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Research Article

Bank Liquidity and the Global Financial Crisis

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We investigate the stochastic dynamics of bank liquidity parameters such as liquid assets and nett cash outflow in relation to the global financial crisis. These parameters enable us to determine the liquidity coverage ratio that is one of the metrics used in ratio analysis to measure bank liquidity. In this regard, numerical results show that bank behavior related to liquidity was highly procyclical during the financial crisis. We also consider a theoretical-quantitative approach to bank liquidity provisioning. In this case, we provide an explicit expression for the aggregate liquidity risk when a locally risk-minimizing strategy is utilized.

1. Introduction

During the global financial crisis (GFC), banks were under severe pressure to maintain adequate liquidity. In general, empirical evidence shows that banks with sufficient liquidity can meet their payment obligations while banks with low liquidity cannot. The GFC highlighted the fact that liquidity risk can proliferate quickly with funding sources dissipating and concerns about asset valuation and capital adequacy realizing. This situation underscores the important relationship between funding risk (involving raising funds to bankroll asset holdings) and market liquidity (involving the efficient conversion of assets into liquid funds at a given price). In response to this, the Basel Committee on Banking Supervision (BCBS) is attempting to develop an international framework for liquidity risk measurement, standards, and monitoring (see, e.g., [1]). Although pre-Basel III regulation establishes procedures for assessing credit, market, and operational risk, it does not provide effective protocols for managing liquidity and systemic risks. The drafting of Basel III represents an effort to address the latter (see, e.g., [2–4]).

Current liquidity risk management procedures can be classified as micro- or macroprudential. In the case of the former, simple liquidity ratios such as credit-to-deposit ratios (nett stable funding ratios), liquidity coverage ratios and the assessment of the gap between short-term liabilities and assets are appropriate to cover the objectives of bank balance sheet analysis. The ratio approach for liquidity risk management is a quantitative international accepted standard for alerting banks about any possible adverse economic downturns. For instance, the credit-to-deposit ratio assesses the relationships between sources and uses of funds held in the bank's portfolio but has limitations which ultimately do not reflect information on market financing with short-term maturity. By contrast, the liquidity coverage ratio (LCR) performs better by ensuring the coverage of some of the immediate liabilities. Since the LCR depends only on bank balance sheet data, it does not take into account the residual maturities on various uses and sources of funds. Also, in a global context, a quantitative approach may not take financial market conditions into account. In this case, a more comprehensive characterization of the bank system's liquidity risk through designed stress testing and constructed contingency plans is considered. The Basel Committee on Banking Supervision suggested best practices related to international liquidity standards. In this case, a well-designed policy monitoring instrument to measure and regulate the dynamics of foreign currency is considered to best take financial market conditions into account. Also, central banks (CBs) have a pivotal role to play in managing liquidity inflows via macroeconomic management of exchange rate and interest rate responses. The modeling of capital markets as well as stock and bond behavior also contribute to the liquidity response for possible stress conditions observed. The above approaches for liquidity analysis take into account the macroprudential liquidity management of banks.

In this paper, in Section 2, we discuss balance sheet items related to liquid assets and nett cash outflow in order to build a stochastic LCR model. Before the GFC, banks were prosperous with high LCRs, high cash inflows, low interest rates, and low nett cash outflows. This was followed by the collapse of liquidity, exploding default rates, and the effects thereof during the GFC. Next, in Section 3, we apply a dynamic provisioning strategy to liquidity risk management. In this case, we address the problem of dynamic liquidity provisioning for a mortgage, Λ, which is an underlying (illiquid) nonmarketable asset, by substituting (liquid) marketable securities, S. In the light of the above, banks prefer to trade in a Treasury bond market because of liquidity reasons. Since the loan process $(\Lambda_t)_{0\leq t\leq T}$ is not completely correlated with the substitute, it creates the market incompleteness. In other words, we will employ non-self-financing strategy to replicate the trading process. Therefore the banks would require that the uncertainty involved over the remaining of the trading period be minimized. In this case, we specifically minimize at each date, the uncertainty over the next infinitesimal period. In the dynamics trading there is always a residual risk emanating from the imperfection of the correlation between the Brownian motions. Due to the no-arbitrage opportunities there are infinitely many equivalent martingale measures so that pricing is directly linked to risk. Therefore, we choose a pricing candidate (equivalent martingale measure) under which the discounted stock price follows a martingale. This equivalent measure is chosen according to a provisioning strategy which ensures that the value of Λ is the value of the replicating portfolio. We also provide a framework for assessing residual aggregate liquidity risk stemming from the application of the above strategy.

