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Research Article

Stability of Jensen-Type Functional Equations on Restricted Domains in a Group and Their Asymptotic Behaviors

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We consider the Hyers-Ulam stability problems for the Jensen-type functional equations in general restricted domains. The main purpose of this paper is to find the restricted domains for which the functional inequality satisfied in those domains extends to the inequality for whole domain. As consequences of the results we obtain asymptotic behavior of the equations.

1. Introduction

The Hyers-Ulam stability problems of functional equations was originated by Ulam in 1960 when he proposed the following question [1].

Let f be a mapping from a group G_1 to a metric group G_2 with metric $d(\cdot,\cdot)$ such that

$$d(f(xy), f(x)f(y)) \le \varepsilon. \tag{1.1}$$

Then does there exist a group homomorphism h and $\delta_\epsilon > 0$ such that

$$d(f(x), h(x)) \le \delta_\epsilon \tag{1.2}$$

for all $x \in G_1$?

One of the first assertions to be obtained is the following result, essentially due to Hyers [2], that gives an answer for the question of Ulam.

Theorem 1.1. *Suppose that S is an additive semigroup, Y is a Banach space, $\epsilon \geq 0$, and $f : S \to Y$ satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon \tag{1.3}$$

for all $x,\ y \in S$. Then there exists a unique function $A : S \to Y$ satisfying

$$A(x+y) = A(x) + A(y) \tag{1.4}$$

for which

$$\|f(x) - A(x)\| \leq \epsilon \tag{1.5}$$

for all $x \in S$.

We call the functions satisfying (1.4) *additive functions*. Perhaps Aoki in 1950 was the first author treating the generalized version of Hyers' theorem [3]. Generalizing Hyers' result he proved that if a mapping $f : X \to Y$ between two Banach spaces satisfies

$$\|f(x+y) - f(x) - f(y)\| \leq \Phi(x, y) \text{ for } x, y \in X \tag{1.6}$$

with $\Phi(x, y) = \epsilon(\|x\|^p + \|y\|^p)$ $(\epsilon \geq 0,\ 0 \leq p < 1)$, then there exists a unique additive function $A : X \to Y$ such that $\|f(x) - A(x)\| \leq 2\epsilon\|x\|^p/(2-2^p)$ for all $x \in X$. In 1951 Bourgin [4, 5] stated that if Φ is symmetric in $\|x\|$ and $\|y\|$ with $\sum_{j=1}^{\infty} \Phi(2^j x, 2^j x)/2^j < \infty$ for each $x \in X$, then there exists a unique additive function $A : X \to Y$ such that $\|f(x) - A(x)\| \leq \sum_{j=1}^{\infty} \Phi(2^j x, 2^j x)/2^j$ for all $x \in X$. Unfortunately, there was no use of these results until 1978 when Rassias [6] dealt with the inequality of Aoki [3]. Following Rassias' result, a great number of papers on the subject have been published concerning numerous functional equations in various directions [6–15]. Among the results, stability problem in a restricted domain was investigated by Skof, who proved the stability problem of inequality (1.3) in a restricted domain [16, 17]. Developing this result, Jung, Rassias, and M. J. Rassias considered the stability problems in restricted domains for some functional equations including the Jensen functional equation [9] and Jensen-type functional equations [13]. We also refer the reader to [18–27] for some related results on Hyers-Ulam stabilities in restricted conditions. The results can be summarized as follows. Let X and Y be a real normed space and a real Banach space, respectively. For fixed $d \geq 0$, if $f : X \to Y$ satisfies the functional inequalities (such as that of Cauchy, quadratic, Jensen, and Jensen type) for all $x, y \in X$ with $\|x\| + \|y\| \geq d$ (which is the case where the inequalities are given by two indeterminate variables x and y), the inequalities hold for all $x, y \in X$. Following the approach in [28] we consider the Jensen-type equation in various restricted domains in an Abelian group. As applications, we obtain the stability problems for the above equations in more general restricted domains than that of the form $\{(x, y) \in X : \|x\| + \|y\| \geq d\}$, which generalizes and refines the stability theorems in [13]. As an application we obtain asymptotic behaviors of the equations.

