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Research Article

Wigner-Ville Distribution Associated with the Linear Canonical Transform

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The linear canonical transform is shown to be one of the most powerful tools for nonstationary signal processing. Based on the properties of the linear canonical transform and the classical Wigner-Ville transform, this paper investigates the Wigner-Ville distribution in the linear canonical transform domain. Firstly, unlike the classical Wigner-Ville transform, a new definition of Wigner-Ville distribution associated with the linear canonical transform is given. Then, the main properties of the newly defined Wigner-Ville transform are investigated in detail. Finally, the applications of the newly defined Wigner-Ville transform in the linear-frequency-modulated signal detection are proposed, and the simulation results are also given to verify the derived theory.

1. Introduction

With the development of the modern signal processing technology for the nonstationary signal processing, a series of novel signal analysis theories and processing tools have been put forward to meet the requirements of modern signal processing, for example, the short-time Fourier transform [1], the wavelet transform (WT) [2], the ambiguity function (AF) [3], the Wigner-Ville distribution (WVD) [4], the the fractional Fourier transform (FRFT), and the linear canonical transform (LCT) [5–7]. Recently, more and more results [8, 9] show that the LCT is one of the most powerful signal processing tools; it receives much interests in signal processing community and has been applied in many fields, such as the time-frequency analysis [10], the filter design [11], the pattern recognition [6], encryption, and watermarking [12]. For more results associated with the LCT, one can refer to [5–7].

The linear-frequency-modulated (LFM) signal is one of the most important nonstationary signals, which is widely used in communications, radar, and sonar system [13–17]. The detection and parameter estimation of LFM signal are important in signal processing community; many methods have been given, such as iterative algorithm [13, 14], the Radon-ambiguity transform [15], the chirp-Fourier transform method [16], and the Wigner-Hough transform [17]. Among these methods, the Wigner-Ville distribution is shown to be an important method in LFM signal detection and parameter estimation; it is also proved to be one of the classical time-frequency representations and has been shown to play an important role in nonstationary signal processing [18, 19]. Based on the properties of the LCT, the FRFT, and the classical WVD, Pei and Ding [10] firstly investigate the WVD associated with the LCT and discuss the relations among the common fractional and canonical operators. Unlike the definition of WVD associated with the LCT in [10], we propose a new kind of WVD definition associated with the LCT in this paper; the main properties and the application of the newly defined WVD in the LFM signal detection are also investigated.

The paper is organized as follows: Section 2 reviews the preliminaries about the LCT, the classical Wigner-Ville, distributions and the relations between them. The new definition of the WVD associated with the LCT is proposed in Section 3; its main properties are also investigated in this section. The applications of the newly defined WVD in the LFM signal detection are proposed in Section 4; the simulation results are also given to show the correctness and effectiveness of the proposed techniques. Section 5 concludes the paper.

2. Preliminary

2.1. *The Linear Canonical Transform (LCT)*

The LCT is the name of a parameterized continuum of transforms which include, as particular cases, most of the integral transforms, such as the Fourier transform, the fractional Fourier transform, and the scaling operator. The LCT of a signal $f(t)$ with parameter A is defined as follows [7]:

$$F_A(u) = L_A[f(t)](u) = \begin{cases} \int_{-\infty}^{\infty} f(t)\sqrt{\dfrac{1}{j2\pi b}}e^{(j/2)((a/b)t^2-(2/b)ut+(d/b)u^2)}dt, & b \neq 0, \\ \sqrt{d}e^{(j/2)cdu^2} f(du), & b = 0, \end{cases} \tag{2.1}$$

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the parameter matrix of LCT satisfying $ad - bc = 1$, that is, $\det(A) = 1$.

The inverse transform of the LCT (ILCT) is given by an LCT having parameter $A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Hence, we can obtain the original signal $x(t)$ from $F^A[x](u)$ via

$$f(t) = F^{A^{-1}}\left[F^A[f](u)\right](t) = \begin{cases} \int_{-\infty}^{\infty} F^A[f](u)\sqrt{\dfrac{j}{2\pi b}}e^{(j/2)(-(d/b)u^2+(2/b)ut-(a/b)t^2)}du, & b \neq 0, \\ \sqrt{a}e^{-(j/2)cat^2} F^A[f](at), & b = 0. \end{cases} \tag{2.2}$$

From the definition of LCT, we can see that, when $b = 0$, the LCT of a signal is essentially a chirp multiplication and it is of no particular interest to our object. Therefore, without loss of generality, we set $b > 0$ in the following sections of the paper.