1.1. Literature Review

The documents formulated in response to the proposed Basel III regulatory framework are among the most topical literature on bank liquidity (see, e.g., [1]). During the GFC, unprecedented levels of liquidity support were required from CBs in order to sustain the financial system and even with such extensive support a number of banks failed, were forced into mergers, or required resolution. The crisis illustrated how quickly and severely liquidity risks can crystallize and certain sources of funding can evaporate, compounding concerns related to the valuation of assets and capital adequacy (see, e.g., [2–4]). A key characteristic of the GFC was the inaccurate and ineffective management of liquidity risk. In recognition of the need for banks to improve their liquidity risk management and control their exposures to such risk, the BCBS issued Principles for Sound Liquidity Risk Management and Supervision in September 2008 (see, e.g., [1]). Supervisors are expected to assess both the adequacy of a bank's liquidity risk management framework and its liquidity risk exposure. In addition, they are required to take prompt action to address the banks risk management deficiencies or excess exposure in order to protect depositors and enhance the overall stability of the financial system. To reinforce these supervisory objectives and efforts, the BCBS has recently focussed on further elevating the resilience of internationally active banks to liquidity stresses across the globe, as well as increasing international harmonization of liquidity risk supervision (see, e.g., [1]). The BCBS hopes to develop internationally consistent regulatory standards for liquidity risk supervision as a cornerstone of a global framework to strengthen liquidity risk management and supervision (see, e.g., [2–4]).

In [5] it is asserted that bank liquidity behavior can be described by straightforward indicators constructed from firm-specific balance sheet data (see, also, [6, 7]). Also, their analysis underscores the relevance of using several indicators of liquidity risk at the same time, given the different leads and lags of the measures with systemic risk. Our study is related to theirs in that we make use of balance sheet items to determine bank behavior. Another similarity is that we make use of data from [6] to formulate conclusions in a numerical quantitative framework (compare with the analysis in Section 3 below).

The contribution [8] studies the role of securitization in bank management. A new index of "bank loan portfolio liquidity" which can be thought of as a weighted average of the potential to securitize loans of a given type, where the weights reflect the composition of a bank loan portfolio. The paper uses this new index to show that by allowing banks to convert illiquid loans into liquid funds, securitization reduces banks holdings of liquid securities and increases their lending ability. Furthermore, securitization provides banks with an additional source of funding and makes bank lending less sensitive to cost of funds shocks. By extension, the securitization weakens the ability of regulators to affect banks lending activity but makes banks more susceptible to liquidity and funding crisis when the securitization market is shutdown. We conduct a similar analysis in Section 4 of this paper where illiquid underlying loans are substituted by liquid marketable securities.

In [9], we use actuarial methods to solve a nonlinear stochastic optimal liquidity risk management problem for subprime originators with deposit inflow rates and marketable securities allocation as controls (see [10]). The main objective is to minimize liquidity risk in the form of funding and credit crunch risk in an incomplete market. In order to accomplish this, we construct a stochastic model that incorporates originator mortgage and deposit reference processes. Finally, numerical examples that illustrate the main modeling and optimization features of the paper are provided. Our work in this paper also has a connection with [9] in that the nexus between funding risk and market liquidity is explored.

However, this paper is an improvement on the aforementioned in that bank balance sheet features play a more prominent role (see, Sections 2, 3, and 4).

1.2. Main Questions and Article Outline

In this subsection, we pose the main questions and provide an outline of the paper.

1.2.1. Main Questions

In this paper on bank liquidity, we answer the following questions.

Question 1 (banking model). Can we model banks' liquid assets and nett cash outflows as well as LCRs in a stochastic framework? (compare with Section 2).

Question 2 (bank liquidity in a numerical quantitative framework). Can we explain and provide numerical examples of bank liquidity dynamics? (refer to Section 3).

Question 3 (bank liquidity in a theoretical quantitative framework). Can we devise a liquidity provisioning strategy in a theoretical quantitative framework? (compare with Section 4).

1.2.2. Paper Outline

The rest of the paper is organized as follows. Section 1 introduces the concept of liquidity risk while providing an appropriate literature review. A stochastic LCR model for bank liquidity is constructed in Section 2. Issues pertaining to bank liquidity in a numerical quantitative framework are discussed in Section 3. Section 4 treats liquidity in a theoretical quantitative manner. Finally, we provide concluding remarks in Section 5.

2. Bank Liquidity Model

In the sequel, we use the notational convention "subscript t or s" to represent (possibly) random processes, while "bracket t or s" is used to denote deterministic processes. The assessment of a bank's relative composition of the stock of high-quality liquid assets (liquid assets) and nett cash outflows, is one of the primary ways of analyzing its liquidity position. In this regard, we consider a measure of liquidity offered by the LCR. Before the GFC, banks were prosperous with high LCRs, high cash inflows, low interest rates, and low nett cash outflows. This was followed by the collapse of liquidity, exploding default rates, and the effects thereof. We make the following assumption to set the space and time index that we consider in our LCR model.