2. Stability of Jensen-Type Functional Equations

Throughout this section, we denote by G, X, and Y, an Abelian group, a real normed space, and a Banach space, respectively. In this section we consider the Hyers-Ulam stability of the Jensen and Jensen-type functional inequalities for the functions $f : G \to Y$

$$\|f(x+y) + f(x-y) - 2f(x)\| \leq \epsilon, \tag{2.1}$$

$$\|f(x+y) - f(x-y) - 2f(y)\| \leq \epsilon \tag{2.2}$$

in restricted domains $U \subset G \times G$.

Inequalities (2.1) and (2.2) were previously treated by J. M. Rassias and M. J. Rassias [13], who proved the Hyers-Ulam stability of the inequalities in the restricted domain $U = \{(x, y) : \|x\| + \|y\| \geq d\},\ d \geq 0$, for the functions $f : X \to Y$:

Theorem 2.1. *Let $d \geq 0$ and $\epsilon > 0$ be fixed. Suppose that $f : X \to Y$ satisfies the inequality*

$$\|f(x+y) + f(x-y) - 2f(x)\| \leq \epsilon \tag{2.3}$$

for all $x, y \in X$, with $\|x\| + \|y\| \geq d$. Then there exists a unique additive function $A : X \to Y$ such that

$$\|f(x) - A(x) - f(0)\| \leq \frac{5}{2}\epsilon \tag{2.4}$$

for all $x \in X$.

Theorem 2.2. *Let $d \geq 0$ and $\epsilon > 0$ be fixed. Suppose that $f : X \to Y$ satisfies the inequality*

$$\|f(x+y) - f(x-y) - 2f(y)\| \leq \epsilon \tag{2.5}$$

for all $x, y \in X$, with $\|x\| + \|y\| \geq d$ and

$$\|f(x) + f(-x)\| \leq 3\epsilon \tag{2.6}$$

for all $x \in X$, with $\|x\| \geq d$. Then there exists a unique additive function $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{33}{2}\epsilon \tag{2.7}$$

for all $x \in X$.

We use the following usual notations. We denote by $G \times G = \{(a_1, a_2) : a_1, a_2 \in G\}$ the product group; that is, for $a = (a_1, a_2),\ b = (b_1, b_2) \in G \times G$, we define $a + b = (a_1 + b_1, a_2 + b_2),\ a - b = (a_1 - b_1, a_2 - b_2)$. For a subset H of $G \times G$ and $a, b \in G \times G$, we define $a + H = \{a + h : h \in H\}$.

For given $x, y \in G$ we denote by $P_{x,y}$, $Q_{x,y}$ the subsets of points of the forms (not necessarily distinct) in $G \times G$, respectively,

$$P_{x,y} = \{(0,0), (x,-x), (y,y), (x+y,-x+y)\},$$
$$Q_{x,y} = \{(-x,x), (y,y), (-x+y,x+y)\}. \tag{2.8}$$

The set $P_{x,y}$ can be viewed as the vertices of rectangles in $G \times G$, and $Q_{x,y}$ can be viewed as a subset of the vertices of rectangles in $G \times G$.

Definition 2.3. Let $U \subset G \times G$. One introduces the following conditions $(J1)$ and $(J2)$ on U. For any $x, y \in G$, there exists a $z \in G$ such that

$$(J1) \quad (0,z) + P_{x,y} = \{(0,z), (x,-x+z), (y,y+z), (x+y,-x+y+z)\} \subset U,$$
$$(J2) \quad (z,0) + Q_{x,y} = \{(-x+z,x), (y+z,y), (-x+y+z,x+y)\} \subset U, \tag{2.9}$$

respectively.

The sets $(0,z)+P_{x,y}$, $(z,0)+Q_{x,y}$ can be understood as the translations of $P_{x,y}$ and $Q_{x,y}$ by $(0,z)$ and $(z,0)$, respectively.

There are many interesting examples of the sets U satisfying some of the conditions $(J1)$ and $(J2)$. We start with some trivial examples.

