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Research Article

Proximal Point Methods for Solving Mixed Variational Inequalities on the Hadamard Manifolds

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We use the auxiliary principle technique to suggest and analyze a proximal point method for solving the mixed variational inequalities on the Hadamard manifold. It is shown that the convergence of this proximal point method needs only pseudomonotonicity, which is a weaker condition than monotonicity. Some special cases are also considered. Results can be viewed as refinement and improvement of previously known results.

1. Introduction

In recent years, much attention has been given to study the variational inequalities and related problems on the Riemannian manifold and the Hadamard manifold. This framework is a useful for the developments of various fields. Several ideas and techniques from the Euclidean space have been extended and generalized to this nonlinear framework. The Hadamard manifolds are examples of hyperbolic spaces and geodesics; see [1–7] and the references therein. Németh [8], Tang et al. [6], and Colao et al. [2] have considered the variational inequalities and equilibrium problems on the Hadamard manifolds. They have studied the existence of a solution of the equilibrium problems under some suitable conditions. To the best of our knowledge, no one has considered the auxiliary principle technique for solving the mixed variational inequalities on the Hadamard manifolds. In this paper, we use the auxiliary principle technique to suggest and analyze a proximal iterative method for solving the mixed variational inequalities. If the nonlinearity in the mixed variational inequalities is an indicator function, then the mixed variational inequalities are equivalent to the variational inequality on the Hadamard manifold. This shows that the results obtained in this paper continue to hold for variational inequalities on the Hadamard manifold, which is due to Tang et al. [6] and Németh [8]. We hope that the technique and idea of this paper may stimulate further research in this area.

2. Preliminaries

We now recall some fundamental and basic concepts needed for a reading of this paper. These results and concepts can be found in the books on the Riemannian geometry [2, 3, 5].

Let M be a simply connected m-dimensional manifold. Given $x \in M$, the tangent space of M at x is denoted by T_xM and the tangent bundle of M by $TM = \cup_{x\in M} T_xM$, which is naturally a manifold. A vector field A on M is a mapping of M into TM which associates to each point $x \in M$, a vector $A(x) \in T_xM$. We always assume that M can be endowed with a Riemannian metric to become a Riemannian manifold. We denote by $\langle,\cdot,\rangle$ the scalar product on T_xM with the associated norm $\|\cdot\|_x$, where the subscript x will be omitted. Given a piecewise smooth curve $\gamma : [a,b] \to M$ joining x to y (i.e., $\gamma(a) = x$ and $\gamma(b) = y$) by using the metric, we can define the length of γ as $L(\gamma) = \int_a^b \|\gamma'(t)\| dt$. Then, for any $x, y \in M$ the Riemannian distance $d(x,y)$, which includes the original topology on M, is defined by minimizing this length over the set of all such curves joining x to y.

Let Δ be the Levi-Civita connection with $(M, \langle\cdot,\cdot\rangle)$. Let γ be a smooth curve in M. A vector field A is said to be parallel along γ if $\Delta_{\gamma'} A = 0$. If γ' itself is parallel along γ, we say that γ is a geodesic and in this case $\|\gamma'\|$ is constant. When $\|\gamma'\| = 1$, γ is said to be normalized. A geodesic joining x to y in M is said to be minimal if its length equals $d(x,y)$.

A Riemannian manifold is complete, if for any $x \in M$ all geodesics emanating from x are defined for all $t \in R$. By the Hopf-Rinow theorem, we know that if M is complete, then any pair of points in M can be joined by a minimal geodesic. Moreover, (M,d) is a complete metric space, and bounded closed subsets are compact.

Let M be complete. Then the exponential map $\exp_x : T_xM \to M$ at x is defined by $\exp_x v = \gamma_v(1,x)$ for each $v \in T_xM$, where $\gamma(\cdot) = \gamma_v(\cdot,x)$ is the geodesic starting at x with velocity v (i.e., $\gamma(0) = x$ and $\gamma'(0) = v$) Then $\exp_x tv = \gamma_v(t,x)$ for each real number t.