It is shown in [5–7] that the FT, FRFT, chirp, and scaling operations are all the special cases of the LCT. Therefore, the LCT can be used to solve some problems that cannot be solved well by these operations [20]. The well-known theories and concepts in the classical Fourier transform domain are generalized to the LCT domain by different researchers. The uniform and nonuniform sampling theories are well studied in the LCT domain and showed that we can obtain the better results compared to the classical ones in the Fourier domain [8, 9, 21–23]. The other concepts, for example, the WVD [10], the convolution and product theories [24, 25], the uncertainty principle [26], the spectral analysis [27], and the eigenfunctions [28], are also proposed and investigated in the LCT domain. The discrete methods and the fast computation of the LCT are investigated in detail in [29–31].

2.2. The Wigner-Ville Distribution (WVD)

The instantaneous autocorrelation function of a signal $f(t)$ is defined as [1]

$$R_f(t,\tau) = f\left(t+\frac{\tau}{2}\right) f^*\left(t-\frac{\tau}{2}\right), \tag{2.3}$$

and the classical WVD of $f(t)$ is defined as the FT of $R_f(t,\tau)$ for τ

$$W(t,w) = \int_{-\infty}^{+\infty} R_f(t,\tau) e^{-jw\tau} d\tau. \tag{2.4}$$

The WVD is one of the most powerful time-frequency analysis tools and has a series of good properties, the main properties of the WVD are listed as follows.

(1) Conjugation symmetry property:

$$W(t,w) = W^*(t,w). \tag{2.5}$$

(2) Time marginal property:

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} W(t,w)dw = \left|f(t)\right|^2. \tag{2.6}$$

(3) Frequency marginal property:

$$\int_{-\infty}^{+\infty} W_f(t,w)dt = |F(w)|^2. \tag{2.7}$$

(4) Energy distribution property:

$$\frac{1}{2\pi}\iint_{-\infty}^{+\infty} W_f(t,w)dt\,dw = \int_{-\infty}^{+\infty} \left|f(t)\right|^2 dt = \langle f(t), f(t)\rangle. \tag{2.8}$$

2.3. The Previous Results about WVD Associated with LCT

With the developments of the FRFT and the LCT, Almeida in [32] and Lohmann in [33] investigate the relationship between the WVD and the FRFT; they show that the WVD of the FRFTed signal can be seen as a rotation of the classical WVD in the time-frequency plane. Along this direction, Pei and Ding discuss the relationship between the classical WVD and the WVD associated with the LCT [10]. In their definition, suppose the LCT of a signal $f(t)$ with parameter A is denoted as $F_A(u) = L_A[(f(t)](u)$; then the WVD associated with the LCT is defined as

$$W_{F_A}(u,v) = \int_{-\infty}^{+\infty} F_A\left(u+\frac{\tau}{2}\right)F_A^*\left(u-\frac{\tau}{2}\right)e^{-jv\tau}d\tau. \tag{2.9}$$

It is shown in [10, 32, 33] that this definition of the WVD associated with the LCT can be seen as the rotation or affine transform of the LCTed signal in the time-frequency plane. If the classical WVD of a signal $f(t)$ is denotes as $W_f(t,w)$ and the newly defined WVD associated in (2.9) is denotes as $W_{F_A}(u,v)$, we have the following result [10]:

$$W_f(t,w) = W_{F_A}(u,v), \tag{2.10}$$

where

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} t \\ w \end{pmatrix}. \tag{2.11}$$

Unlike the WVD definition in (2.9) associated with the LCT, we propose a new kind of definition for WVD in the LCT domain and the potential applications in the LFM signal detection are also proposed in the following sections.

3. The New Definition and Properties of WVD Associated with LCT

3.1. The New Definition of WVD

Based on the properties of the LCT and the form of the classical WVD definition associated with the Fourier transform, we give a new definition of WVD by the LCT of instantaneous autocorrelation function $R_f(t,\tau)$. In other words, we take place of the kernel of FT with the kernel of LCT to get a new kind of WVD associated with the LCT as follows.

