Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2012, Article ID 629149, 13 pages
doi:10.1155/2012/629149

Research Article

On Multivalued Nonexpansive Mappings in $\mathbb{R}$-Trees

K. Samanmit and B. Panyanak

Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand

Correspondence should be addressed to B. Panyanak, banchap@chiangmai.ac.th

Received 25 April 2012; Accepted 20 June 2012

Academic Editor: Hong-Kun Xu

The relationships between nonexpansive, weakly nonexpansive, $*$-nonexpansive, proximally nonexpansive, proximally continuous, almost lower semicontinuous, and ε-semicontinuous mappings in $\mathbb{R}$-trees are studied. Convergence theorems for the Ishikawa iteration processes are also discussed.

1. Introduction

A mapping t on a subset E of a Banach space $(X, \|\cdot\|)$ is said to be *nonexpansive* if

$$\|t(x) - t(y)\| \leq \|x - y\|, \quad \forall x, y \in E. \tag{1.1}$$

A point x in E is called a *fixed point* of t if $x = t(x)$. The existence of fixed points for nonexpansive mappings in Banach spaces was studied independently by three authors in 1965 (see Browder [1], Göhde [2], and Kirk [3]). They showed that every nonexpansive mapping defined on a bounded closed convex subset of a uniformly convex Banach space always has a fixed point. Since then many researchers generalized the concept of nonexpansive mappings in different directions and also studied the fixed point theory for various types of generalized nonexpansive mappings.

Browder-Göhde-Kirk's result was extended to multivalued nonexpansive mappings by Lim [4] in 1974. Husain and Tarafdar [5] and Husain and Latif [6] introduced the concepts of weakly nonexpansive and $*$-nonexpansive multivalued mappings and studied the existence of fixed points for such mappings in uniformly convex Banach spaces. In 1991, Xu [7] pointed out that a weakly nonexpansive multivalued mapping must be nonexpansive and thus the main results of Husain-Tarafdar and Husain-Latif on weakly nonexpansive multivalued mappings are special cases of those of Lim [4]. Xu [7] also showed that $*$-nonexpansiveness is different from nonexpansiveness for multivalued mappings. In 1995, Lopez Acedo and Xu [8] introduced the concept of proximally nonexpansive multivalued mappings and proved that it coincides with the concept of $*$-nonexpansive mappings when the mappings take compact values.

In 2009, Shahzad and Zegeye [9] proved strong convergence theorems of the Ishikawa iteration for quasi-nonexpansive multivalued mappings satisfying the endpoint condition. They also constructed a modified Ishikawa iteration for proximally nonexpansive mappings and proved strong convergence theorems of the proposed iteration without the endpoint condition. Puttasontiphot [10] gave the analogous results of Shahzad and Zegeye in complete CAT(0) spaces. However, there is not any result in linear or nonlinear spaces concerning the convergence of Ishikawa iteration for quasi-nonexpansive multivalued mappings which completely removes the endpoint condition.

In this paper, motivated by the above results, we obtain the relationships between nonexpansive, weakly nonexpansive, $*$-nonexpansive, and proximally nonexpansive mappings in a nice subclass of CAT(0) spaces, namely, $\mathbb{R}$-trees. We also introduce a condition on mappings which is much more general than the endpoint condition and prove strong convergence theorems of a modified Ishikawa iteration for quasi-nonexpansive multivalued mappings satisfying such condition.

2. Preliminaries

Let (X, d) be a metric space and let $\emptyset \neq E \subseteq X$, $x \in X$. The distance from x to E is defined by

$$\operatorname{dist}(x, E) = \inf\{d(x, y) : y \in E\}. \tag{2.1}$$

The set E is called *proximal* if for each $x \in X$, there exists an element $y \in E$ such that $d(x, y) = \operatorname{dist}(x, E)$. Let $\varepsilon > 0$ and $x_0 \in X$. We will denote the open ball centered at x_0 with radius ε by $B(x_0, \varepsilon)$, the closed *ε-hull* of E by $N_\varepsilon(E) = \{x \in X : \operatorname{dist}(x, E) \leq \varepsilon\}$, and the family of nonempty subsets of E by 2^E. Let $H(\cdot, \cdot)$ be the Hausdorff distance on 2^E, that is,

$$H(A, B) = \max\left\{\sup_{a\in A} \operatorname{dist}(a, B), \sup_{b\in B} \operatorname{dist}(b, A)\right\}, \quad A, B \in 2^E. \tag{2.2}$$

Let $T : E \to 2^E$ be a multivalued mapping. For each $x \in E$, we let

$$P_{T(x)}(x) := \{u \in T(x) : d(x, u) = \operatorname{dist}(x, T(x))\}. \tag{2.3}$$

In the case of $P_{T(x)}(x)$ is a singleton; we will assume, without loss of generality, that $P_{T(x)}(x)$ is a point in E. A point $x \in E$ is called a *fixed point* of T if $x \in T(x)$. A point $x \in E$ is called an *endpoint* of T if x is a fixed point of T and $T(x) = \{x\}$. We will denote by $\operatorname{Fix}(T)$ the set of all fixed points of T and by $\operatorname{End}(T)$ the set of all endpoints of T. We see that for each mapping T, $\operatorname{End}(T) \subseteq \operatorname{Fix}(T)$ and the converse is not true in general. A mapping T is said to satisfy the *endpoint condition* if $\operatorname{End}(T) = \operatorname{Fix}(T)$.