A complete simply connected Riemannian manifold of nonpositive sectional curvature is called a *Hadamard manifold*. Throughout the remainder of this paper, we always assume that M is an m-manifold Hadamard manifold.

We also recall the following well-known results, which are essential for our work.

Lemma 2.1 (see [5]). *Let $x \in M$. Then $\exp_x : T_xM \to M$ is a diffeomorphism, and for any two points $x, y \in M$, there exists a unique normalized geodesic joining x to y, $\gamma_{x,y}$, which is minimal.*

So from now on, when referring to the geodesic joining two points, we mean the unique minimal normalized one. Lemma 2.1 says that M is diffeomorphic to the Euclidean space R^m. Thus M has the same topology and differential structure as R^m. It is also known that the Hadamard manifolds and Euclidean spaces have similar geometrical properties. Recall that a geodesic triangle $\triangle(x_1, x_2,$ and $x_3)$ of a Riemannian manifold is a set consisting of three points x_1, x_2, x_3 and three minimal geodesics joining these points.

Lemma 2.2 (see (comparison Theorem for Triangles [2, 3, 5])). *Let $\triangle(x_1, x_2, x_3)$ be a geodesic triangle. Denote, for each $i = 1, 2, 3 \pmod 3$, by $\gamma_i : [0, l_i] \to M$ the geodesic joining x_i to x_{i+1}, and*

$\alpha_i; = L(\gamma_i'(0), -\gamma_l'(i-1)(li-1))$, the angle between the vectors $\gamma_i'(0)$ and $-\gamma_{i-1}'(l_{i-1})$, and $l_i; = L(\gamma_i)$. Then

$$\alpha_1 + \alpha_2 + \alpha_3 \leq \pi \quad , \tag{2.1}$$

$$l_l^2 + l_{i+1}^2 - 2L_i l_{i+1} \cos \alpha_{i+1} \leq l_{i-1}^2. \tag{2.2}$$

In terms of the distance and the exponential map, the inequality (2.2) *can be rewritten as*

$$d^2(x_i, x_{i+1}) + d^2(x_{i+1}, x_{i+2}) - 2\left\langle \exp_{x_{i+1}}^{-1} x_i, \exp_{x_{i+1}}^{-1} x_{i+2} \right\rangle \leq d^2(x_{i-1}, x_i), \tag{2.3}$$

since

$$\left\langle \exp_{x_{i+1}}^{-1} x_i, \exp_{x_{i+1}}^{-1} x_{i+2} \right\rangle = d(x_i, x_{i+1}) d(x_{i+1}, x_{i+2}) \cos \alpha_{i+1}. \tag{2.4}$$

Lemma 2.3 (see [5]). *Let $\triangle(x, y, z)$ be a geodesic triangle in a Hadamard manifold M. Then, there exist $x', y', z' \in R^2$ such that*

$$d(x, y) = \|x' - y'\|, \qquad d(y, z) = \|y' - z'\|, \qquad d(z, x) = \|z' - x'\|. \tag{2.5}$$

The triangle $\triangle(x', y', z')$ is called the comparison triangle of the geodesic triangle $\triangle(x, y, z)$, which is unique up to isometry of M.

From the law of cosines in inequality (2.3), we have the following inequality, which is a general characteristic of the spaces with nonpositive curvature [5]:

$$\left\langle \exp_x^{-1} y, \exp_x^{-1} z \right\rangle + \left\langle \exp_y^{-1} x, \exp_y^{-1} z \right\rangle \geq d^2(x, y). \tag{2.6}$$

From the properties of the exponential map, we have the following known result.