Definition 3.1. Suppose the kernel of the LCT with parameter A is $K_A(t,u)$; then the WVD of a signal $f(t)$ associated with the LCT is defined as

$$W_A^f(t,u) = \int_{-\infty}^{+\infty} R_f(t,\tau)K_A(u,\tau)d\tau \tag{3.1}$$

with $K_A(t,u) = \sqrt{1/j2\pi b}e^{j(d/2b)u^2}e^{j(a/2b)\tau^2 - j(u\tau/b)}$ and $R_f(t,\tau) = f(t+\tau/2)f^*(t-\tau/2)$. The parameters a, b, c, d are the real numbers satisfying $ad - bc = 1$.

In order to make different from the existing results about the WVD, we denote the WVD associated with the LCT for parameter $A = (a, b; c, d)$ by $W_A^f(t, u)$ and simplified as the WDL of $f(t)$.

The LCT of a signal $f(t)$ can be looked as the affine transform of the signal in the time-frequency plane; so the WDL of a signal can be interpreted as the affine transform of the instantaneous autocorrelation function $R_f(t, \tau)$ of this signal in the time-frequency plane. Some of the important properties are investigated in the following subsection.

3.2. The Properties

Suppose the WDL of a signal $f(t)$ is denoted as $W_A^f(t, u)$, then the following important properties of WDL can be obtained.

(1) Conjugation symmetry property: the WDL of $f^*(t)$ is

$$W_A^{f^*}(t, u) = \left[W_{A^{-1}}^f(t, u)\right]^* \tag{3.2}$$

and the WDL of $f(-t)$ is $W_A^f(-t, u)$.

(2) Shifting property: if we remark $f'(t) = f(t - t_0)$, then

$$W_A^{f'}(t, u) = W_A^f(t - t_0, u) \tag{3.3}$$

and the WDL of $g(t) = f(t) \cdot e^{jwt}$ is

$$W_A^g(t, u) = e^{j(duw+(dbw^2/2))} W_A^f(t, u + wb). \tag{3.4}$$

(3) Limited support: if $f(t) = 0, |t| > t_0$, then $W_A^f(t, u) = 0, |t| > t_0$.

(4) Inverse property: the signal $f(t)$ can be expressed by the WDL of $f(t)$ as:

$$f(t) = \frac{1}{f^*(0)} \int_{-\infty}^{+\infty} W_A^f\left(\frac{t}{2}, u\right) K_{A^{-1}}(t, u) du. \tag{3.5}$$

Proof. From the definition of WDL for a signal $f(t)$, we know

$$W_A^f(t, u) = \int_{-\infty}^{+\infty} f\left(t + \frac{\tau}{2}\right) f^*\left(t - \frac{\tau}{2}\right) K_A(u, \tau) d\tau. \tag{3.6}$$

By the inverse transform of the LCT, we obtain the instantaneous autocorrelation function $R_f(t, \tau)$ as follows:

$$f\left(t + \frac{\tau}{2}\right) f^*\left(t - \frac{\tau}{2}\right) = \int_{-\infty}^{+\infty} W_A^f(t, u) K_{A^{-1}}(\tau, u) du. \tag{3.7}$$

Letting $\tau/2 = t$, (3.7) will reduce to

$$f(2t)f^*(0) = \int_{-\infty}^{+\infty} W_A^f(t,u) K_{A^{-1}}(2t,u)\,du, \tag{3.8}$$

and the final result can be obtained by letting $2t = s$

$$f(s) = \frac{1}{f^*(0)} \int_{-\infty}^{+\infty} W_A^f\left(\frac{s}{2},u\right) K_{A^{-1}}(s,u)\,du. \tag{3.9}$$

□

(5) Moyal formula

$$\iint_{-\infty}^{+\infty} W_A^f(t,u)\left[W_A^g(t,u)\right]^* dt, du = \left|\int_{-\infty}^{+\infty} f(t)g^*(t)\,dt\right|^2 = |\langle f,g\rangle|^2. \tag{3.10}$$