Example 2.4. Let G be a real normed space. For $d \geq 0$, $x_0, y_0 \in G$, let

$$U = \{(x,y) \in G \times G \ : k\|x\| + s\|y\| \geq d\},$$
$$V = \{(x,y) \in G \times G \ : \|kx + sy\| \geq d\}. \tag{2.10}$$

Then U satisfies $(J1)$ if $s > 0$, $(J2)$ if $k > 0$ and V satisfies $(J1)$ if $s \neq 0$, $(J2)$ if $k \neq 0$.

Example 2.5. Let G be a real inner product space. For $d \geq 0$, $x_0, y_0 \in G$

$$U = \{(x,y) \in G \times G \ : \langle x_0, x\rangle + \langle y_0, y\rangle \geq d\}. \tag{2.11}$$

Then U satisfies $(J1)$ if $y_0 \neq 0$, $(J2)$ if $x_0 \neq 0$.

Example 2.6. Let G be the group of nonsingular square matrices with the operation of matrix multiplication. For $k, s \in \mathbb{R}$, $\delta, d \geq 0$, let

$$U = \left\{(P_1, P_2) \in G \times G \ : |\det P_1|^k |\det P_2|^s \leq \delta\right\},$$
$$V = \left\{(P_1, P_2) \in G \times G \ : |\det P_1|^k |\det P_2|^s \geq d\right\}. \tag{2.12}$$

Then both U and V satisfy $(J1)$ if $s \neq 0$, $(J2)$ if $k \neq 0$.

In the following one can see that if $P_{x,y}$ and $Q_{x,y}$ are replaced by arbitrary subsets of four points (not necessarily distinct) in $G\times G$, respectively, then the conditions become stronger; that is, there are subsets U_1 and U_2 which satisfy the conditions $(J1)$ and $(J2)$, respectively, but U_1 and U_2 fail to fulfill the following conditions (2.13) and (2.14), respectively. For any subset $\{X_1, X_2, X_3, X_4\}$ of points (not necessarily distinct) in $G \times G$, there exists a $z \in G$ such that

$$(0, z) + \{X_1, X_2, X_3, X_4\} \subset U_1, \tag{2.13}$$

$$(z, 0) + \{X_1, X_2, X_3, X_4\} \subset U_2, \tag{2.14}$$

respectively.

Now we give examples of U_1 and U_2 which satisfy $(J1)$ and $(J2)$, respectively, but not (2.13) and (2.14), respectively.

Example 2.7. Let $G = \mathbb{Z}$ be the group of integers. Enumerate

$$\mathbb{Z} \times \mathbb{Z} = \{(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n), \ldots\} \subset \mathbb{R}^2 \tag{2.15}$$

such that

$$|a_1| + |b_1| \le |a_2| + |b_2| \le \cdots \le |a_n| + |b_n| \le \cdots, \tag{2.16}$$

and let $P_n = \{(0, 0), (a_n, -a_n), (b_n, b_n), (a_n + b_n, -a_n + b_n)\}$, $n = 1, 2, \ldots$. Then it is easy to see that $U = \bigcup_{n=1}^{\infty}((0, 2^n) + P_n)$ satisfies the condition $(J1)$. Now let $P = \{(p_1, q_1), (p_2, q_2)\} \subset \mathbb{Z} \times \mathbb{Z}$ with $|q_2 - q_1| \le |p_2 - p_1|$, $p_1p_2 > 0$. Then $(0, z) + P$ is not contained in U for all $z \in \mathbb{Z}$. Indeed, for any choices of $(x_n, y_n) \in P_n + (0, 2^n)$, $n = 1, 2, \ldots$, we have $y_m - y_n > |x_m - x_n|$ for all $m > n$, $m, n = 1, 2, \ldots$. Thus, if $(0, z)+P \subset U$ for some $z \in \mathbb{Z}$, then $P+(0, z) \subset (0, 2^n)+P_n$ for some $n \in \mathbb{N}$. Thus, it follows from the condition $q_2-q_1 \le |p_2-p_1|$ that the line segment joining the points of $P + (-z, z)$ intersects the line $x = 0$ in $\mathbb{R}^2$, which contradicts the condition $p_1p_2 > 0$. Similarly, let $Q_n = \{(-a_n, a_n), (b_n, b_n), (-a_n+b_n, a_n+b_n)\}$. Then it is easy to see that $U = \bigcup_{n=1}^{\infty}((2^n, 0)+Q_n)$ satisfies the condition $(J2)$ but not (2.14).