Lemma 2.4 (see [5]). *Let $x_0 \in M$ and $\{x_n\} \subset M$ such that $x_n \to x_0$. Then the following assertions hold.*

(i) *For any $y \in M$,*

$$\exp_{x_n}^{-1} y \longrightarrow \exp_{x_0}^{-1} y, \qquad \exp_y^{-1} x_n \longrightarrow \exp_y^{-1} x_0. \tag{2.7}$$

(ii) *If $\{v_n\}$ is a sequence such that $v_n \in T_{x_n}M$ and $v_n \to v_0$, then $v_0 \in T_{x_0}M$.*

(iii) *Given the sequences $\{u_n\}$ and $\{v_n\}$ satisfying $u_n, v_n \in T_{x_n}M$, if $u_n \to u_0$ and $v_n \to v_0$, with $u_0, v_0 \in T_{x_0}M$, then*

$$\langle u_n, v_n \rangle \longrightarrow \langle u_0, v_0 \rangle. \tag{2.8}$$

A subset $K \subseteq M$ is said to be convex if for any two points $x, y \in K$, the geodesic joining x and y is contained in K, K; that is, if $\gamma : [a, b] \to M$ is a geodesic such that $x = \gamma(a)$ and $y = \gamma(b)$, then $\gamma((1-t)a + tb) \in K$, for all $t \in [0, 1]$. From now on $K \subseteq M$ will denote a nonempty, closed and convex set, unless explicitly stated otherwise.

A real-valued function f defined on K is said to be convex if, for any geodesic γ of M, the composition function $f \circ \gamma : R \to R$ is convex; that is,

$$(f \circ \gamma)(ta + (1-t)b) \leq t(f \circ \gamma)(a) + (1-t)(f \circ \gamma)(b), \quad \forall a, b \in R, \ t \in [0, 1]. \tag{2.9}$$

The subdifferential of a function $f : M \to R$ is the set-valued mapping $\partial f : M \to 2^{TM}$ defined as

$$\partial f(x) = \left\{ u \in T_x M : \left\langle u, \exp_x^{-1} y \right\rangle \leq f(y) - f(x), \forall y \in M \right\}, \quad \forall x \in M, \tag{2.10}$$

and its elements are called subgradients. The subdifferential $\partial f(x)$ at a point $x \in M$ is a closed and convex (possibly empty) set. Let $D(\partial f)$ denote the domain of ∂f defined by

$$D(\partial f) = \{ x \in M : \partial f(x) \neq \emptyset \}. \tag{2.11}$$

The existence of subgradients for convex functions is guaranteed by the following proposition; see [7].

Lemma 2.5 (see [5, 7]). *Let M be a Hadamard manifold and $f : M \to R$ convex. Then, for any $x \in M$, the subdifferential $\partial f(x)$ of f at x is nonempty; that is, $D(\partial f) = M$.*

For a given single-valued vector field $T : M \to TM$ and a real-valued function $f : M \to R$, we consider the problem of finding $u \in M$ such that

$$\left\langle Tu, \exp_u^{-1} v \right\rangle + f(v) - f(u) \geq 0, \quad \forall v \in M, \tag{2.12}$$

which is called the mixed variational inequality. This problem was considered by Colao et al. [2]. They proved the existence of a solution of problem (2.12) using the KKM maps. For the applications, formulation, and other aspects of the mixed variational inequalities in the linear setting, see [8–16].

We remark that if the function f is an indicator of a closed and convex set K in M, then problem (2.12) is equivalent to finding $u \in K$ such that

$$\left\langle Tu, \exp_x^{-1} v \right\rangle \geq 0, \quad \forall v \in K, \tag{2.13}$$

which is called the variational inequality on the Hadamard manifolds. Németh [8], Colao et al. [2] and Udrişte [7] studied variational inequalities on the Hadamard manifold from different point of views. In the linear setting, variational inequalities have been studied extensively; see [8–25] and the references therein.

Definition 2.6. An operator T is said to be speudomonotone with respect a mapping f, if and only if

$$\left\langle T(u), \exp_u^{-1} v \right\rangle + f(v) - f(u) \geq 0 \Longrightarrow \left\langle T(v), \exp_v^{-1} u \right\rangle + f(v) - f(u) \geq 0, \quad \forall u, v \in M. \tag{2.14}$$

3. Main Results

We now use the auxiliary principle technique of Glowinski et al. [9] to suggest and analyze an implicit iterative method for solving the mixed variational inequality (2.12) on the Hadamard manifold.