Proof. From the definition of the WDL, we obtain

$$\begin{aligned}
&\iint_{-\infty}^{+\infty} W_A^f(t,u)\left[W_A^g(t,u)\right]^* dt, du \\
&= \frac{1}{2\pi|b|}\iiint_{-\infty}^{+\infty} f\left(t+\frac{\tau}{2}\right) f^*\left(t-\frac{\tau}{2}\right) e^{-ju\tau/b + ja\tau^2/2b}\,d\tau \\
&\quad\times \int_{-\infty}^{+\infty} g^*\left(t+\frac{\tau'}{2}\right) g\left(t-\frac{\tau'}{2}\right) e^{(ju\tau'/b)-(ja\tau'^2/2b)}\,d\tau'\,dt\,du \\
&= \frac{1}{2\pi|b|}\iint_{-\infty}^{+\infty} f\left(t+\frac{\tau}{2}\right) f^*\left(t-\frac{\tau}{2}\right) e^{ja\tau^2/2b}\,d\tau \\
&\quad\times \int_{-\infty}^{+\infty} g^*\left(t+\frac{\tau'}{2}\right) g\left(t-\frac{\tau'}{2}\right) e^{-ja\tau'^2/2b}\,d\tau'\,dt \int_{-\infty}^{+\infty} e^{ju(\tau'-\tau)/b}\,du \\
&= \iint_{-\infty}^{+\infty} f\left(t+\frac{\tau}{2}\right) f^*\left(t-\frac{\tau}{2}\right) e^{ja\tau^2/2b}\,d\tau \\
&\quad\times \int_{-\infty}^{+\infty} g^*\left(t+\frac{\tau'}{2}\right) g\left(t-\frac{\tau'}{2}\right) e^{-ja\tau'^2/2b}\,d\tau'\,\delta(\tau-\tau')\,dt \\
&= \int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} f\left(t+\frac{\tau}{2}\right) f^*\left(t-\frac{\tau}{2}\right) g^*\left(t+\frac{\tau}{2}\right) g\left(t-\frac{\tau}{2}\right) dt\right] d\tau.
\end{aligned} \tag{3.11}$$

Let $\mu = t - \tau/2$; then the above equation reduces to the final result:

$$= \int_{-\infty}^{+\infty} f(\mu+\tau)g^*(\mu+\tau)\,d\tau \left[\int_{-\infty}^{+\infty} f(\mu)g^*(\mu)\,d\mu\right]^* = |\langle f,g\rangle|^2. \tag{3.12}$$

□

(6) The relationship between the classical WVD and WDL from the definition of LCT, it is easy to verify that when the parameter A reduces to $A = (0,1;-1,0)$, the WDL reduces to the classical WVD. In this sense, the WDL can be seen as the generalization of the classical WVD to the LCT domain:

$$W_A^f(t,u) = \sqrt{-j}W(t,u). \tag{3.13}$$

4. Applications of the WDL

The newly defined WDL is applied in the LFM signal detection in this section, the one- and two- component LFM signals are analyzed with the WDL in the LCT domain, and the simulation results are also proposed to verify the derived results.

4.1. One-Component LFM

If the LFM signal is modeled as $f(t) = e^{j(w_0 t+mt^2/2)}$; w_0, m represent the initial frequency and frequency rate of $f(t)$, respectively. From the definition of the WDL, the WDL of $f(t)$ is

$$\begin{aligned}
W_A^f(t,u) &= \int_{-\infty}^{+\infty} f\left(t+\frac{\tau}{2}\right) f^*\left(t-\frac{\tau}{2}\right) K_A(u,\tau)d\tau \\
&= \frac{1}{\sqrt{j2\pi b}} \int_{-\infty}^{+\infty} e^{[j(w_0\tau+mt\tau)]} e^{(j(d/2b)u^2 - j(u\tau/b)+j(a/2b)\tau^2)} d\tau \\
&= \frac{1}{\sqrt{j2\pi b}} e^{j(\frac{d}{2b})u^2} \int_{-\infty}^{+\infty} e^{[j(w_0+mt-u/b)\tau]} e^{j(a/2b)\tau^2} d\tau \\
&= \sqrt{\frac{2\pi}{jb}} e^{j(d/2b)u^2} \delta\left(\frac{u}{b} - (mt+w_0)\right), \quad a=0 \\
&= \frac{1}{\sqrt{j2\pi b}} e^{j(\frac{d}{2b})u^2} e^{[-j(b/2a)(mt+w_0-u/b)]^2} \\
&\quad \times \int_{-\infty}^{+\infty} e^{[(j(\frac{a}{2b})(\tau+(mt+w_0-u/b)/(a/b)))^2]} d\tau, \quad a \neq 0.
\end{aligned} \tag{4.1}$$

We can see from this equation that if we choose the especial parameter, the WDL of $f(t)$ will produce an impulse in (t,u) plane. From this fact, we propose the following algorithm for the detection and estimation of the of LFM signal by WDL.