Assumption 2.1 (filtered probability space and time index). Throughout, we assume that we are working with a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\{\mathcal{F}_t\}_{t\geq 0}$ on a time index set $[0, T]$. We assume that the aforementioned space satisfies the usual conditions. Under $\mathbb{P}$, $\{W_t;\ 0 \leq t \leq T, W_0 = 0\}$ is an $\mathcal{F}_t$-Brownian motion.

Furthermore, we are able to produce a system of stochastic differential equations that provide information about the stock of high-quality liquid assets (liquid assets) at time t with

$x^1 : \Omega \times [0,T] \rightarrow \mathbb{R}^+$ denoted by x_t^1 and nett cash outflows at time t with $x^2 : \Omega \times [0,T] \rightarrow \mathbb{R}^+$ denoted by x_t^2 and their relationship. The dynamics of liquid assets, x_t^1, is stochastic in nature because it depends in part on the stochastic rates of return on assets and cash inflow and outflow (see [9] for more details) and the securitization market. Also, the dynamics of the nett cash outflow, x_t^2, is stochastic because its value has a reliance on cash inflows as well as liquidity and market risk that have randomness associated with them. Furthermore, for $x : \Omega \times [0,T] \rightarrow \mathbb{R}^2$ we use the notation x_t to denote

$$x_t = \begin{bmatrix} x_t^1 \\ x_t^2 \end{bmatrix}, \tag{2.1}$$

and represent the LCR with $l : \Omega \times [0,T] \rightarrow \mathbb{R}^+$ by

$$l_t = \frac{x_t^1}{x_t^2}. \tag{2.2}$$

It is important for banks that l_t in (2.2) has to be sufficiently high to ensure high bank liquidity.

2.1. Liquid Assets

In this section, we discuss the stock of high-quality liquid assets constituted by cash, CB reserves, marketable securities, and government/CB bank debt issued.

2.1.1. Description of Liquid Assets

The first component of stock of high-quality liquid assets is *cash* that is made up of banknotes and coins. According to [1], a *CB reserve* should be able to be drawn down in times of stress. In this regard, local supervisors should discuss and agree with the relevant CB the extent to which CB reserves should count toward the stock of liquid assets.

Marketable securities represent claims on or claims guaranteed by sovereigns, CBs, noncentral government public sector entities (PSEs), the Bank for International Settlements (BIS), the International Monetary Fund (IMF), the European Commission (EC), or multilateral development banks. This is conditional on all the following criteria being met. These claims are assigned a 0% risk weight under the Basel II standardized approach. Also, deep repo-markets should exist for these securities and that they are not issued by banks or other financial service entities.

Another category of stock of high-quality liquid assets refers to *government/CB bank debt issued in domestic currencies* by the country in which the liquidity risk is being taken by the bank's home country (see, e.g., [1, 4]).

2.1.2. Dynamics of Liquid Assets

In this section, we consider

$$dh_t = r_t^h dt + \sigma_t^h dW_t^h, \qquad h(t_0) = h_0, \tag{2.3}$$

where the stochastic processes $h : \Omega \times [0,T] \to \mathbb{R}^+$ are the *return per unit of liquid assets*, $r^h \to \mathbb{R}^+$ is the rate of return per liquid asset unit, the scalar $\sigma^h : T \to \mathbb{R}$ is the volatility in the rate of returns, and $W^h : \Omega \times [0,T] \to \mathbb{R}$ is standard Brownian motion. Before the GFC, risky asset returns were much higher than those of riskless assets, making the former a more attractive but much riskier investment. It is inefficient for banks to invest all in risky or riskless securities with asset allocation being important. In this regard, it is necessary to make the following assumption to distinguish between risky (e.g., marketable securities and government/CB bank debt) and riskless assets (cash) for future computations.

Assumption 2.2 (liquid assets). Suppose from the outset that liquid assets are held in the financial market with $n + 1$ asset classes. One of these assets is riskless (cash) while the assets $1, 2, \ldots, n$ are risky.

The risky liquid assets evolve continuously in time and are modelled using an n-dimensional Brownian motion. In this multidimensional context, the *asset returns in the kth liquid asset class per unit of the kth class* is denoted by $y_t^k, k \in \mathbb{N}_n = \{0, 1, 2, \ldots, n\}$ where $y : \Omega \times [0,T] \to \mathbb{R}^{n+1}$. Thus, the return per liquid asset unit is

$$y = \left(C(t), y_t^1, \ldots, y_t^n\right), \tag{2.4}$$

where $C(t)$ represents the return on cash and $y_t^1, \ldots, y_t^n$ represents the risky return. Furthermore, we can model y as

$$dy_t = r_t^y dt + \Sigma_t^y dW_t^y, \qquad y(t_0) = y_0, \tag{2.5}$$

where $r^y : T \to \mathbb{R}^{n+1}$ denotes the rate of liquid asset returns, $\Sigma_t^y \in \mathbb{R}^{(n+1)\times n}$ is a matrix of liquid asset returns, and $W^y : \Omega \times [0,T] \to \mathbb{R}^n$ is standard Brownian motion. Notice that there are only n scalar Brownian motions due to one of the liquid assets being riskless.