Definition 2.1. Let E be a nonempty subset of a metric space (X,d) and $T : E \to 2^E$. Then T is said to be

(i) *nonexpansive* if $H(T(x),T(y)) \leq d(x,y)$ for all $x,y \in E$;

(ii) *quasi-nonexpansive* if $\mathrm{Fix}(T) \neq \emptyset$ and

$$H(T(x),T(p)) \leq d(x,p) \quad \forall x \in E,\ p \in \mathrm{Fix}(T); \tag{2.4}$$

(iii) *weakly nonexpansive* if for each $x,y \in E$ and $u_x \in T(x)$, there exists $u_y \in T(y)$ such that

$$d(u_x,u_y) \leq d(x,y); \tag{2.5}$$

(iv) $*$-*nonexpansive* if for each $x,y \in E$ and $u_x \in P_{T(x)}(x)$, there exists $u_y \in P_{T(y)}(y)$ such that

$$d(u_x,u_y) \leq d(x,y); \tag{2.6}$$

(v) *proximally nonexpansive* if the map $F : E \to 2^E$ defined by $F(x) := P_{T(x)}(x)$ is nonexpansive;

(vi) *proximally continuous* if the map $F(x) := P_{T(x)}(x)$ is continuous;

(vii) *almost lower semicontinuous* if given $\varepsilon > 0$, for each $x \in E$ there is an open neighborhood U of x such that

$$\bigcap_{y\in U} N_\varepsilon(T(y)) \neq \emptyset; \tag{2.7}$$

(viii) ε-*semicontinuous* if given $\varepsilon > 0$, for each $x \in E$ there is an open neighborhood U of x such that

$$T(y) \cap N_\varepsilon(T(x)) \neq \emptyset \quad \forall y \in U. \tag{2.8}$$

The following facts can be found in [7, 8].

Proposition 2.2. *Let E be a nonempty subset of a metric space (X,d) and $T : E \to 2^E$ be a multivalued mapping. Then the following statements hold:*

(i) *if T is weakly nonexpansive, then T is nonexpansive;*

(ii) *if T is $*$-nonexpansive and T takes nonempty proximal values, then T is proximally nonexpansive;*

(iii) *the converses of (i) and (ii) hold if T takes compact values.*

For any pair of points x, y in a metric space (X, d), a *geodesic path* joining these points is an isometry c from a closed interval $[0, l]$ to X such that $c(0) = x$ and $c(l) = y$. The image of c is called a *geodesic segment* joining x and y. If there exists exactly one geodesic joining x and y we denote by $[x, y]$ the geodesic joining x and y. For $x, y \in X$ and $\alpha \in [0, 1]$, we denote the point $z \in [x, y]$ such that $d(x, z) = \alpha d(x, y)$ by $(1 - \alpha)x \oplus \alpha y$. The space (X, d) is said to be a *geodesic space* if every two points of X are joined by a geodesic, and X is said to be *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset E of X is said to be *convex* if E includes every geodesic segment joining any two of its points, and E is said to be *gated* if for any point $x \notin E$ there is a unique point y_x such that for any $z \in E$,

$$d(x, z) = d(x, y_x) + d(y_x, z). \tag{2.9}$$

The point y_x is called the *gate* of x in E. From the definition of y_x we see that it is also the unique nearest point of x in E. The set E is called *geodesically bounded* if there is no geodesic ray in E, that is, an isometric image of $[0, \infty)$. We will denote by $\mathcal{P}(E)$ the family of nonempty proximinal subsets of E, by $\mathcal{CC}(E)$ the family of nonempty closed convex subsets of E, and by $\mathcal{KC}(E)$ the family of nonempty compact convex subsets of E.

Definition 2.3. An $\mathbb{R}$-tree (sometimes called *metric tree*) is a geodesic metric space X such that:

(i) there is a unique geodesic segment $[x, y]$ joining each pair of points $x, y \in X$;

(ii) if $[y, x] \cap [x, z] = \{x\}$, then $[y, x] \cup [x, z] = [y, z]$.

By (i) and (ii) we have

(i) if $u, v, w \in X$, then $[u, v] \cap [u, w] = [u, z]$ for some $z \in X$.

An $\mathbb{R}$-tree is a special case of a CAT(0) space. For a thorough discussion of these spaces and their applications, see [11]. Notice also that a metric space X is a complete $\mathbb{R}$-tree if and only if X is hyperconvex with unique metric segments, see [12]. For more about hyperconvex spaces and fixed point theorems in hyperconvex spaces, see [13]. We now collect some basic properties of $\mathbb{R}$-trees.