Theorem 2.8. *Let $U \subset G\times G$ satisfy the condition $(J1)$ and $\epsilon \ge 0$. Suppose that $f : G \to Y$ satisfies* (2.1) *for all $(x, y) \in U$. Then there exists an additive function $A : G \to Y$ such that*

$$\|f(x) - A(x) - f(0)\| \le 2\epsilon \tag{2.17}$$

for all $x \in G$.

Proof. For given $x, y \in G$, choose a $z \in G$ such that $(0, z) + P_{x,y} \subset U$. Replacing x by $x + y$, y by $-x + y + z$; x by x, y by $-x + z$; x by y, y by $y + z$; x by 0, y by z in (2.1), respectively, we have

$$\begin{aligned} \|f(2y + z) + f(2x - z) - 2f(x + y)\| &\le \epsilon, \\ \|f(z) + f(2x - z) - 2f(x)\| &\le \epsilon, \end{aligned}$$

$$\begin{aligned}\|f(2y+z)+f(-z)-2f(y)\| &\le \epsilon,\\ \|f(z)+f(-z)-2f(0)\| &\le \epsilon.\end{aligned} \tag{2.18}$$

From (2.18), using the triangle inequality and dividing the result by 2, we have

$$|f(x+y)-f(x)-f(y)+f(0)| \le 2\epsilon \tag{2.19}$$

for all $x, y \in G$. From (2.19), using Theorem 1.1, we get the result. □

Let $d \ge 0$, $s \in \mathbb{R}$, and let $U = \{(x,y) : \|x\| + s\|y\| \ge d\}$. Then U satisfies the condition $(J1)$. Thus, as a direct consequence of Theorem 2.8, we obtain the following (cf. Theorem 2.1).

Corollary 2.9. *Let $d \ge 0$, $s \in \mathbb{R}$. Suppose that $f : X \to Y$ satisfies inequality* (2.1) *for all $x, y \in X$, with $\|x\| + s\|y\| \ge d$. Then there exists a unique additive function $A : X \to Y$ such that*

$$\|f(x)-A(x)-f(0)\| \le 2\epsilon \tag{2.20}$$

for all $x \in X$.

Theorem 2.10. *Let $U \subset G\times G$ satisfy the condition $(J2)$ and $\epsilon \ge 0$. Suppose that $f : G \to Y$ satisfies* (2.2) *for all $(x,y) \in U$. Then there exists a unique additive function $A : G \to Y$ such that*

$$\|f(x)-A(x)\| \le \frac{3}{2}\epsilon \tag{2.21}$$

for all $x \in G$.

Proof. For given $x, y \in G$, choose $z \in G$ such that $(z,0)+Q_{x,y} \subset U$. Replacing x by $-x+y+z$, y by $x+y$; x by $-x+z$, y by x; x by $y+z$, y by y in (2.2), respectively, we have

$$\begin{aligned}\|f(2y+z)-f(-2x+z)-2f(x+y)\| &\le \epsilon,\\ \|f(z)-f(-2x+z)-2f(x)\| &\le \epsilon,\\ \|f(2y+z)-f(z)-2f(y)\| &\le \epsilon.\end{aligned} \tag{2.22}$$

From (2.22), using the triangle inequality and dividing the result by 2, we have

$$|f(x+y)-f(x)-f(y)| \le \frac{3}{2}\epsilon. \tag{2.23}$$

Now by Theorem 1.1, we get the result. □

Let $d \ge 0$, $k \in \mathbb{R}$, and let $U = \{(x,y) : k\|x\| + \|y\| \ge d\}$. Then U satisfies the condition $(J2)$. Thus, as a direct consequence of Theorem 2.10, we generalize and refine Theorem 2.2 as follows.