Step 1. Compute the WDL of a signal.

Step 2. Search for the peak values in the time-frequency plane, then estimate the instantaneous frequency.

Step 3. Apply the least-squares ap proximation to the instantaneous frequency and obtain the final estimation value.

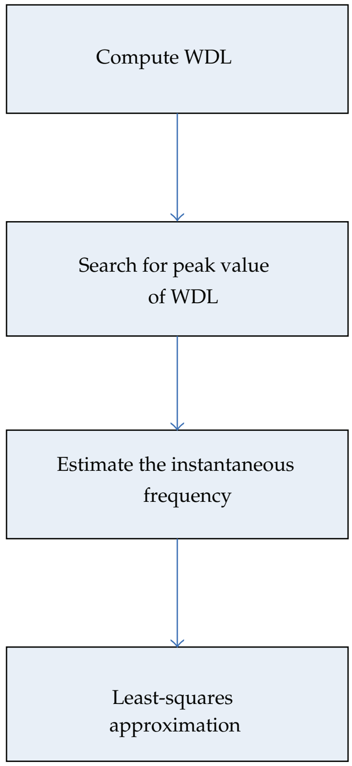

Figure 1: Detection algorithm diagram of instantaneous frequency.

The diagram of the LFM signal detection can be summarized in Figure 1.

4.2. Bicomponent Signal

When the processing signal is modeled as a bicomponent finite-length signal as follows.

$$f(t) = \begin{cases} e^{j(w_0 t+(k_0 t^2/2))} + e^{j(w_1 t + k_1 t^2/2)}, & |t| < \dfrac{T}{2}, \\ 0, & |t| \geq \dfrac{T}{2}, \end{cases} \tag{4.2}$$

this signal can be expressed as $f(t) = f_1(t) + f_2(t)$, and the WDL of $f(t)$ can be represented by the WDL of $f_1(t)$ and $f_2(t)$ as follows:

$$\begin{aligned} W_A^f(t,u) &= \int_{-\infty}^{+\infty} \left(f_1\left(t+\frac{\tau}{2}\right) + f_2\left(t+\frac{\tau}{2}\right)\right)\left(f_1\left(t-\frac{\tau}{2}\right) + f_2\left(t-\frac{\tau}{2}\right)\right) K_A(t,\tau) d\tau \\ &= W_A^{f_1}(t,u) + W_A^{f_2}(t,u) + 2\,\mathrm{Re}\left[W_A^{f_1,f_2}(t,u)\right]. \end{aligned} \tag{4.3}$$

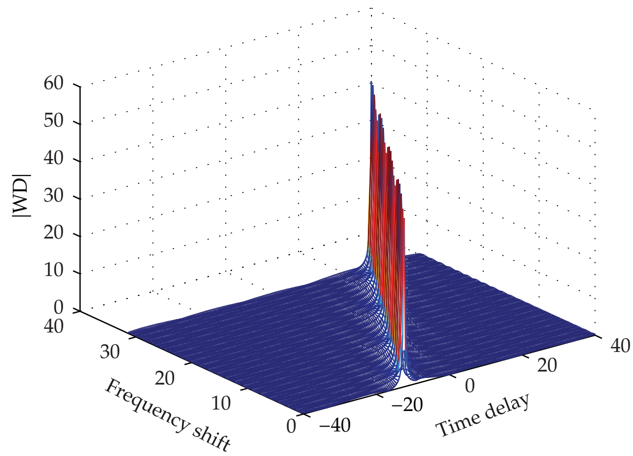

Figure 2: The WDL of $f(t)$.

The first two terms represent the autoterms of the signal, whereas the rest is the cross-term. If the parame (a, b, c, d) are chosen to be special numbers, the graph of WDL for signal $f(t)$ will be composed of the WDL of $f_1(t)$ and $f_2(t)$, respectively.

4.3. Simulation Results

4.3.1. The WDL of One-Component LFM

The simulations are performed to verify the derived results; a finite-length LFM signal as follows is chosen:

$$f(t) = e^{j(w_0 t + m_0 t^2/2)}, \quad |t| < \frac{T}{2}, \tag{4.4}$$

and $T = 40, w_0 = 10, m_0 = 0.8$.

The magnitude of $|W_A^f(t, u)|$ is plotted in Figure 2, and the projection of $W_A^f(t, u)$ onto time-frequency plane is plotted in Figure 3.