Lemma 2.4. *Let X be a complete $\mathbb{R}$-tree. Then the following statements hold:*

(i) *[14, page 1048] the gate subsets of X are precisely its closed and convex subsets;*

(ii) *[11, page 176] if E is a closed convex subset of X, then, for each $x \in X$, there exists a unique point $P_E(x) \in E$ such that*

$$d(x, P_E(x)) = \operatorname{dist}(x, E); \tag{2.10}$$

(iii) *[11, page 176] if E is closed convex and if x' belong to $[x, P_E(x)]$, then $P_E(x') = P_E(x)$;*

(iv) *[15, Lemma 3.1] if A and B are closed convex subsets of X, then, for any $u \in X$,*

$$d(P_A(u), P_B(u)) \leq H(A, B); \tag{2.11}$$

(v) *[16, Lemma 3.2] if E is closed convex, then, for any $x, y \in X$, one has either*

$$P_E(x) = P_E(y) \tag{2.12}$$

or

$$d(x,y) = d(x, P_E(x)) + d(P_E(x), P_E(y)) + d(P_E(y), y); \tag{2.13}$$

(vi) *[17, Lemma 2.5] if $x, y, z \in X$ and $\alpha \in [0,1]$, then*

$$d^2((1-\alpha)x \oplus \alpha y, z) \leq (1-\alpha)d^2(x,z) + \alpha d^2(y,z) - \alpha(1-\alpha)d^2(x,y); \tag{2.14}$$

(vii) *[18, Proposition 1] if E is a closed convex subset of X and $T : E \to \mathcal{CC}(E)$ is a quasi-nonexpansive mapping, then* $\mathrm{Fix}(T)$ *is closed and convex.*

3. Results in $\mathbb{R}$-Trees

In general metric spaces, the concepts of nonexpansive and $*$-nonexpansive multivalued mappings are different (see Examples 5.1 and 5.2). But, if we restrict ourself to an $\mathbb{R}$-tree we can show that every nonexpansive mapping with nonempty closed convex values is a $*$-nonexpansive mapping. The following lemma is very crucial.

Lemma 3.1. *Let E be a nonempty closed convex subset of a complete $\mathbb{R}$-tree X and $v \notin E$. If $v \in [P_E(v), u]$ for some $u \in X$, then $P_E(v) = P_E(u)$.*

Proof. By Lemma 2.4(iii), $P_E(x) = P_E(v)$ for all $x \in [P_E(v), v]$. Then for $z \in E$, we have

$$d(z,x) = d(z, P_E(v)) + d(P_E(v), x) \quad \forall x \in [P_E(v), v]. \tag{3.1}$$

This implies that $P_E(v)$ is the gate of z in $[P_E(v), v]$ for all $z \in E$. Since $v \in [P_E(v), u]$, then v is the gate of u in $[P_E(v), v]$. By Lemma 2.4(v), for each $z \in E$ we have

$$\begin{aligned} d(u,z) &= d(u,v) + d(v, P_E(v)) + d(P_E(v), z) \\ &= d(u, P_E(v)) + d(P_E(v), z) \\ &\geq d(u, P_E(v)). \end{aligned} \tag{3.2}$$

Hence $P_E(v) = P_E(u)$ as desired. □

Proposition 3.2. *Let E be a nonempty subset of a complete $\mathbb{R}$-tree X and $T : E \to 2^E$ be a multivalued mapping. If T takes closed and convex values, then the following statements hold:*

(i) *T is weakly nonexpansive if and only if T is nonexpansive;*

(ii) *T is $*$-nonexpansive if and only if T is proximally nonexpansive;*

(iii) *if T is nonexpansive, then T is proximally nonexpansive;*

(iv) *if T is proximally nonexpansive, then T is proximally continuous;*

(v) *if T is proximally continuous, then T is almost lower semicontinuous;*

(vi) *if T is almost lower semicontinuous, then T is ε-semicontinuous.*

Proof. (i) ($\Rightarrow$) Follows from Proposition 2.2(i). ($\Leftarrow$): let $x, y \in E$ and $u_x \in T(x)$. Choose $u_y = P_{T(y)}(u_x)$. Then

$$\begin{aligned} d(u_x, u_y) &= \operatorname{dist}(u_x, T(y)) \\ &\le H(T(x), T(y)) \\ &\le d(x, y). \end{aligned} \tag{3.3}$$

(ii) ($\Rightarrow$) Follows from Proposition 2.2(ii). ($\Leftarrow$): for each $x \in E$, we let $u_x = P_{T(x)}(x)$. Then

$$d(u_x, u_y) = d(P_{T(x)}(x), P_{T(y)}(y)) \le d(x, y). \tag{3.4}$$

This means T is $*$-nonexpansive.

(iii) We let $x, y \in E$ and divide the proof to 3 cases.

Case 1. $P_{T(x)}(x), P_{T(y)}(y) \in [x, y]$. Then $d(P_{T(x)}(x), P_{T(y)}(y)) \le d(x, y)$.