Corollary 2.11. *Let $d \geq 0$, $k \in \mathbb{R}$. Suppose that $f : X \to Y$ satisfies inequality (2.2) for all x, y, with $k\|x\| + \|y\| \geq d$. Then there exists a unique additive function $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \leq \frac{3}{2}\epsilon \tag{2.24}$$

for all $x \in X$.

Remark 2.12. Corollary 2.11 refines Theorem 2.2 in both the bounds and the condition (2.6).

Now we discuss other possible restricted domains. We assume that G is a 2-divisible Abelian group. For given $x, y \in G$, we denote by $R_{x,y},\ S_{x,y} \subset G \times G$,

$$\begin{aligned} R_{x,y} &= \left\{(x,x), (x,y), \left(\frac{x-y}{2}, \frac{x-y}{2}\right), \left(\frac{x-y}{2}, \frac{-x+y}{2}\right)\right\}, \\ S_{x,y} &= \left\{(x,x), (y,x), \left(\frac{x-y}{2}, \frac{x-y}{2}\right), \left(\frac{-x+y}{2}, \frac{x-y}{2}\right)\right\}. \end{aligned} \tag{2.25}$$

One can see that $R_{x,y}$ and $S_{x,y}$ consist of the vertices of parallelograms in $G \times G$, respectively.

Definition 2.13. Let $U \subset G \times G$. One introduces the following conditions $(J3), (J4)$ on U. For any $x, y \in G$, there exists a $z \in G$ such that

$$\begin{aligned} &(J3) \quad (z,-z) + R_{x,y} = \left\{(x+z, x-z), (x+z, y-z), \left(\frac{x-y}{2}+z, \frac{x-y}{2}-z\right),\right. \\ &\qquad\qquad \left.\left(\frac{x-y}{2}+z, \frac{-x+y}{2}-z\right)\right\} \subset U, \\ &(J4) \quad (z,-z) + S_{x,y} = \left\{(x+z, x-z), (y+z, x-z), \left(\frac{x-y}{2}+z, \frac{x-y}{2}-z\right),\right. \\ &\qquad\qquad \left.\left(\frac{-x+y}{2}+z, \frac{x-y}{2}-z\right)\right\} \subset U, \end{aligned} \tag{2.26}$$

respectively.

Example 2.14. Let G be a real normed space. For $k, s, d \in \mathbb{R}$, let

$$\begin{aligned} U &= \{(x,y) \in G \times G : k\|x\| + s\|y\| \geq d\}, \\ V &= \{(x,y) \in G \times G : \|kx + sy\| \geq d\}. \end{aligned} \tag{2.27}$$

Then U satisfies $(J3)$ and $(J4)$ if $k + s > 0$, and V satisfies $(J3)$ and $(J4)$ if $k \neq s$.

Example 2.15. Let G be a real inner product space. For $d \geq 0$, $x_0, y_0 \in G$,

$$U = \{(x,y) \in G \times G : \langle x_0, x\rangle + \langle y_0, y\rangle \geq d\}. \tag{2.28}$$

Then U satisfies $(J3), (J4)$ if $x_0 \neq y_0$.

Example 2.16. Let G be the group of nonsingular square matrices with the operation of matrix multiplication. For $k, s \in \mathbb{R},\ \delta, d \geq 0$, let

$$\begin{aligned} U &= \left\{ (P_1, P_2) \in G \times G : |\det P_1|^k |\det P_2|^s \leq \delta \right\}, \\ V &= \left\{ (P_1, P_2) \in G \times G : |\det P_1|^k |\det P_2|^s \geq d \right\}. \end{aligned} \tag{2.29}$$

Then U and V satisfy both $(J3)$ and $(J4)$ if $k \neq s$.

From now on, we assume that G is a 2-divisible Abelian group.

Theorem 2.17. *Let $U \subset G \times G$ satisfy the condition $(J3)$ and $\epsilon \geq 0$. Suppose that $f : G \to Y$ satisfies* (2.1) *for all $(x, y) \in U$. Then there exists a unique additive function $A : G \to Y$ such that*

$$\| f(x) - A(x) - f(0) \| \leq 4\epsilon \tag{2.30}$$

for all $x \in G$.