We can see from Figures 2 and 3 that the WDL of $f(t)$ has the energy accumulation property. Energy is accumulated in a straight line of the plane (t, u), which is the same as discussed before.

4.3.2. The Parameter Estimation of One LFM

Suppose the signal $f(t)$ is added with the white Gaussian noise; then it can be modeled as

$$f(t) = e^{j(w_0 t + m_0 t^2/2)} + n(t), \quad |t| < \frac{T}{2}. \tag{4.5}$$

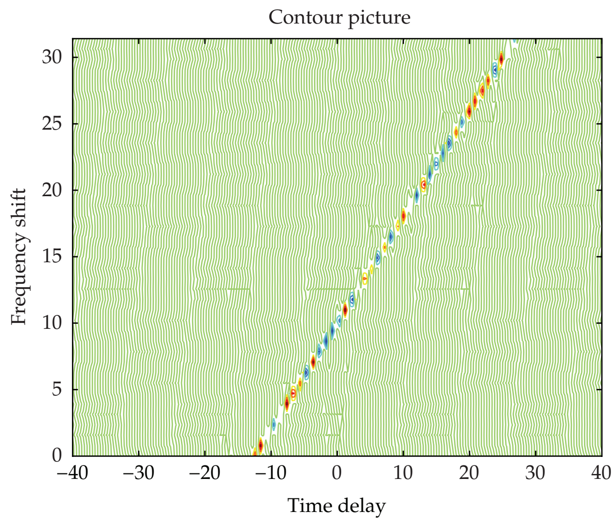

Figure 3: The contour picture of WDL of $f(t)$.

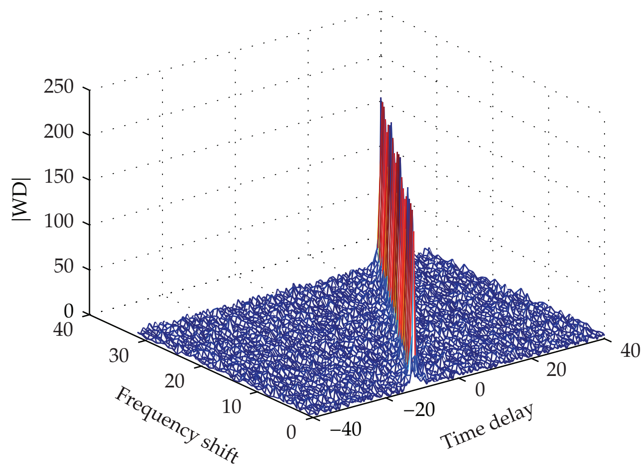

Figure 4: The WDL of $f(t)$ with SNR = 5 dB.

the initial parameters are set as $w_0 = 10$, $m_0 = 0.8$, and the length of signal $T = 40$. The magnitude of the WDL of $f(t)$ and the contour picture of the above signal is plotted in Figures 4 and 5, respectively.

Applying the parameter estimation algorithm as shown in Figure 1, search for the peak value in the time-frequency plane of WDL, we can obtain the instantaneous frequency as shown in Figure 6. Applying least-squares approximation to the instantaneous frequency and obain the ultimate instantaneous frequency estimation value $\widehat{m}_0 = 0.808$, $\widehat{w}_0 = 9.8918$.

4.3.3. Comparison with the Classical WVD

In order to compare the WVD with the WDL, we investigate the performance of peak value estimating method of them for the signal $f(t)$ added with noise. The contour picture of WVD and WDL of $f(t)$ with SNR = −5 dB is plotted in Figures 7 and 8, respectively.

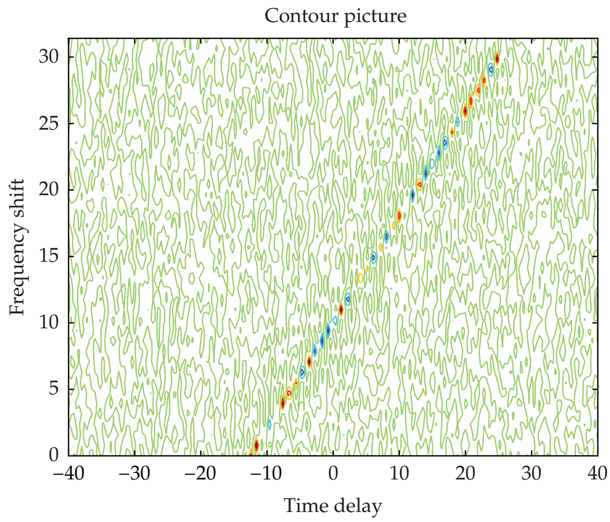

Figure 5: The contour of WVD of $f(t)$ with SNR = 5 dB.