Case 2. $P_{T(x)}(x) \notin [x, y], P_{T(y)}(y) \in [x, y]$ or vice versa. Let $u \in [P_{T(y)}(y), y]$. Then by Lemma 2.4(iii), $P_{T(y)}(y) = P_{T(y)}(u)$. We claim that $P_{T(x)}(x) = P_{T(x)}(u)$. Let v be the gate of $P_{T(x)}(x)$ in $[x, y]$. Then $v \neq P_{T(x)}(x)$. Since $v \in [x, P_{T(x)}(x)]$, then by Lemma 2.4(iii) we have $P_{T(x)}(v) = P_{T(x)}(x)$. This implies that $v \in [P_{T(x)}(v), u]$. Since $v \notin T(x)$, by Lemma 3.1 we have

$$P_{T(x)}(x) = P_{T(x)}(v) = P_{T(x)}(u). \tag{3.5}$$

By Lemma 2.4(iv),

$$\begin{aligned} d(P_{T(x)}(x), P_{T(y)}(y)) &= d(P_{T(x)}(u), P_{T(y)}(u)) \\ &\le H(T(x), T(y)) \\ &\le d(x, y). \end{aligned} \tag{3.6}$$

Case 3. $P_{T(x)}(x) \notin [x, y]$ and $P_{T(y)}(y) \notin [x, y]$. Let v and w be the gates of $P_{T(x)}(x)$ and $P_{T(y)}(y)$ in $[x, y]$, respectively. Since $v \in [P_{T(x)}(x), x]$ and $w \in [P_{T(y)}(y), y]$, then

$$P_{T(x)}(x) = P_{T(x)}(v), \qquad P_{T(y)}(y) = P_{T(y)}(w). \tag{3.7}$$

Let $u \in [v, w]$. Then by Lemma 3.1, we have

$$P_{T(x)}(v) = P_{T(x)}(u), \qquad P_{T(y)}(w) = P_{T(y)}(u). \tag{3.8}$$

By (3.7), we have

$$P_{T(x)}(x) = P_{T(x)}(u), \qquad P_{T(y)}(y) = P_{T(y)}(u). \tag{3.9}$$

By Lemma 2.4(iv),

$$\begin{aligned} d\left(P_{T(x)}(x), P_{T(y)}(y)\right) &= d\left(P_{T(x)}(u), P_{T(y)}(u)\right) \\ &\leq H\left(T(x), T(y)\right) \\ &\leq d(x, y). \end{aligned} \tag{3.10}$$

(iv) Follows from the fact that nonexpansiveness implies continuity.

(v) Given $\varepsilon > 0$ and let $x_0 \in E$. Since the map $F(x) = P_{T(x)}(x)$ is single valued continuous, then there exists $\delta > 0$ such that

$$d\left(P_{T(x)}(x), P_{T(x_0)}(x_0)\right) < \varepsilon \quad \forall x \in B(x_0, \delta). \tag{3.11}$$

Let $U = B(x_0, \delta)$. Then U is an open neighborhood of x_0. Since

$$\operatorname{dist}\left(P_{T(x_0)}(x_0), T(x)\right) \leq d\left(P_{T(x_0)}(x_0), P_{T(x)}(x)\right) < \varepsilon \quad \forall x \in U, \tag{3.12}$$

then

$$P_{T(x_0)}(x_0) \in \bigcap_{x \in U} N_{\varepsilon}(T(x)). \tag{3.13}$$

Therefore, T is almost lower semicontinuous.

(vi) See [19, page 114]. □

The following result can be found in [19, Theorem 4].

Proposition 3.3. *Let X be a complete $\mathbb{R}$-tree, E a nonempty closed convex geodesically bounded subset of X, and $T : E \to CC(E)$ an ε-semicontinuous mapping. Then T has a fixed point.*

As a consequence of Propositions 3.2 and 3.3, we obtain the following.

Corollary 3.4. *Let E be a nonempty closed convex geodesically bounded subset of a complete $\mathbb{R}$-tree X and $T : E \to CC(E)$ be a multivalued mapping. Then T has a fixed point if one of the following statements holds:*

(i) *T is weakly nonexpansive;*

(ii) *T is nonexpansive;*

(iii) *T is $*$-nonexpansive;*

(iv) *T is proximally nonexpansive;*

(v) *T is proximally continuous;*

(vi) *T is almost lower semicontinuous.*

4. Convergence Theorems

Let E be a nonempty convex subset of an $\mathbb{R}$-tree X,$T : E \to \mathcal{P}(E)$ a multivalued mapping and $\{\alpha_n\}$,$\{\beta_n\} \subseteq [0,1]$.

(A) The sequence of Ishikawa iterates [9] is defined by $x_1 \in E$,

$$y_n = \beta_n z_n \oplus (1-\beta)x_n, \quad n \geq 1, \tag{4.1}$$

where $z_n \in P_{T(x_n)}(x_n)$, and

$$x_{n+1} = \alpha_n z'_n \oplus (1-\alpha_n)x_n, \quad n \geq 1, \tag{4.2}$$

where $z'_n \in P_{T(y_n)}(y_n)$.