Proof. For given $x, y \in G$, choose a $z \in G$ such that $(z, -z) + R_{x,y} \subset U$. Replacing x by $x + z$, y by $x - z$; x by $x + z$, y by $y - z$; x by $(x - y)/2 + z$, y by $(x - y)/2 - z$; x by $(x - y)/2 + z$, y by $(-x + y)/2 - z$ in (2.1), respectively, we have

$$\begin{aligned} &\| f(2x) + f(2z) - 2f(x + z) \| \leq \epsilon, \\ &\| f(x + y) + f(x - y + 2z) - 2f(x + z) \| \leq \epsilon, \\ &\left\| f(x - y) + f(2z) - 2f\left(\frac{x - y}{2} + z\right) \right\| \leq \epsilon, \\ &\left\| f(0) + f(x - y + 2z) - 2f\left(\frac{x - y}{2} + z\right) \right\| \leq \epsilon. \end{aligned} \tag{2.31}$$

From (2.31), using the triangle inequality, we have

$$| f(2x) - f(x + y) - f(x - y) + f(0) | \leq 4\epsilon \tag{2.32}$$

for all $x, y \in G$. Replacing x by $(x + y/2)$, y by $(x - y/2)$ in (2.32), we have

$$| f(x + y) - f(x) - f(y) + f(0) | \leq 4\epsilon \tag{2.33}$$

for all $x, y \in G$. From (2.33), using Theorem 1.1, we get the result. □

Let $d \geq 0,\ k, s \in \mathbb{R}$ with $k + s > 0$, and let $U = \{(x, y) : k\|x\| + s\|y\| \geq d\}$. Then U satisfies the conditions $(J3)$ and $(J4)$. Thus, as a direct consequence of Theorem 2.17 we generalize Theorem 2.1 as follows.

Corollary 2.18. *Let $d \ge 0$, $k, s \in \mathbb{R}$ with $k+s>0$. Suppose that $f : X \to Y$ satisfies the inequality* (2.1) *for all x, y, with $k\|x\| + s\|y\| \ge d$. Then there exists a unique additive function $A : X \to Y$ such that*

$$\|f(x) - A(x) - f(0)\| \le 4\epsilon \tag{2.34}$$

for all $x \in X$.

Theorem 2.19. *Let $U \subset G\times G$ satisfy the condition $(J4)$ and $\epsilon \ge 0$. Suppose that $f : G \to Y$ satisfies* (2.2) *for all $(x, y) \in U$. Then there exists a unique additive function $A : G \to Y$ such that*

$$\|f(x) - A(x)\| \le 4\epsilon \tag{2.35}$$

for all $x \in G$.

Proof. For given $x, y \in G$, choose $z \in G$ such that $(z, -z) + S_{x,y} \subset U$. Replacing x by $x+z$, y by $x-z$; x by $y+z$, y by $x-z$; x by $(x-y)/2+z$, y by $(x-y)/2-z$; x by $(-x+y)/2+z$, y by $(x-y)/2-z$ in (2.2), respectively, we have

$$\begin{gathered}\|f(2x) - f(2z) - 2f(x-z)\| \le \epsilon,\\ \|f(x+y) - f(-x+y+2z) - 2f(x-z)\| \le \epsilon,\\ \left\|f(x-y) - f(2z) - 2f\left(\frac{x-y}{2}+z\right)\right\| \le \epsilon,\\ \left\|f(0) - f(-x+y+2z) - 2f\left(\frac{x-y}{2}+z\right)\right\| \le \epsilon.\end{gathered} \tag{2.36}$$

From (2.36), using the triangle inequality, we have

$$|f(2x) - f(x+y) - f(x-y) + f(0)| \le 4\epsilon \tag{2.37}$$

for all $x, y \in G$. Replacing x by $(x+y)/2$, y by $(x-y)/2$ in (2.37) and using Theorem 1.1, we get the result. □

As a direct consequence of Theorem 2.19, we have the following.

Corollary 2.20. *Let $d \ge 0$, $k, s \in \mathbb{R}$ with $k+s>0$. Suppose that $f : X \to Y$ satisfies the inequality* (2.2) *for all x, y, with $k\|x\| + s\|y\| \ge d$. Then there exists a unique additive function $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \le 4\epsilon \tag{2.38}$$

for all $x \in X$.

