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Research Article

Characterizations of Asymptotic Cone of the Solution Set of a Composite Convex Optimization Problem

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We characterize the asymptotic cone of the solution set of a convex composite optimization problem. We then apply the obtained results to study the necessary and sufficient conditions for the nonemptiness and compactness of the solution set of the problem. Our results generalize and improve some known results in literature.

1. Introduction

In this paper, we consider the following extended-valued convex composite optimization problem:

$$\begin{aligned} \min \quad & f(g(x)), \\ \text{s.t.} \quad & x \in S, \end{aligned} \tag{CCOP}$$

where $S \subset R^n$ is closed and convex. The outer function $f : R^m \to R \cup \{+\infty\}$ is a convex function; denote by $\operatorname{dom} f$ the effective domain of f, that is, $\operatorname{dom} f = \{x \in R^m \mid f(x) < +\infty\}$. The inner function $g : R^n \to R^m$ is a vector-valued function such that $g(S) \subseteq \operatorname{dom} f$. It is known that convex composite optimization model provides a unifying framework for the convergence behaviour of some algorithms. Moreover, it is also a convenient tool for the study of first- and second-order optimality conditions in constrained optimization. The study of convex composite optimization model has recently received a great deal of attention in the literature; see, for example, [1–6] and the references therein.

On the other hand, the main idea of some existent algorithms (such as proximal point algorithm [7], Tikhonov-type regularization algorithm [8], and viscosity approximate methods [9]) is computing a sequence of subproblems instead of the original one. Thus, the study of nonemptiness and compactness of solution set of the subproblems is significant in both theory and methodology. What is worth noting is that it is an important condition to guarantee the boundedness of the sequences generated by the algorithms for optimization problems and variational inequality problems (see, e.g., [10–15]). Finding sufficient conditions, in particular, necessary and sufficient conditions, which are easy to verify, for the nonemptiness and compactness of the solution set of optimization problems becomes an interesting issue. It is known that asymptotic analysis is a powerful tool to study some properties of a set. We may investigate the nonemptiness and compactness of the solution set based on the asymptotic description of the functions and sets.

However to the best of our knowledge, there is no literature that has been published to study the asymptotic cone of the solution set of problem (CCOP). Motivated by these situations, in this paper we firstly try to investigate the asymptotic cone of the solution set of the problem (CCOP). The paper is organized as follows. In Section 2, we present some basic assumptions and notations needed to describe the class of convex composite optimization problem. In Section 3, we provide the main results of this paper. In Section 4, we draw a conclusion.

2. Preliminaries

To begin, we must develop our basic definitions and assumptions, to describe the class of convex composite optimization problem which this paper will consider.

Definition 2.1 (see [14]). Let K be a nonempty set in R^n. Then the asymptotic cone of the set K, denoted by K^∞, is the set of all vectors $d \in R^n$ that are limits in the direction of the sequence $\{x_k\} \subset K$, namely:

$$K^\infty = \left\{ d \in R^n \mid \exists t_k \longrightarrow +\infty,\ x_k \in K,\ \lim_{k \to +\infty} \frac{x_k}{t_k} = d \right\}. \tag{2.1}$$

In the case that K is convex and closed, then, for any $x_0 \in K$,

$$K^\infty = \{ d \in R^n \mid x_0 + td \in K,\ \forall t > 0 \}. \tag{2.2}$$

Definition 2.2 (see [16]). A subset K of R^n is said to be bounded if to every neighborhood V of 0 in R^n corresponds to a number $s > 0$ such that $K \subset tV$ for every $t > s$.

Lemma 2.3 (see [14]). *A nonempty set $K \subset R^n$ is bounded if and only if its asymptotic cone is just the zero cone: $K^\infty = \{0\}$.*

Definition 2.4 (see [14]). For any given function $\phi : R^n \to R \cup \{+\infty\}$, the asymptotic function of ϕ is defined as the function ϕ^∞ such that $\operatorname{epi} \phi^\infty = (\operatorname{epi} \phi)^\infty$, where

$\operatorname{epi}\phi = \{(x,t) \in R^n \times R \mid \phi(x) \le t\}$ is the epigraph of ϕ. Consequently, we can give the analytic representation of the asymptotic function ϕ^∞:

$$\phi^\infty(d) = \inf\left\{ \liminf_{k\to+\infty} \frac{\phi(t_k d_k)}{t_k} : t_k \longrightarrow +\infty,\ d_k \longrightarrow d \right\}. \tag{2.3}$$

When ϕ is a proper convex and lower semicontinuous (lsc in short) function, we have

$$\phi^\infty(d) = \sup\{f(x+d) - f(x) \mid x \in \operatorname{dom}\phi\} \tag{2.4}$$

or equivalently

$$\begin{gathered} \phi^\infty(d) = \lim_{t\to+\infty} \frac{\phi(x+td) - \phi(x)}{t} = \sup_{t>0} \frac{\phi(x+td) - \phi(x)}{t}, \quad \forall d \in \operatorname{dom}\phi, \\ \phi^\infty(d) = \lim_{t\to 0^+} t\phi\left(t^{-1}d\right), \quad \forall d \in \operatorname{dom}\phi. \end{gathered} \tag{2.5}$$

For the indicator function δ_K of a nonempty set K, we have that $\delta_K^\infty = \delta_{K^\infty}$.

