s)}{2kT}, & t \le s \le 2kT. \end{cases}
\end{gathered} \tag{2.10}$$

For each open bounded set $\Omega \subset C_{2kT}$, from the above formula, it is easy to see that the mapper N is L-compact on $\overline{\Omega}$.

Step 1. For each $k \in \mathbf{N}$, let $\Omega_1 = \{x \in C^1_{2kT} : Lx = \lambda Nx, \lambda \in (0,1)\}$, that is,

$$\Omega_1 = \left\{x \in C^1_{2kT} : x''(t) + \lambda g(x'(t)) + \lambda h(x(t)) = \lambda f_k(t), \lambda \in (0,1)\right\}. \tag{2.11}$$

We will show that Ω_1 is bounded in C^1_{2kT}. Suppose that $u \in \Omega_1$, then

$$u''(t) + \lambda g(u'(t)) + \lambda h(u(t)) = \lambda f_k(t), \quad \lambda \in (0,1). \tag{2.12}$$

Multiplying both sides of (2.12) by $u'(t)$ and integrating on the interval $[-kT, kT]$, we have from assumption (A2) that

$$m\int_{-kT}^{kT} |u'(t)|^{l_0+1}dt \le -\int_{-kT}^{kT} u'(t)g(u'(t))dt = -\int_{-kT}^{kT} f_k(t)u'(t)dt. \tag{2.13}$$

By using hólder inequality, we get

$$\|u'\|_{l_0+1}^{l_0+1} \le \frac{1}{m}\|f_k\|_{(l_0+1)/l_0} \cdot \|u'\|_{l_0+1}, \tag{2.14}$$

which together with the conclusion of Remark 2.7 shows

$$\|u'\|_{l_0+1} \le \left(\frac{1}{m}\|f_k\|_{(l_0+1)/l_0}\right)^{1/l_0} := \beta_1. \tag{2.15}$$

Clearly, β_1 is a constant independent of k and λ.

Multiplying both sides of (2.12) by $u''(t)$ and integrating on the interval $[-kT, kT]$, we have

$$\|u''\|_2^2 - \lambda\int_{-kT}^{kT}(u'(t))^2 h'(u(t))dt = \lambda\int_{-kT}^{kT} f_k(t)u''(t)dt. \tag{2.16}$$

It follows from (2.16) and assumption (A3) that

$$\|u''\|_2^2 \le \int_{-kT}^{kT}\left|f_k(t)u''(t)\right|dt \le \|f_k\|_2 \cdot \|u''\|_2, \tag{2.17}$$

which implies

$$\|u''\|_2 \le \|f_k\|_2 := \beta_2. \tag{2.18}$$

By using Lemma 2.2, we have

$$\begin{aligned}\|u'\|_\infty &\le T^{-1/(l_0+1)}\|u'\|_{l_0+1} + T^{1/2}\|u''\|_2 \\ &\le T^{-1/(l_0+1)}\beta_1 + T^{1/2}\beta_2 \\ &:= \beta.\end{aligned} \tag{2.19}$$

Clearly, β is a constant independent of k and λ.

On the other hand, multiplying both sides of (2.12) by $u(t)$ and integrating on the interval $[-kT, kT]$, we have

$$-\int_{-kT}^{kT}\left|u'(t)\right|^2 dt + \lambda\int_{-kT}^{kT} h(u(t))u(t)dt = -\lambda\int_{-kT}^{kT} g\left(u'(t)\right)u(t)dt + \lambda\int_{-kT}^{kT} f_k(t)u(t)dt. \tag{2.20}$$

It follows from assumption (A2) that

$$\lambda n\int_{-kT}^{kT}(u(t))^{l_1+1}dt + \int_{-kT}^{kT}(u'(t))^2 dt \le \lambda m_1\int_{-kT}^{kT}\left|u'(t)\right|^{l_0}|u(t)|dt + \lambda\int_{-kT}^{kT}\left|f_k(t)\right||u(t)|dt, \tag{2.21}$$