Recall that a multivalued mapping $T : E \to \mathcal{P}(E)$ is said to satisfy *Condition (I)* if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) > 0$ for $r \in (0,\infty)$ such that

$$\operatorname{dist}(x, T(x)) \geq f(\operatorname{dist}(x, \operatorname{Fix}(T))) \quad \forall x \in E. \tag{4.3}$$

The mapping T is called *hemicompact* if for any sequence $\{x_n\}$ in E such that

$$\lim_{n\to\infty} \operatorname{dist}(x_n, T(x_n)) = 0, \tag{4.4}$$

there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $q \in E$ such that $\lim_{k\to\infty} x_{n_k} = q$.

The following theorems are consequences of [10, Theorems 3.6 and 3.7].

Theorem 4.1. *Let X be a complete $\mathbb{R}$-tree, E a nonempty closed convex subset of X, and $T : E \to \mathcal{P}(E)$ a proximally nonexpansive mapping with $\operatorname{Fix}(T) \neq \emptyset$. Let $\{x_n\}$ be the Ishikawa iterates defined by (A). Assume that T satisfies condition (I) and $\alpha_n, \beta_n \in [a,b] \subset (0,1)$. Then $\{x_n\}$ converges to a fixed point of T.*

Theorem 4.2. *Let X be a complete $\mathbb{R}$-tree, E a nonempty closed convex subset of X, and $T : E \to \mathcal{P}(E)$ a proximally nonexpansive mapping with $\operatorname{Fix}(T) \neq \emptyset$. Let $\{x_n\}$ be the Ishikawa iterates defined by (A). Assume that T is hemicompact and (i) $0 \leq \alpha_n, \beta_n < 1$; (ii) $\beta_n \to 0$; (iii) $\sum \alpha_n \beta_n = \infty$. Then $\{x_n\}$ converges to a fixed point of T.*

As consequences of Proposition 3.2, Theorems 4.1 and 4.2, we obtain the following.

Corollary 4.3. *Let X be a complete $\mathbb{R}$-tree, E a nonempty closed convex subset of X, and $T : E \to \mathcal{CC}(E)$ a nonexpansive mapping with $\operatorname{Fix}(T) \neq \emptyset$. Let $\{x_n\}$ be the Ishikawa iterates defined by (A). Assume that T satisfies condition (I) and $\alpha_n, \beta_n \in [a,b] \subset (0,1)$. Then $\{x_n\}$ converges to a fixed point of T.*

Corollary 4.4. *Let X be a complete $\mathbb{R}$-tree, E a nonempty closed convex subset of X, and $T : E \to \mathcal{CC}(E)$ a nonexpansive mapping with $\operatorname{Fix}(T) \neq \emptyset$. Let $\{x_n\}$ be the Ishikawa iterates defined by (A). Assume that T is hemicompact and (i) $0 \leq \alpha_n, \beta_n < 1$; (ii) $\beta_n \to 0$ and (iii) $\sum \alpha_n \beta_n = \infty$. Then $\{x_n\}$ converges to a fixed point of T.*

Definition 4.5. Let E be a nonempty subset of a complete $\mathbb{R}$-tree and $T : E \to CC(E)$ be a multivalued mapping for which $\operatorname{Fix}(T) \neq \emptyset$. We say that $u \in E$ is a key of T if, for each $x \in \operatorname{Fix}(T)$, x is the gate of u in $T(x)$. We say that T satisfies the *gate condition* if T has a key in E.

It follows from the definitions that the endpoint condition implies the gate condition and the converse is not true. Example 5.3 shows that there is a nonexpansive mapping satisfying the gate condition but does not satisfy the endpoint condition.

Motivated by the above results, we introduce a modified Ishikawa iteration as follows: let E be a nonempty convex subset of an $\mathbb{R}$-tree $X, T : E \to CC(E)$ a multivalued mapping, and $\{\alpha_n\}, \{\beta_n\} \subseteq [0,1]$. Fix $u \in E$.

(B) The sequence of Ishikawa iterates is defined by $x_1 \in E$,

$$y_n = \beta_n z_n \oplus (1-\beta_n) x_n, \quad n \geq 1, \tag{4.5}$$

where z_n is the gate of u in $T(x_n)$, and

$$x_{n+1} = \alpha_n z_n' \oplus (1-\alpha_n) x_n, \quad n \geq 1, \tag{4.6}$$

where z_n' is the gate of u in $T(y_n)$.

Recall that a sequence $\{x_n\}$ in a metric space (X,d) is said to be *Fejér monotone* with respect to a subset E of X if

$$d(x_{n+1}, p) \leq d(x_n, p) \quad \forall p \in E,\ n \geq 1. \tag{4.7}$$

The following fact can be found in [20].