Example 2.5. Let Q be a symmetric $n \times n$ positive matrix and $f(x) := (1 + \langle x, Qx\rangle)^{1/2}$. Then,

$$f^\infty(d) = \langle d, Qd\rangle^{1/2}. \tag{2.6}$$

Definition 2.6 (see [17]). The function $\phi : R^n \to R \cup \{+\infty\}$ is said to be coercive if its asymptotic function $\phi^\infty(d) > 0$, for all $d \neq 0 \in R^n$, and it is said to be countercoercive if its asymptotic function $\phi^\infty(d) = -\infty$, for some $d \neq 0 \in R^n$.

Since the inner function is vector valued, we may define some partial order in objective and decision spaces. Let $C = R^m_+ \subset R^m$. We define, for any $y_1, y_2 \in R^m$,

$$\begin{gathered} y_1 \le_C y_2 \quad \text{iff } y_2 - y_1 \in C, \\ y_1 \not\le_{\operatorname{int} C} y_2 \quad \text{iff } y_2 - y_1 \notin \operatorname{int} C. \end{gathered} \tag{2.7}$$

Through these partial orders, we introduce some definitions in vector optimization theory.

Definition 2.7 (see [18]). Let $K \subset R^n$ be convex, and a map $F : K \to R^m$ is said to be C-convex if

$$F\left((1-\lambda)x + \lambda y\right) \le_C (1-\lambda)F(x) + \lambda F(y) \tag{2.8}$$

for any $x, y \in K$ and $\lambda \in [0,1]$. F is said to be strictly C-convex if

$$F\left((1-\lambda)x + \lambda y\right) \le_{\operatorname{int} C} (1-\lambda)F(x) + \lambda F(y) \tag{2.9}$$

for any $x, y \in K$ with $x \neq y$ and $\lambda \in (0,1)$.

Definition 2.8 (see [18]). A map $f : K \subset R^m \to R \cup \{+\infty\}$ is said to be C-monotone if, for any $x, y \in K$ and $x \leq_C y$. There holds

$$f(x) \leq f(y). \tag{2.10}$$

Next we give an example to show the C-convex and C-monotone of a function.

Example 2.9. Let $X = R^3_+$, $Y = R^2$, $C = R^2_+$, and $g : X \to Y$ is defined by

$$g(\cdot) = \begin{pmatrix} \sum_{i=1}^{3}(x_i + 1) \\ \frac{1}{2}\|x\|^2 \end{pmatrix}. \tag{2.11}$$

Then, g is a C-convex function. Let $f : R^2 \to R$ be defined by

$$f(\cdot) = e^{x_1} + e^{x_2}. \tag{2.12}$$

Then, f is C-monotone.

3. Main Results

Before discussing the main results, we propose the following proposition for continuity and convexity of the composite function.

Proposition 3.1. *In the problem* (CCOP), *one assumes f is proper, lsc, convex, and C-monotone, and g is continuous and C-convex. Then, the composite function $f(g(\cdot))$ is proper, lsc, and convex.*

Proof. Since f is proper and lsc, g is continuous, we derive $f(g(\cdot))$ is proper and lsc by virtue of Proposition 1.40 in [17]. Next we will check the convexity of $f(g(\cdot))$. Let $x_1, x_2 \in S$ and $\lambda \in [0,1]$. By the C-convexity of g, we have

$$g(\lambda x_1 + (1-\lambda)x_2) \leq_C \lambda g(x_1) + (1-\lambda)g(x_2). \tag{3.1}$$

By the C-monotonicity of f, we have from (3.5) that

$$f(g(\lambda x_1 + (1-\lambda)x_2)) \leq f(\lambda g(x_1) + (1-\lambda)g(x_2)). \tag{3.2}$$

From the convexity of f, we know

$$f(\lambda g(x_1) + (1-\lambda)g(x_2)) \leq \lambda f(g(x_1)) + (1-\lambda)f(g(x_2)). \tag{3.3}$$

Combining (3.2) with (3.3), we may conclude the composite function $f(g(\cdot))$ is convex. The proof is complete. □

We denote

$$S_1 = \{u \in R^n \mid f(g(\lambda u + y)) - f(g(y)) \le 0,\ \forall \lambda > 0,\ y \in S\}. \tag{3.4}$$