3. Asymptotic Behavior of the Equations

In this section we discuss asymptotic behaviors of the equations which gives refined versions of the results in [13].

Using Theorems 2.8 and 2.17, we have the following (cf. [13]).

Theorem 3.1. *Let U satisfy $(J1)$ or $(J3)$. Suppose that $f : X \to Y$ satisfies the asymptotic condition*

$$\|f(x+y) + f(x-y) - 2f(x)\| \longrightarrow 0 \tag{3.1}$$

as $\|x\| + \|y\| \to \infty$, $(x, y) \in U$. Then there exists a unique additive function $A : X \to Y$ such that

$$f(x) = A(x) + f(0) \tag{3.2}$$

for all $x \in X$.

Proof. By the condition (3.1), for each $n \in \mathbb{N}$, there exists $d_n > 0$ such that

$$\|f(x+y) + f(x-y) - 2f(x)\| \le \frac{1}{n} \tag{3.3}$$

for all $(x, y) \in U$ with $\|x\| + \|y\| \ge d_n$. Let $U_0 = U \cap \{(x, y) : \ \|x\| + \|y\| \ge d_n\}$. Then U_0 satisfies both the conditions $(J1)$ and $(J3)$. By Theorems 2.8 and 2.17, there exists a unique additive function $A_n : X \to Y$ such that

$$\|f(x) - A_n(x) - f(0)\| \le \frac{2}{n} \quad \text{or} \quad \frac{4}{n} \tag{3.4}$$

for all $x \in X$. Putting $n = m$ in (3.4) and using the triangle inequality, we have

$$\|A_n(x) - A_m(x)\| \le 8 \tag{3.5}$$

for all $x \in X$. Using the additivity of A_n, A_m, we have $A_n = A_m$ for all $n, m \in \mathbb{N}$. Letting $n \to \infty$ in (3.4), we get the result. □

Corollary 3.2. *Let $k, s \in \mathbb{R}$ satisfy one of the conditions: $s > 0$, $k + s > 0$. Suppose that $f : X \to Y$ satisfies the condition*

$$\|f(x+y) + f(x-y) - 2f(x)\| \longrightarrow 0 \tag{3.6}$$

as $k\|x\| + s\|y\| \to \infty$. Then there exists a unique additive function $A : X \to Y$ such that

$$f(x) = A(x) + f(0) \tag{3.7}$$

for all $x \in X$.

Using Theorems 2.10 and 2.19, we have the following (cf. [13]).

Theorem 3.3. *Let U satisfy $(J2)$ or $(J4)$. Suppose that $f : X \to Y$ satisfies the condition*

$$\|f(x+y) - f(x-y) - 2f(y)\| \longrightarrow 0 \tag{3.8}$$

as $\|x\| + \|y\| \to \infty, (x, y) \in U$. Then f is an additive function.

Corollary 3.4. *Let $k, s \in \mathbb{R}$ satisfy one of the conditions: $k > 0,\ k + s > 0$. Suppose that $f : X \to Y$ satisfies the condition*

$$\|f(x+y) - f(x-y) - 2f(y)\| \longrightarrow 0 \tag{3.9}$$

as $k\|x\| + s\|y\| \to \infty$. Then f is an additive function.

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References

[1] S. M. Ulam, *A Collection of Mathematical Problems*, Interscience Publishers, New York, NY, USA, 1960.

[2] D. H. Hyers, "On the stability of the linear functional equation," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 27, pp. 222–224, 1941.

[3] T. Aoki, "On the stability of the linear transformation in Banach spaces," *Journal of the Mathematical Society of Japan*, vol. 2, pp. 64–66, 1950.

[4] D. G. Bourgin, "Classes of transformations and bordering transformations," *Bulletin of the American Mathematical Society*, vol. 57, pp. 223–237, 1951.

[5] D. G. Bourgin, "Multiplicative transformations," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 36, pp. 564–570, 1950.

[6] T. M. Rassias, "On the stability of the linear mapping in Banach spaces," *Proceedings of the American Mathematical Society*, vol. 72, no. 2, pp. 297–300, 1978.