Proposition 4.6. *Let (X,d) be a complete metric space, E be a nonempty closed subset of X, and $\{x_n\}$ be Fejér monotone with respect to E. Then $\{x_n\}$ converges to some $p \in E$ if and only if $\lim_{n\to\infty} \operatorname{dist}(x_n, E) = 0$.*

Lemma 4.7. *Let E be a nonempty closed convex subset of a complete $\mathbb{R}$-tree X and $T : E \to CC(E)$ be a quasi-nonexpansive mapping satisfying the gate condition. Let u be a key of T and let $\{x_n\}$ be the Ishikawa iterates defined by (B). Then $\{x_n\}$ is Fejér monotone with respect to* $\operatorname{Fix}(T)$ *and $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in \operatorname{Fix}(T)$.*

Proof. Let $p \in \operatorname{Fix}(T)$. For each n, we have

$$\begin{aligned} d(y_n, p) &= d(\beta_n z_n \oplus (1-\beta_n) x_n, p) \\ &\leq \beta_n d(z_n, p) + (1-\beta_n) d(x_n, p) \\ &= \beta_n d(P_{T(x_n)}(u), P_{T(p)}(u)) + (1-\beta_n) d(x_n, p) \\ &\leq \beta_n H(T(x_n), T(p)) + (1-\beta_n) d(x_n, p) \\ &\leq \beta_n d(x_n, p) + (1-\beta_n) d(x_n, p) \\ &\leq d(x_n, p), \end{aligned} \tag{4.8}$$

$$
\begin{aligned}
d(x_{n+1}, p) &= d(\alpha_n z'_n \oplus (1-\alpha_n)x_n, p) \\
&\leq \alpha_n d(z'_n, p) + (1-\alpha_n)d(x_n, p) \\
&= \alpha_n d(P_{T(y_n)}(u), P_{T(p)}(u)) + (1-\alpha_n)d(x_n, p) \\
&\leq \alpha_n H(T(y_n), T(p)) + (1-\alpha_n)d(x_n, p) \\
&\leq \alpha_n d(y_n, p) + (1-\alpha_n)d(x_n, p) \\
&\leq d(x_n, p).
\end{aligned}
\tag{4.9}
$$

This shows that $\{x_n\}$ is Fejér monotone with respect to $\mathrm{Fix}(T)$. Notice from (4.9) that $d(x_n, p) \leq d(x_1, p)$ for all $n \geq 1$. This implies that $\{d(x_n, p)\}_{n=1}^{\infty}$ is bounded and decreasing. Hence $\lim_{n\to\infty} d(x_n, p)$ exists. □

Theorem 4.8. *Let E be a nonempty closed convex subset of a complete $\mathbb{R}$-tree X and $T : E \to CC(E)$ be a quasi-nonexpansive mapping satisfying the gate condition. Let u be a key of T and let $\{x_n\}$ be the Ishikawa iterates defined by (B). Assume that T satisfies condition (I) and $\alpha_n, \beta_n \in [a, b] \subset (0, 1)$. Then $\{x_n\}$ converges to a fixed point of T.*

Proof. Let $p \in \mathrm{Fix}(T)$. By Lemma 2.4(vi), we have

$$
\begin{aligned}
d^2(x_{n+1}, p) &= d^2(\alpha_n z'_n \oplus (1-\alpha_n)x_n, p) \\
&\leq (1-\alpha_n)d^2(x_n, p) + \alpha_n d^2(z'_n, p) - \alpha_n(1-\alpha_n)d^2(x_n, z'_n) \\
&= (1-\alpha_n)d^2(x_n, p) + \alpha_n d^2(P_{T(y_n)}(u), P_{T(p)}(u)) - \alpha_n(1-\alpha_n)d^2(x_n, z'_n) \\
&\leq (1-\alpha_n)d^2(x_n, p) + \alpha_n H^2(T(y_n), T(p)) - \alpha_n(1-\alpha_n)d^2(x_n, z'_n) \\
&\leq (1-\alpha_n)d^2(x_n, p) + \alpha_n d^2(y_n, p), \\
d^2(y_n, p) &= d^2(\beta_n z_n \oplus (1-\beta_n)x_n, p) \\
&\leq (1-\beta_n)d^2(x_n, p) + \beta_n d^2(z_n, p) - \beta_n(1-\beta_n)d^2(x_n, z_n) \\
&= (1-\beta_n)d^2(x_n, p) + \beta_n d^2(P_{T(x_n)}(u), P_{T(p)}(u)) - \beta_n(1-\beta_n)d^2(x_n, z_n) \\
&\leq (1-\beta_n)d^2(x_n, p) + \beta_n H^2(T(x_n), T(p)) - \beta_n(1-\beta_n)d^2(x_n, z_n) \\
&\leq (1-\beta_n)d^2(x_n, p) + \beta_n d^2(x_n, p) - \beta_n(1-\beta_n)d^2(x_n, z_n) \\
&\leq d^2(x_n, p) - \beta_n(1-\beta_n)d^2(x_n, z_n) \\
&\leq d^2(x_n, p) - \beta_n(1-\beta_n)d^2(x_n, z_n).
\end{aligned}
\tag{4.10}
$$

Thus, by (4.10) we have

$$
d^2(x_{n+1}, p) \leq (1-\alpha_n)d^2(x_n, p) + \alpha_n d^2(x_n, p) - \alpha_n\beta_n(1-\beta_n)d^2(x_n, z_n).
\tag{4.11}
$$