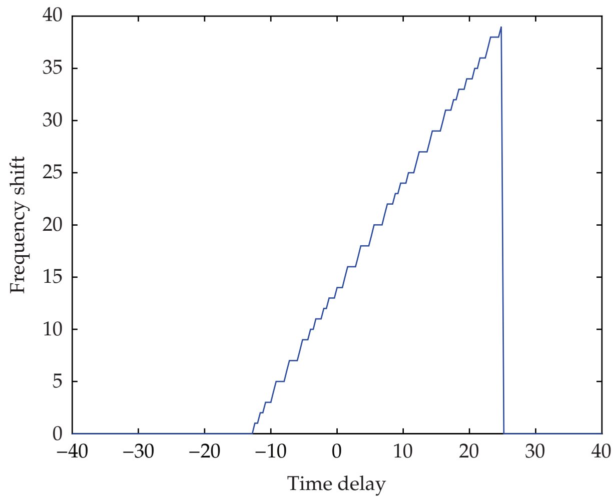

Figure 6: Search for the peak value in the time-frequency plane of WDL.

From Figures 7 and 8, we can obtain better results by the WDL under the low SNR circumstance as we discussed before.

5. Conclusion

Based on the LCT and the classical WVD theory, this paper proposes a new kind of definition of WVD associated with the LCT, namely WDL, which can be seen as the generalization of classical WVD to the LCT domain. Its main properties are derived in detail, and the applications of the WDL in the detection the parameters of the LFM signals are investigated. The simulations are also performed to verify the derived results. The future works will be the

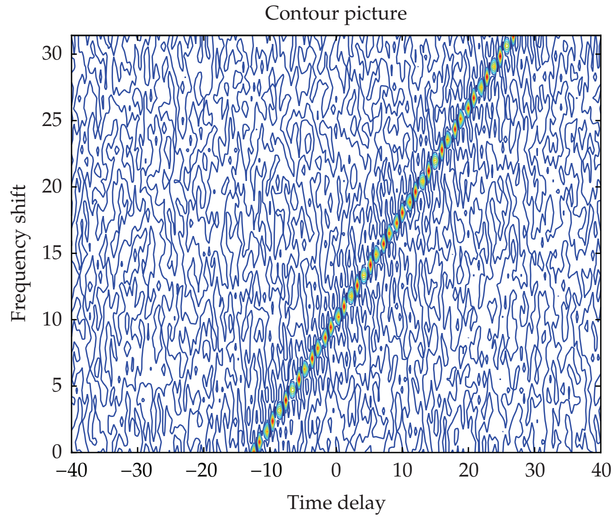

Figure 7: The contour picture of WVD of $f(t)$SNR = −5 dB.

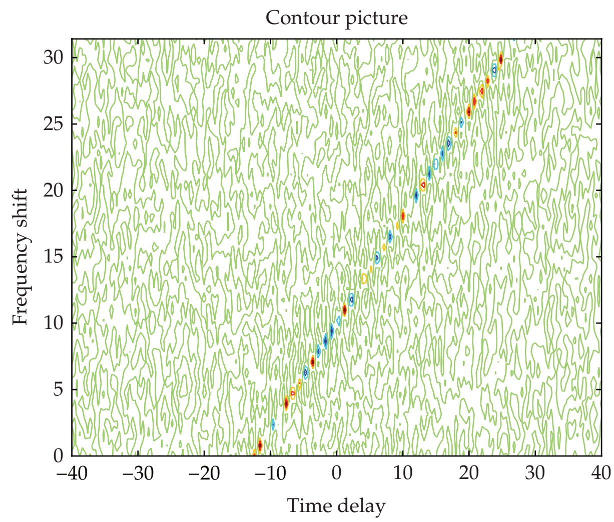

Figure 8: The contour picture of WDL of $f(t)$ with SNR = −5 dB.

applications of the newly defined WDL in the nonstationary signal processing and the study of the marginal properties for Cohen's class along this direction.

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