which together with (2.19) and $l_0 \geq l_1$ results in

$$\begin{aligned}
n\int_{-kT}^{kT}|u(t)|^{l_1+1}dt &\leq m_1\int_{-kT}^{kT}\left|u'(t)\right|^{l_0}\cdot|u(t)|dt+\int_{-kT}^{kT}\left|f_k(t)u(t)\right|dt \\
&\leq m_1\left(\int_{-kT}^{kT}\left|u'(t)\right|^{l_0(l_1+1)/l_1}dt\right)^{l_1/(l_1+1)}\cdot\left(\int_{-kT}^{kT}|u(t)|^{l_1+1}dt\right)^{1/(l_1+1)} \\
&\quad+\left(\int_{-kT}^{kT}\left|f_k(t)\right|^{(l_1+1)/l_1}dt\right)^{l_1/(l_1+1)}\cdot\left(\int_{-kT}^{kT}|u(t)|^{l_1+1}dt\right)^{1/(l_1+1)} \\
&= m_1\left(\int_{-kT}^{kT}\left|u'(t)\right|^{l_0+1}\left|u'(t)\right|^{(l_0-l_1)/l_1}dt\right)^{l_1/(l_1+1)}\cdot\|u\|_{l_1+1}+\left\|f_k\right\|_{(l_1+1)/l_1}\|u\|_{l_1+1} \\
&\leq m_1\beta^{(l_0-1)/(l_1+1)}\left(\int_{-kT}^{kT}\left|u'(t)\right|^{l_0+1}dt\right)^{l_1/(l_1+1)}\|u\|_{l_1+1}+\left\|f_k\right\|_{(l_1+1)/l_1}\|u\|_{l_1+1} \\
&= m_1\beta^{(l_0-l_1)/(l_1+1)}\left(\left\|u'\right\|_{l_0+1}\right)^{l_1(l_0+1)/(l_1+1)}\|u\|_{l_1+1}+\left\|f_k\right\|_{(l_1+1)/l_1}\|u\|_{l_1+1} \\
&\leq m_1\beta^{(l_0-1)/(l_1+1)}\beta_1^{l_1(l_0+1)/(l_1+1)}\|u\|_{l_1+1}+\left\|f_k\right\|_{(l_1+1)/l_1}\|u\|_{l_1+1}
\end{aligned} \tag{2.22}$$

that is,

$$\|u\|_{l_1+1}^{l_1+1} \leq \frac{1}{n}\left[m_1\beta^{(l_0-l_1)/(l_1+1)}\beta_1^{l_1(l_0+1)/(l_1+1)}+\left\|f_k\right\|_{(l_1+1)/l_1}\right]\|u\|_{l_1+1}. \tag{2.23}$$

Therefore

$$\|u\|_{l_1+1} \leq \left(\frac{1}{n}\right)^{1/l_1}\left[m_1\beta^{(l_0-l_1)/(l_1+1)}\beta_1^{l_1(l_0+1)/(l_1+1)}+\left\|f_k\right\|_{(l_1+1)/l_1}\right]^{1/l_1} := \alpha_1, \tag{2.24}$$

where α_1 is a constant independent of k and λ. By using Lemma 2.2 again, we get

$$\begin{aligned}
\|u\|_\infty &\leq T^{-1/(l_1+1)}\|u\|_{l_1+1}+T^{l_0/(l_0+1)}\left\|u'\right\|_{l_0+1} \\
&\leq T^{-1/(l_1+1)}\alpha_1+T^{l_0/(l_0+1)}\beta_1 := \alpha.
\end{aligned} \tag{2.25}$$

Obviously, α is a constant independent of k and λ. Therefore, if $u \in \Omega_1$, then by (2.19) we see that

$$\|u\|_{C_{2kT}^1} = \max\left\{\|u\|_\infty, \left\|u'\right\|_\infty\right\} \leq \max\{\alpha, \beta\}+\sup_{t\in R}\left|f(t)\right| := \widetilde{M}. \tag{2.26}$$

Clearly, $\widetilde{M} > 0$ is a constant independent of k and λ; that is, Ω_1 is uniformly bounded for all $k \in \mathbf{N}$ and $\lambda \in (0,1)$.

Step 2. From assumptions (A2) and (A3), we see that there must be a constant $M_1 > 0$ such that $-h(M_1) + \overline{f_k} > 0$ and $-h(-M_1) + \overline{f_k} < 0$. Set $\Omega_2 = \{u(t) \in C^1_{2kT} : \|u\|_{C^1_{2kT}} < M\}$, where $M = \max\{M_1, \widetilde{M}\}$. We will show that $Nu \notin \operatorname{Im} L$, for all $u \in \partial\Omega_2 \cap \operatorname{Ker} L$.