[7] G. L. Forti, "The stability of homomorphisms and amenability, with applications to functional equations," *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, vol. 57, pp. 215–226, 1987.

[8] D. H. Hyers, G. Isac, and T. M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser Boston, Boston, Mass, USA, 1998.

[9] S.-M. Jung, "Hyers-Ulam-Rassias stability of Jensen's equation and its application," *Proceedings of the American Mathematical Society*, vol. 126, no. 11, pp. 3137–3143, 1998.

[10] K.-W. Jun and H.-M. Kim, "Stability problem for Jensen-type functional equations of cubic mappings," *Acta Mathematica Sinica*, vol. 22, no. 6, pp. 1781–1788, 2006.

[11] G. H. Kim and Y. H. Lee, "Boundedness of approximate trigonometric functional equations," *Appled Mathematics Letters*, vol. 31, pp. 439–443, 2009.

[12] C. G. Park, "Hyers-Ulam-Rassias stability of homomorphisms in quasi-Banach algebras," *Bulletin des Sciences Mathématiques*, vol. 132, no. 2, pp. 87–96, 2008.

[13] J. M. Rassias and M. J. Rassias, "On the Ulam stability of Jensen and Jensen type mappings on restricted domains," *Journal of Mathematical Analysis and Applications*, vol. 281, no. 2, pp. 516–524, 2003.

[14] J. M. Rassias, "On the Ulam stability of mixed type mappings on restricted domains," *Journal of Mathematical Analysis and Applications*, vol. 276, no. 2, pp. 747–762, 2002.

[15] J. M. Rassias, "On approximation of approximately linear mappings by linear mappings," *Journal of Functional Analysis*, vol. 46, no. 1, pp. 126–130, 1982.

[16] F. Skof, "Sull'approssimazione delle applicazioni localmente δ−additive," *Atti della Accademia delle Scienze di Torino Classe di Scienze Fisiche, Matematiche e Naturali*, vol. 117, pp. 377–389, 1983.
[17] F. Skof, "Local properties and approximation of operators," *Rendiconti del Seminario Matematico e Fisico di Milano*, vol. 53, pp. 113–129, 1983.
[18] B. Batko, "Stability of an alternative functional equation," *Journal of Mathematical Analysis and Applications*, vol. 339, no. 1, pp. 303–311, 2008.
[19] B. Batko, "On approximation of approximate solutions of Dhombres' equation," *Journal of Mathematical Analysis and Applications*, vol. 340, no. 1, pp. 424–432, 2008.
[20] J. Brzdęk, "On stability of a family of functional equations," *Acta Mathematica Hungarica*, vol. 128, no. 1-2, pp. 139–149, 2010.
[21] J. Brzdęk, "On the quotient stability of a family of functional equations," *Nonlinear Analysis*, vol. 71, no. 10, pp. 4396–4404, 2009.
[22] J. Brzdęk, "On a method of proving the Hyers-Ulam stability of functional equations on restricted domains," *The Australian Journal of Mathematical Analysis and Applications*, vol. 6, pp. 1–10, 2009.
[23] J. Brzdęk and J. Sikorska, "A conditional exponential functional equation and its stability," *Nonlinear Analysis*, vol. 72, no. 6, pp. 2923–2934, 2010.
[24] S.-M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis*, vol. 48, Springer, New York, NY, USA, 2011.
[25] J. Sikorska, "Exponential functional equation on spheres," *Applied Mathematics Letters*, vol. 23, no. 2, pp. 156–160, 2010.
[26] J. Sikorska, "On two conditional Pexider functional equations and their stabilities," *Nonlinear Analysis*, vol. 70, no. 7, pp. 2673–2684, 2009.
[27] J. Sikorska, "On a Pexiderized conditional exponential functional equation," *Acta Mathematica Hungarica*, vol. 125, no. 3, pp. 287–299, 2009.
[28] J. Chung, "Stability of functional equations on restricted domains in a group and their asymptotic behaviors," *Computers & Mathematics with Applications*, vol. 60, no. 9, pp. 2653–2665, 2010.

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