For a given $u \in M$ satisfying (2.12), consider the problem of finding $w \in M$ such that

$$\left\langle \rho T w + \left(\exp_u^{-1} w\right), \exp_w^{-1} v \right\rangle + f(v) - f(w) \geq 0, \quad \forall v \in M, \tag{3.1}$$

which is called the auxiliary mixed variational inequality on the Hadamard manifolds. We note that if $w = u$, then w is a solution of the mixed variational inequality (2.12). This observation enable to suggest and analyzes the following proximal point method for solving the mixed variational inequality (2.12).

Algorithm 3.1. For a given u_0, compute the approximate solution by the iterative scheme:

$$\left\langle \rho T u_{n+1} + \left(\exp_{u_n}^{-1} u_{n+1}\right), \exp_{u_{n+1}}^{-1} v \right\rangle + f(v) - f(u_{n+1}) \geq 0, \quad \forall v \in M, \tag{3.2}$$

Algorithm 3.1 is called the implicit (proximal point) iterative method for solving the mixed variational inequality on the Hadamard manifold.

If $M = R^n$, then Algorithm 3.1 collapses to the following algorithm:

Algorithm 3.2. For a given $u_0 \in R^n$, find the approximate solution u_{n+1} by the iterative scheme.

$$\langle \rho T u_{n+1} + u_{n+1} - u_n, v - u_{n+1} \rangle + \rho f(v) - f(u_{n+1}) \geq 0, \quad \forall v \in R^n, \tag{3.3}$$

which is known as the proximal pint method for solving the mixed variational inequalities. For the convergence analysis of Algorithm 3.2, see [11, 12].

If f is the indicator function of a closed and convex set K in M, then Algorithm 3.1 reduces to the following method, which is due to Tang et al. [6].

Algorithm 3.3. For a given $u_0 \in K$, compute the approximate solution by the iterative scheme

$$\left\langle \rho T u_{n+1} + \left(\exp_{u_n}^{-1} u_{n+1}\right), \exp_{u_{n+1}}^{-1} v \right\rangle \geq 0, \quad \forall v \in K. \tag{3.4}$$

We would like to mention that Algorithm 3.1 can be rewritten in the following equivalent form.

Algorithm 3.4. For a given $u_0 \in M$, compute the approximate solution by the iterative scheme:

$$\begin{aligned} &\left\langle \rho T u_n + \exp_{u_n}^{-1} y_n, \exp_{y_n}^{-1} v \right\rangle + \rho f(v) - \rho f(y_n) \geq 0 \quad \forall v \in M, \\ &\left\langle T y_n + \exp_{u_n}^{-1} u_{n+1}, \exp_{u_{n+1}}^{-1} v \right\rangle + \rho f(v) - \rho f(u_{n+1}), \quad \forall v \in M, \end{aligned} \tag{3.5}$$

which is called the extraresolvent method for solving the mixed variational inequalities on the Hadamard manifolds.

In a similar way, one can obtain several iterative methods for solving the variational inequalities on the Hadamard manifold.

We now consider the convergence analysis of Algorithm 3.1, and this is the main motivation of our next result.

Theorem 3.5. *Let T be a pseudomonotone vector field. Let u_n be the approximate solution of the mixed variational inequality* (2.12) *obtained from Algorithm 3.1; then*

$$d^2(u_{n+1}, u) + d^2(u_{n+1}, u_n) \leq d^2(u_n, u), \tag{3.6}$$

where $u \in M$ is the solution of the mixed variational inequality (2.12).