This implies that

$$a^2(1-b)d^2(x_n,z_n) \le \alpha_n\beta_n(1-\beta_n)d^2(x_n,z_n) \le d^2(x_n,p) - d^2(x_{n+1},p) \tag{4.12}$$

and so

$$\sum_{n=1}^{\infty} a^2(1-b)d^2(x_n,z_n) < \infty. \tag{4.13}$$

Thus, $\lim_{n\to\infty} d^2(x_n,z_n) = 0$. Also $\operatorname{dist}(x_n,T(x_n)) \le d(x_n,z_n) \to 0$ as $n \to \infty$. Since T satisfies condition (I), we have $\lim_{n\to\infty} d(x_n,\operatorname{Fix}(T)) = 0$. By Lemma 4.7, $\{x_n\}$ is Fejér monotone with respect to $\operatorname{Fix}(T)$. The conclusion follows from Proposition 4.6. □

As a consequence of Proposition 3.2 and Theorem 4.8, we obtain the following.

Corollary 4.9. *Let E be a nonempty closed convex subset of a complete $\mathbb{R}$-tree X and $T : E \to CC(E)$ be a nonexpansive mapping satisfying the gate condition. Let u be a key of T and let $\{x_n\}$ be the Ishikawa iterates defined by (B). Assume that T satisfies condition (I) and $\alpha_n, \beta_n \in [a,b] \subset (0,1)$. Then $\{x_n\}$ converges to a fixed point of T.*

Theorem 4.10. *Let E be a nonempty closed convex subset of a complete $\mathbb{R}$-tree X and $T : E \to CC(E)$ be a quasi-nonexpansive mapping satisfying the gate condition. Let u be a key of T and let $\{x_n\}$ be the Ishikawa iterates defined by (B). Assume that T is hemicompact and continuous and (i) $0 \le \alpha_n, \beta_n < 1$; (ii) $\beta_n < 1$ and (iii) $\sum \alpha_n\beta_n = \infty$. Then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. As in the proof of Theorem 4.8, we obtain

$$\lim_{n\to\infty} \operatorname{dist}(x_n,T(x_n)) = 0. \tag{4.14}$$

Since T is hemicompact, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to q$ for some $q \in E$. Since T is continuous, then

$$\operatorname{dist}(q,T(q)) \le d(q,x_{n_k}) + \operatorname{dist}(x_{n_k},T(x_{n_k})) + H(T(x_{n_k}),T(q)) \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \tag{4.15}$$

This implies that $q \in T(q)$. By Lemma 4.7, $\lim_{n\to\infty} d(x_n,q)$ exists and hence q is the limit of $\{x_n\}$ itself. □

Corollary 4.11. *Let E be a nonempty closed convex subset of a complete $\mathbb{R}$-tree X and $T : E \to CC(E)$ be a nonexpansive mapping satisfying the gate condition. Let u be a key of T and let $\{x_n\}$ be the Ishikawa iterates defined by (B). Assume that T is hemicompact and (i) $0 \le \alpha_n, \beta_n < 1$; (ii) $\beta_n < 1$; (iii) $\sum \alpha_n\beta_n = \infty$. Then $\{x_n\}$ converges strongly to a fixed point of T.*

5. Examples

Example 5.1 (see [7] (A nonexpansive mapping which is not $*$-nonexpansive)). Let E be the triangle in the Euclidean plane with vertexes $O(0,0), A(1,0), B(0,1)$. Let $T : E \to \mathcal{KC}(E)$ be given by

$$T(x,y) = \text{the segment joining } (0,1) \text{ and } (x,0). \tag{5.1}$$

Then for each $(x_1,y_1),(x_2,y_2) \in E$, we have

$$H(T(x_1,y_1),T(x_2,y_2)) = |x_1 - x_2| \le d((x_1,y_1),(x_2,y_2)). \tag{5.2}$$

Therefore, T is nonexpansive.

For each $(x,y) \in E$, we denote by $u_{(x,y)}$ the point in $T(x,y)$ nearest to (x,y). Thus, for $(x,y) \in E$ with $0 < x, y < 1$ we have

$$\left|u_{(x,y)} - u_{(1,0)}\right| > d((x,y),(1,0)). \tag{5.3}$$

This implies that T is not $*$-nonexpansive.

Example 5.2 (see [7] (A $*$-nonexpansive mapping which is not nonexpansive)). Let $E = [0,\infty)$ and $T : E \to \mathcal{KC}(E)$ be defined by

$$T(x) = [x, 2x] \quad \forall x \in E. \tag{5.4}$$

Then $u_x = x$ for every $x \in E$. This implies that T is $*$-nonexpansive. However, we have

$$H(T(x),T(y)) = H([x,2x],[y,2y]) = 2|x-y|. \tag{5.5}$$

This shows that T is not nonexpansive.

Example 5.3. Let $E = [0,1]$ and $T : E \to \mathcal{CC}(E)$ be defined by $T(x) = [0,x]$ for $x \in E$. Then $H(T(x),T(y)) = |x-y|$ for all $x, y \in E$. This implies that T is nonexpansive. We see that $\mathrm{Fix}(T) = [0,1]$ and $u = 1$ is a key of T. Since $\mathrm{End}(T) = \{0\}$, then T does not satisfy the endpoint condition.