In fact, by assumption (A2), we see that $g(0) = 0$, and if $u \in \partial\Omega_2 \cap \operatorname{Ker} L$, then $u(t) \equiv M$ or $u \equiv -M$. So

$$QN(u) = -\frac{1}{2kT}\int_{-kT}^{kT}\left[h(M) - f_k(t)\right]dt = -h(M) + \overline{f_k} \geq -h(M_1) + \overline{f_k} > 0, \quad \forall u(t) \equiv M,$$
$$QN(u) = -\frac{1}{2kT}\int_{-kT}^{kT}\left[h(-M) - f_k(t)\right]dt = -h(-M) + \overline{f_k} \leq -h(-M_1) + \overline{f_k} < 0, \quad \forall u(t) \equiv -M, \tag{2.27}$$

where $\overline{f_k} = (1/2kT)\int_{-kT}^{kT} f_k(t)dt$. This implies that $Nu \notin \operatorname{Im} L$, for all $u \in \partial\Omega_2 \cap \operatorname{Ker} L$.

Step 3. Set $J : \operatorname{Im} Q \to \operatorname{Ker} L$, $Jx = x$, we will show $\deg\{JQN, \Omega_2 \cap \operatorname{Ker} L, 0\} \neq 0$.

Let $H(x,\mu) = \mu x + (1-\mu)JQNx$, for all $x \in \Omega_2 \cap \operatorname{Ker} L$, when $x \in \partial(\Omega_2 \cap \operatorname{Ker} L)$, we have $x = \pm M$ and

$$\begin{aligned} H(M,\mu) &= \mu M + (1-\mu)JQN(M) = \mu M + (1-\mu)QN(M) > 0, \\ H(-M,\mu) &= -\mu M + (1-\mu)JQN(-M) = -\mu M + (1-\mu)QN(-M) < 0. \end{aligned} \tag{2.28}$$

So for all $\mu \in [0,1]$, $H(\partial(\Omega_2 \cap \operatorname{Ker} L), \mu) \neq 0$, and then

$$\begin{aligned} \deg\{JQN, \Omega_2 \cap \operatorname{Ker} L, 0\} &= \deg\{H(\cdot,0), \Omega_2 \cap \operatorname{Ker} L, 0\} \\ &= \deg\{H(\cdot,1), \Omega_2 \cap \operatorname{Ker} L, 0\} = \deg\{I, \Omega_2 \cap \operatorname{Ker} L, 0\} \\ &= 1. \end{aligned} \tag{2.29}$$

Therefore, by Lemma 2.5, (1.4) has a $2kT$-periodic solution $u_k \in \overline{\Omega_2}$. □

Remark 2.8. Suppose that all the conditions in Lemma 2.6 hold. We see that for each $k \in \mathbf{N}$, (1.4) has a $2kT$-periodic solution $u_k \in \Omega_2$. This implies that

$$\|u_k\|_\infty \leq M, \qquad \left\|u_k'\right\|_\infty \leq M. \tag{2.30}$$

Furthermore, as same as the proof of step 1 in Lemma 2.6 with replacing $u(t)$ by $u_k(t)$, we have

$$\|u_k\|_{l_1+1} \leq \alpha_1, \qquad \left\|u_k'\right\|_{l_0+1} \leq \beta_1, \tag{2.31}$$

where α_1 and β_1 are two positive constants independent of $k \in \mathbf{N}$.

Lemma 2.9 (see [12]). *Let $u_k \in C^1_{2kT}$ be the $2kT$-periodic solution for* (1.4) *and satisfies* (2.30) *and* (2.31) *for all $k \in \mathbf{N}$. Then there exists a function $u_0 \in C^1(R, R)$ such that for each interval $[c, d] \subset R$, there is a subsequence $\{u_{k_j}\}$ of $\{u_k\}_{k\in\mathbf{N}}$ with $u'_{k_j}(t) \to u'_0(t)$ uniformly on $[c, d]$.*

3. Main Results

Theorem 3.1. *Suppose that assumptions (A1), (A2), and (A3) in Lemma 2.6 hold. Then* (1.1) *has a unique homoclinic solution.*

Proof. Since assumptions (A1), (A2), onsisting of Kuratowski operations we used following principles and (A3) in Lemma 2.6 hold, by using Lemma 2.6, we see that (1.4) has a $2kT$-periodic solution $u_k(t)$ satisfying (2.30) and (2.31) for each $k \in \mathbf{N}$. It follows from Lemma 2.9 that there exists a $u_0 \in C^1(R, R)$ such that for each interval $[c, d] \subset R$, there is a subsequence $\{u_{k_j}\}$ of $\{u_k\}_{k\in\mathbf{N}}$ satisfying $u'_{k_j}(t) \to u'_0(t)$ uniformly on $[c, d]$. Below, we will show that $u_0(t)$ is just a unique homoclinic solution to (1.1).

Step 1. We show that u_0 is a solution of (1.1).