Proof. Let $u \in M$ be a solution of the mixed variational inequality (). Then, by using the pseudomonotonicity of the vector filed, $T(u)$, we have

$$\left\langle \rho T(v), \exp_u^{-1} v \right\rangle + \rho f(v) - \rho f(u) \leq 0, \quad \forall v \in M. \tag{3.7}$$

Taking $v = u_{n+1}$ in (3.7), we have

$$\left\langle \rho T(u_{n+1}), \exp_u^{-1} u_{n+1} \right\rangle + \rho f(u_{n+1}) - \rho f(u) \leq 0. \tag{3.8}$$

Taking $v = u$ in (3.2), we have

$$\left\langle \rho T u_{n+1} + \left(\exp_{u_n}^{-1} u_{n+1}\right), \exp_{u_{n+1}}^{-1} u \right\rangle + f(u) - f(u_{n+1}) \geq 0. \tag{3.9}$$

From (3.8) and (3.9), we have

$$\left\langle \exp_{u_{n+1}}^{-1} u_n, \exp_{u_{n+1}}^{-1} u \right\rangle \leq 0. \tag{3.10}$$

For the geodesic triangle $\triangle(u_n, u_{n+1}, u)$ the inequality (3.10) can be written as,

$$d^2(u_{n+1}, u) + d^2(u_{n+1}, u_n) - \left\langle \exp_{u_{n+1}}^{-1} u_n, \exp_{u_{n+1}}^{-1} u \right\rangle \leq d^2(u_n, u). \tag{3.11}$$

Thus, from (3.10) and (3.11), we obtained inequality (3.6), the required result. □

Theorem 3.6. *Let $u \in M$ be solution of* (2.12), *and let u_{n+1} be the approximate solution obtained from Algorithm 3.1; then* $\lim_{n\to\infty}(u_{n+1}) = u$.

Proof. Let $\hat{u} \in M$ be a solution of (2.12). Then, from (3.6), it follows that the sequence $\{u_n\}$ is bounded and

$$\sum_{n=0}^{\infty} d^2(u_{n+1}, u_n) \leq d^2(u_0, u); \tag{3.12}$$

it follows that

$$\lim_{n\to\infty} d(u_{n+1}, u_n) = 0. \tag{3.13}$$

Let $\hat{u}$ be a cluster point of $\{u_n\}$. Then there exists a subsequence $\{u_{n_i}\}$ such that $\{u_{u_i}\}$ converges to $\hat{u}$. Replacing u_{n+1} by u_{n_i} in (3.2), taking the limit, and using (3.13), we have

$$\langle T\hat{u}, \exp_{\hat{u}}^{-1} v\rangle + f(v) - f(\hat{u}) \geq 0, \quad \forall v \in M. \tag{3.14}$$

This shows that $\hat{u} \in M$ solves (2.12) and

$$d^2(u_{n+1}\hat{u}) \leq d^2(u_n, \hat{u}), \tag{3.15}$$

which implies that the sequence $\{u_n\}$ has unique cluster point and $\lim_{n\to\infty} u_n = \hat{u}$ is a solution of (2.12), the required result. □

4. Conclusion

We have used the auxiliary principle technique to suggest and analyzed a proximal point iterative method for solving the mixed quasi-variational inequalities on the Hadamard manifolds. Some special cases are also discussed. Convergence analysis of the new proximal point method is proved under weaker conditions. Results obtained in this paper may stimulate further research in this area. The implementation of the new method and its comparison with other methods is an open problem. The ideas and techniques of this paper may be extended for other related optimization problems.

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References

[1] D. Azagra, J. Ferrera, and F. López-Mesas, "Nonsmooth analysis and Hamilton-Jacobi equations on Riemannian manifolds," *Journal of Functional Analysis*, vol. 220, no. 2, pp. 304–361, 2005.