6. Questions

It is not clear that the gate condition in Theorems 4.8 and 4.10 can be omitted. We finish the paper with the following questions.

Question 1. Let E be a nonempty closed convex subset of a complete $\mathbb{R}$-tree X and $T : E \to \mathcal{CC}(E)$ be a quasi-nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$. Let $\{x_n\}$ be the Ishikawa iterates defined by (B). Assume that T satisfies condition (I) and $\alpha_n, \beta_n \in [a,b] \subset (0,1)$. Does $\{x_n\}$ converge to a fixed point of T?

Question 2. Let E be a nonempty closed convex subset of a complete $\mathbb{R}$-tree X and $T : E \to \mathcal{CC}(E)$ be a quasi-nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$. Let $\{x_n\}$ be the Ishikawa iterates defined by (B). Assume that T is hemicompact and continuous and (i) $0 \leq \alpha_n, \beta_n < 1$; (ii) $\beta_n < 1$; (iii) $\sum \alpha_n \beta_n = \infty$. Does $\{x_n\}$ converge to a fixed point of T?

Acknowledgments

This research was supported by the Faculty of Science, Chiang Mai University, Chiang Mai, Thailand. The first author also thanks the Graduate School of Chiang Mai University, Thailand.

References

[1] F. E. Browder, "Nonexpansive nonlinear operators in a Banach space," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 54, pp. 1041–1044, 1965.

[2] D. Göhde, "Zum Prinzip der kontraktiven Abbildung," *Mathematische Nachrichten*, vol. 30, pp. 251–258, 1965.

[3] W. A. Kirk, "A fixed point theorem for mappings which do not increase distances," *The American Mathematical Monthly*, vol. 72, pp. 1004–1006, 1965.

[4] T. C. Lim, "A fixed point theorem for multivalued nonexpansive mappings in a uniformly convex Banach space," *Bulletin of the American Mathematical Society*, vol. 80, pp. 1123–1126, 1974.

[5] T. Husain and E. Tarafdar, "Fixed point theorems for multivalued mappings of nonexpansive type," *Yokohama Mathematical Journal*, vol. 28, no. 1-2, pp. 1–6, 1980.

[6] T. Husain and A. Latif, "Fixed points of multivalued nonexpansive maps," *Mathematica Japonica*, vol. 33, no. 3, pp. 385–391, 1988.

[7] H. K. Xu, "On weakly nonexpansive and $*$-nonexpansive multivalued mappings," *Mathematica Japonica*, vol. 36, no. 3, pp. 441–445, 1991.

[8] G. Lopez Acedo and H. K. Xu, "Remarks on multivalued nonexpansive mappings," *Soochow Journal of Mathematics*, vol. 21, no. 1, pp. 107–115, 1995.

[9] N. Shahzad and H. Zegeye, "On Mann and Ishikawa iteration schemes for multi-valued maps in Banach spaces," *Nonlinear Analysis A*, vol. 71, no. 3-4, pp. 838–844, 2009.

[10] T. Puttasontiphot, "Mann and Ishikawa iteration schemes for multivalued mappings in CAT(0) spaces," *Applied Mathematical Sciences*, vol. 4, no. 61–64, pp. 3005–3018, 2010.

[11] M. Bridson and A. Haefliger, *Metric Spaces of Non-Positive Curvature*, Springer, Berlin, Germany, 1999.

[12] W. A. Kirk, "Hyperconvexity of $\mathbb{R}$-trees," *Fundamenta Mathematicae*, vol. 156, no. 1, pp. 67–72, 1998.

[13] M. A. Khamsi and W. A. Kirk, *An Introduction to Metric Spaces and Fixed Point Theory*, Pure and Applied Mathematics, Wiley-Interscience, New York, NY, USA, 2001.

[14] R. Espinola and W. A. Kirk, "Fixed point theorems in $\mathbb{R}$-trees with applications to graph theory," *Topology and its Applications*, vol. 153, no. 7, pp. 1046–1055, 2006.

[15] J. T. Markin, "Fixed points, selections and best approximation for multivalued mappings in $\mathbb{R}$-trees," *Nonlinear Analysis A*, vol. 67, no. 9, pp. 2712–2716, 2007.

[16] A. G. Aksoy and M. A. Khamsi, "A selection theorem in metric trees," *Proceedings of the American Mathematical Society*, vol. 134, no. 10, pp. 2957–2966, 2006.

[17] S. Dhompongsa and B. Panyanak, "On Δ-convergence theorems in CAT(0) spaces," *Computers & Mathematics with Applications*, vol. 56, no. 10, pp. 2572–2579, 2008.

[18] J. T. Markin, "Fixed points for generalized nonexpansive mappings in $\mathbb{R}$-trees," *Computers & Mathematics with Applications*, vol. 62, no. 12, pp. 4614–4618, 2011.

[19] B. Piatek, "Best approximation of coincidence points in metric trees," *Annales Universitatis Mariae Curie-Skodowska A*, vol. 62, pp. 113–121, 2008.

[20] H. H. Bauschke and P. L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, Springer, New York, NY, USA, 2011.