In view of $u_{k_j}(t)$ being a $2k_jT$-periodic solution to (1.4), we have

$$u''_{k_j}(t) + g\left(u'_{k_j}(t)\right) + h\left(u_{k_j}(t)\right) = f_{k_j}(t), \quad \text{for } t \in \left[-k_jT, k_jT\right],\ j \in \mathbf{N}. \tag{3.1}$$

Take $a, b \in R$ such that $a < b$, there exists $j_0 \in \mathbf{N}$ such that for all $j > j_0$

$$u''_{k_j}(t) + g\left(u'_{k_j}(t)\right) + h\left(u_{k_j}(t)\right) = f(t), \quad \text{for } t \in [a, b]. \tag{3.2}$$

Integrating (3.2) from a to $t \in [a, b]$, we have

$$u'_{k_j}(t) - u'_{k_j}(a) = \int_a^t \left[-g\left(u'_{k_j}(s)\right) - h\left(u_{k_j}(s)\right) + f(s)\right] ds, \quad \text{for } t \in [a, b]. \tag{3.3}$$

Since Lemma 2.9 shows that $u_{k_j} \to u_0$ uniformly on $[a, b]$ and $u'_{k_j} \to u'_0$ uniformly on $[a, b]$ as $j \to \infty$, let $j \to \infty$ in (3.3), we get

$$u'_0(t) - u'_0(a) = \int_a^t \left[-g(u'_0(s)) - h(u_0(s)) + f(s)\right] ds, \quad \text{for } t \in [a, b]. \tag{3.4}$$

In view of a and b are arbitrary, (3.4) shows that $u_0(t)$ is a solution of (1.1).

Step 2. We prove that $u_0(t) \to 0$, as $t \to \pm\infty$.

Obviously, for every $i \in \mathbf{N}$, there exists $j_i \in \mathbf{N}$ such that for all $j > j_i$, we have

$$\begin{aligned}\int_{-iT}^{iT}\left[\left|u_{k_j}(t)\right|^{l_1+1}+\left|u'_{k_j}(t)\right|^{l_0+1}\right]dt &\le \int_{-k_jT}^{k_jT}\left[\left|u_{k_j}(t)\right|^{l_1+1}+\left|u'_{k_j}(t)\right|^{l_0+1}\right]dt\\ &\le \alpha_1^{l_1+1}+\beta_1^{l_0+1}\\ &:= M_2.\end{aligned} \tag{3.5}$$

It follows that

$$\begin{aligned}&\int_{-\infty}^{+\infty}\left[|u_0(t)|^{l_1+1}+\left|u'_0(t)\right|^{l_0+1}\right]dt\\ &\quad= \lim_{i\to+\infty}\int_{-iT}^{iT}\left[|u_0(t)|^{l_1+1}+\left|u'_0(t)\right|^{l_0+1}\right]dt\\ &\quad= \lim_{i\to+\infty}\lim_{j\to+\infty}\int_{-iT}^{iT}\left[\left|u_{k_j}(t)\right|^{l_1+1}+\left|u'_{k_j}(t)\right|^{l_0+1}\right]dt\\ &\quad\le M_2,\end{aligned} \tag{3.6}$$

and then

$$\int_{|t|\ge r}\left[|u_0(t)|^{l_1+1}+\left|u'_0(t)\right|^{l_0+1}\right]dt \longrightarrow 0, \quad \text{as } r \longrightarrow +\infty, \tag{3.7}$$

which yields

$$\int_{|t|\ge r}|u_0(t)|^{l_1+1}dt \longrightarrow 0, \qquad \int_{|t|\ge r}\left|u'_0(t)\right|^{l_0+1}dt \longrightarrow 0, \quad \text{as } r \longrightarrow +\infty. \tag{3.8}$$

By using Lemma 2.1, as $t \to \pm\infty$,

$$\begin{aligned}|u_0(t)| \le{}& (2a)^{-1/(l_1+1)}\left(\int_{t-a}^{t+a}|u_0(s)|^{l_1+1}ds\right)^{1/(l_1+1)}\\ &+ a\cdot(2a)^{-1/(l_0+1)}\left(\int_{t-a}^{t+a}\left|u'_0(s)\right|^{l_0+1}ds\right)^{1/(l_0+1)} \longrightarrow 0.\end{aligned} \tag{3.9}$$

So we have $u_0(t) \to 0$, as $t \to \pm\infty$.