[2] V. Colao, G. Lopez, G. Marino, and V. Martin-Marquez, “Equilibrium problems in Hadamard manifolds,” *Journal of Mathematical Analysis and Applications*, vol. 388, pp. 61–77, 2012.
[3] M. P. do Carmo, *Riemannian Geometry*, Mathematics: Theory & Applications, Birkhäuser, Boston, Mass, USA, 1992.
[4] O. P. Ferreira and P. R. Oliveira, “Proximal point algorithm on Riemannian manifolds,” *Optimization*, vol. 51, no. 2, pp. 257–270, 2002.
[5] T. Sakai, *Riemannian Geometry*, vol. 149, American Mathematical Society, Providence, RI, USA, 1996.
[6] G. Tang, L. W. Zhou, and N. J. Huang, “The proximal point algorithm for pseudomontone variational inequalities on Hadamard manifolds,” *Optimization Letters*. In press.
[7] C. Udrişte, *Convex Functions and Optimization Methods on Riemannian Manifolds*, vol. 297, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1994.
[8] S. Z. Németh, “Variational inequalities on Hadamard manifolds,” *Nonlinear Analysis. Theory, Methods & Applications*, vol. 52, no. 5, pp. 1491–1498, 2003.
[9] R. Glowinski, J.-L. Lions, and R. Trémolières, *Numerical Analysis of Variational Inequalities*, vol. 8 of *Studies in Mathematics and Its Applications*, North-Holland, Amsterdam, The Netherlands, 1981.
[10] M. A. Noor, “General variational inequalities,” *Applied Mathematics Letters*, vol. 1, no. 2, pp. 119–122, 1988.
[11] M. A. Noor, “New approximation schemes for general variational inequalities,” *Journal of Mathematical Analysis and Applications*, vol. 251, no. 1, pp. 217–229, 2000.
[12] M. Aslam Noor, “Some developments in general variational inequalities,” *Applied Mathematics and Computation*, vol. 152, no. 1, pp. 199–277, 2004.
[13] M. A. Noor, “Fundamentals of mixed quasi variational inequalities,” *International Journal of Pure and Applied Mathematics*, vol. 15, no. 2, pp. 137–258, 2004.
[14] M. A. Noor, “Fundamentals of equilibrium problems,” *Mathematical Inequalities & Applications*, vol. 9, no. 3, pp. 529–566, 2006.
[15] M. A. Noor, “Extended general variational inequalities,” *Applied Mathematics Letters*, vol. 22, no. 2, pp. 182–186, 2009.
[16] M. A. Noor, “On an implicit method for nonconvex variational inequalities,” *Journal of Optimization Theory and Applications*, vol. 147, no. 2, pp. 411–417, 2010.
[17] M. A. Noor, “Auxiliary principle technique for solving general mixed variational inequalities,” *Journal of Advanced Mathematical Studies*, vol. 3, no. 2, pp. 89–96, 2010.
[18] M. A. Noor, “Some aspects of extended general variational inequalities,” *Abstract and Applied Analysis*, vol. 2012, Article ID 303569, 16 pages, 2012.
[19] M. A. Noor, K. I. Noor, and T. M. Rassias, “Some aspects of variational inequalities,” *Journal of Computational and Applied Mathematics*, vol. 47, no. 3, pp. 285–312, 1993.
[20] M. A. Noor, K. I. Noor, and E. Al-Said, “Iterative methods for solving nonconvex equilibrium variational inequalities,” *Applied Mathematics & Information Sciences*, vol. 6, no. 1, pp. 65–69, 2012.
[21] Y. Yao, Y. C. Liou, and S. M. Kang, “Two-step projection methods for a system of variational inequality problems in Banach spaces,” *Journal of Global Optimization*. In press.
[22] Y. Yao and N. Shahzad, “Strong convergence of a proximal point algorithm with general errors,” *Optimization Letters*, vol. 6, no. 4, pp. 621–628, 2012.
[23] Y. Yao, M. A. Noor, and Y. C. Liou, “Strong convergence of a modified extra-gradient method to the minimum-norm solution of variational inequalities,” *Abstract and Applied Analysis*, vol. 2012, Article ID 817436, 9 pages, 2012.
[24] Y. Yao, R. Chen, and Y. C. Liou, “A unified implicit algorithm for solving the triple-hierarchical constrained optimization problem,” *Mathematical and Computer Modelling*, vol. 55, pp. 1506–1515, 2012.
[25] Y. Yao, M. A. Noor, Y. C. Liou, and S. M. Kang, “Iterative algorithms for general multi-valued variational inequalities,” *Abstract and Applied Analysis*, vol. 2012, Article ID 768272, 10 pages, 2012.