Theorem 3.2. *Let the assumptions in Proposition 3.1 hold. Further one assumes the solution set $\overline{X} \neq \emptyset$. Then, the asymptotic cone of $\overline{X}$ can be formulated as follows:*

$$\overline{X}^{\infty} = S_1 \cap S^{\infty}. \tag{3.5}$$

Proof. $\Rightarrow$ For any $u \in \overline{X}^{\infty}$, obviously $u \in S^{\infty}$. From the definition of asymptotic cone, we know there exist some sequences $\{x_k\} \subset \overline{X}$ and $\{t_k\} \subset R$ with $t_k \to +\infty$ such that $\lim_{k\to+\infty}(x_k/t_k) = u$. By the fact of $x_k \in \overline{X}$, one has

$$f(g(y)) - f(g(x_k)) \ge 0, \quad \forall y \in S. \tag{3.6}$$

Since $f(g(x))$ is convex, for any fixed $\lambda > 0$ when t_k is sufficiently large, we get

$$f\left(g\left(\left(1 - \frac{\lambda}{t_k}\right)y + \frac{\lambda}{t_k}x_k\right)\right) \le \left(1 - \frac{\lambda}{t_k}\right)f(g(y)) + \frac{\lambda}{t_k}f(g(x_k)), \quad \forall y \in S. \tag{3.7}$$

Combining (3.6) with (3.7), we have

$$f\left(g\left(\left(1 - \frac{\lambda}{t_k}\right)y + \frac{\lambda}{t_k}x_k\right)\right) \le f(g(y)), \quad \forall y \in S. \tag{3.8}$$

Taking limit in (3.8) as $k \to \infty$, we obtain

$$f(g(y + \lambda u)) \le f(g(y)), \quad \forall y \in S. \tag{3.9}$$

That is, $u \in S_1$ and $\overline{X}^{\infty} \subseteq S_1 \cap S^{\infty}$.

$\Leftarrow$ For any $d \in S_1 \cap S^{\infty}$. By the assumption that $\overline{X}$ is nonempty, we have

$$\overline{x} + t_k d \in S, \quad \forall t_k > 0, \tag{3.10}$$

where $\overline{x} \in \overline{X}$ is fixed and $t_k \to +\infty$. For any $y \in S$, we know

$$f(g(y)) - f(g(\overline{x} + t_k d)) = f(g(y)) - f(g(\overline{x})) + f(g(\overline{x})) - f(g(\overline{x} + t_k d)). \tag{3.11}$$

Since $\overline{x} \in \overline{X}$, it is easy to check that

$$f(g(y)) - f(g(\overline{x})) \ge 0, \quad \forall y \in S, \tag{3.12}$$

and by the definition of S_1, we have

$$f(g(\overline{x})) - f(g(\overline{x} + t_k d)) \geq 0. \tag{3.13}$$

Combining (3.11) and (3.12) with (3.13), one has

$$f(g(y)) - f(g(\overline{x} + t_k d)) \geq 0, \quad \forall y \in S. \tag{3.14}$$

Clearly, (3.14) means $\overline{x} + t_k d \in \overline{X}$. We denote $x_k = \overline{x} + t_k d$, and it follows that

$$\lim_{k \to +\infty} \frac{x_k}{t_k} = \lim_{k \to +\infty} \frac{\overline{x} + t_k d}{t_k} = d. \tag{3.15}$$

Hence, $d \in \overline{X}^{\infty}$. The proof is complete. □

Corollary 3.3. *Let assumptions in Proposition 3.1 hold. Then, the solution set* $\overline{X}$ *is nonempty and compact if and only if*

$$S_1 \cap S^{\infty} = \{0\}. \tag{3.16}$$

Proof. The necessity part follows from the statements in Theorem 3.2 and in Lemma 2.3. Now we prove the sufficiency. We may define a function $\varphi : R^n \to R \cup \{+\infty\}$ as $\varphi(x) = f(g(x))$. Clearly, φ is proper, lsc, and convex. By virtue of Proposition 3.1.3 of [14], we know the coercivity of φ is a sufficient condition for the nonemptiness and compactness of $\overline{X}$. From (3.4), for all $y \in S$ we have

$$\begin{gathered} \{u \in S^{\infty} \mid f(g(\lambda u + y)) - f(g(y)) \leq 0,\ \forall \lambda > 0\} = \{0\}, \\ \left\{u \in S^{\infty} \mid \lim_{\lambda \to +\infty} \frac{f(g(\lambda u + y)) - f(g(y))}{\lambda} \leq 0\right\} = \{0\}. \end{gathered} \tag{3.17}$$

Consequently

$$\{u \in S^{\infty} \mid \varphi^{\infty}(u) \leq 0\} = \{0\}. \tag{3.18}$$

This is $\varphi^{\infty}(u) > 0$, for all $u \neq 0$ and φ is coercive. Thus, $\overline{X}$ is nonempty and compact. The proof is complete. □

4. Conclusion

In this paper, we characterized the asymptotic cone of the solution set of a convex composite optimization problem (CCOP). We obtained the analytical expression of the asymptotic cone of the solution set. Furthermore, we studied the necessary and sufficient conditions for the nonemptiness and compactness of the solution set of the problem via the analytical expression of the asymptotic cone. Our results generalized some known results in [14] and firstly studied the compactness of the solution set of convex composite optimization problems.

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