Step 3. We will show that

$$u'_0(t) \longrightarrow 0, \quad \text{as } t \longrightarrow \pm\infty. \tag{3.10}$$

From the Remark 2.8 and Lemma 2.9, we have

$$|u_0(t)| \le M, \qquad |u_0'(t)| \le M, \quad \text{for all } t \in R, \tag{3.11}$$

which together with (1.1) implies that

$$\|u_0''\|_\infty \le g_M + h_M + \sup_{t\in R}|f(t)|, \tag{3.12}$$

where $g_M = \max_{|x|\le M}|g(x)|$ and $h_M = \max_{|x|\le M}|h(x)|$. If $u_0'(t) \nrightarrow 0$, as $t \to \pm\infty$, then there exist a $\varepsilon_0 \in (0, 1/2)$ and a sequence $\{t_k\}$ such that

$$\begin{gathered} |t_1| < |t_2| < |t_3| < \cdots, \qquad |t_k| + 1 < |t_{k+1}|, \quad k \in \mathbf{N}, \\ |u_0'(t_k)| \ge 2\varepsilon_0, \qquad k \in \mathbf{N}. \end{gathered} \tag{3.13}$$

From this, we have for $t \in [t_k, t_k + \varepsilon_0/(1 + M_1)]$

$$|u_0'(t)| = \left|u_0'(t_k) + \int_{t_k}^{t} u_0''(s)ds\right| \ge |u_0'(t_k)| - \int_{t_k}^{t} |u_0''(s)|ds \ge \varepsilon_0. \tag{3.14}$$

It follows that

$$\int_{-\infty}^{+\infty} |u_0'(t)|^{l_0+1}dt \ge \sum_{k=1}^{\infty}\int_{t_k}^{t_k+\varepsilon_0/(1+M_1)} |u_0'(t)|^{l_0+1}dt = \infty, \tag{3.15}$$

which contradicts (3.6), and so (3.10) holds.

Step 4. Finally, we will prove that (1.1) possesses a unique homoclinic solution. In order to do it, let $u(t) = u_1(t) - u_2(t)$, where $u_1(t)$ and $u_2(t)$ are two arbitrary homoclinic solutions of (1.1). Then

$$u(t) \longrightarrow 0 \quad \text{as } t \longrightarrow \pm\infty. \tag{3.16}$$

We will show that

$$u(t) \equiv 0. \tag{3.17}$$

If (3.17) does not hold, then there must be a $t^* \in R$ such that

$$u(t^*) > 0 \tag{3.18}$$

or

$$u(t^*) < 0. \tag{3.19}$$

If $u(t^*) > 0$, then from (3.16), we see that there is a constant $X > 0$ such that $t^* \in (-X, X)$ and $u(t) < u(t^*)/2$ for $t \in (-\infty, X) \cup (X, +\infty)$. Let $t^{**} \in [-X, X]$ such that $u(t^{**}) = \max_{t\in[-X,X]} u(t)$, then

$$u(t^{**}) \geq u(t^*) > 0, \tag{3.20}$$

$$u(t^{**}) \geq u(t^*) > \sup_{t\in(-\infty,X)\cup(X,+\infty)} u(t), \tag{3.21}$$

that is,

$$u(t^{**}) = \max_{t\in R} u(t). \tag{3.22}$$

So $u'(t^{**}) = 0$ and $u''(t^{**}) \leq 0$, and then from (1.1), we see

$$-[h(u_1(t^{**})) - h(u_2(t^{**}))] = u_1''(t^{**}) - u_2''(t^{**}) = u''(t^{**}) \leq 0. \tag{3.23}$$

By using the condition (A3), we have that

$$u(t^{**}) = u_1(t^{**}) - u_2(t^{**}) \leq 0, \tag{3.24}$$

which contradicts to (3.20). This contradiction implies that (3.18) does not hold. Similarly, we can prove that (3.19) does not hold, either. So $u(t) \equiv 0$.

As an application, we consider the following example:

$$u''(t) - m\left(u'(t)\right)^3 - n(u(t)) = \frac{e^{t/2}}{e^{-t} + e^{t}}, \tag{3.25}$$

where $m, n > 0$ are constants and, $f(t) = e^{t/2}/(e^{-t} + e^{t})$. Corresponding to (1.1), we can chose $l_0 = 3$ and $l_1 = 1$ such that assumptions (A2) and (A3) hold. Furthermore, by the direct calculation, we can easily obtain that

$$\begin{aligned} \int_R |f(t)|^{(l_1+1)/l_1} dt &= \int_{-\infty}^{+\infty} |f(t)|^2 dt = \frac{\pi}{4} < \infty, \\ \int_R |f(t)|^{(l_0+1)/l_0} dt &= \int_{-\infty}^{+\infty} |f(t)|^{4/3} dt = \frac{3}{2} < \infty. \end{aligned} \tag{3.26}$$

This implies that assumption (A1) also holds. So by applying Theorem 3.1, we know that (3.25) possesses a unique homoclinic solution. □

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