We assume that the investment strategy $\pi : T \to \mathbb{R}^{n+1}$ is outside the simplex

$$S = \left\{\pi \in \mathrm{R}^{n+1} : \pi = \left(\pi^0, \ldots, \pi^n\right)^T, \pi^0 + \cdots + \pi^n = 1, \pi^0 \geq 0, \ldots \pi^n \geq 0\right\}. \tag{2.6}$$

In this case, short selling is possible. The *liquid asset return* is then $h : \Omega \times R \to \mathbb{R}^+$, where the dynamics of h can be written as

$$dh_t = \pi_t^T dy_t = \pi_t^T r_t^y dt + \pi_t^T \Sigma_t^y dW_t^y. \tag{2.7}$$

This notation can be simplified as follows. We denote

$$
\begin{aligned}
r^C(t) &= r^{y^0}(t), r^C : T \longrightarrow \mathbb{R}^+, \text{ the rate of return on cash,} \\
r_t^y &= \left(r^C(t), \widetilde{r_t^y}^T + r^C(t) 1_n \right)^T, \quad \tilde{r}^y : T \longrightarrow \mathbb{R}^n, \\
\pi_t &= \left(\pi_t^0, \tilde{\pi}_t^T \right)^T = \left(\pi_t^0, \pi_t^1, \dots, \pi_t^k \right)^T, \quad \tilde{\pi} : T \longrightarrow \mathbb{R}^k , \\
\Sigma_t^y &= \begin{pmatrix} 0 & \cdots & 0 \\ & \tilde{\Sigma}_t^y & \end{pmatrix}, \quad \tilde{\Sigma}_t^y \in \mathbb{R}^{n \times n}, \\
\tilde{C}_t &= \tilde{\Sigma}_t^y \widetilde{\Sigma_t^y}^T. \text{ Then, we have that} \\
\pi_t^T r_t^y &= \pi_t^0 r^C(t) + \tilde{\pi}_t^{jT} \tilde{r}_t^y + \tilde{\pi}_t^{jT} r^C(t) 1_n = r^C(t) + \tilde{\pi}_t^T \tilde{r}_t^y, \\
\pi_t^T \Sigma_t^y dW_t^y &= \tilde{\pi}_t^T \tilde{\Sigma}_t^y dW_t^y, \\
dh_t &= \left[r^C(t) + \tilde{\pi}_t^T \tilde{r}_t^y \right] dt + \tilde{\pi}_t^T \tilde{\Sigma}_t^y dW_t^y, \qquad h(t_0) = h_0.
\end{aligned}
\tag{2.8}
$$

2.2. Nett Cash Outflows

In this section, we discuss nett cash outflows arising from cash outflows and inflows.

2.2.1. Description of Nett Cash Outflows

Cash outflows are constituted by retail deposits, unsecured wholesale funding secured funding and additional liabilities (see, e.g., [1]). The latter category includes requirements about liabilities involving derivative collateral calls related to a downgrade of up to 3 notches, market valuation changes on derivatives transactions, valuation changes on posted noncash or non-high-quality sovereign debt collateral securing derivative transactions, asset backed commercial paper (ABCP), special investment vehicles (SIVs), conduits, special purpose vehicles (SPVs), and the currently undrawn portion of committed credit and liquidity facilities.

Cash inflows are made up of amounts receivable from retail counterparties, amounts receivable from wholesale counterparties, receivables in respect of repo and reverse repo transactions backed by illiquid assets, and securities lending/borrowing transactions where illiquid assets are borrowed as well as other cash inflows.

According to [1], *nett cash inflows* is defined as cumulative expected cash outflows minus cumulative expected cash inflows arising in the specified stress scenario in the time period under consideration. This is the nett cumulative liquidity mismatch position under the stress scenario measured at the test horizon. *Cumulative expected cash outflows* are calculated by multiplying outstanding balances of various categories or types of liabilities by assumed percentages that are expected to roll off and by multiplying specified draw-down amounts to various off-balance sheet commitments. *Cumulative expected cash inflows* are calculated by multiplying amounts receivable by a percentage that reflects expected inflow under the stress scenario.

2.2.2. Dynamics of Nett Cash Outflows

Essentially, mortgagors are free to vary their cash inflow rates. Roughly speaking, this rate should be reduced for high LCRs and increased beyond the normal rate when LCRs are too low. In the sequel, the stochastic process $u^1 : \Omega \times [0, T] \to \mathbb{R}^+$ is the *normal cash inflow rate per nett cash inflow unit* whose value at time t is denoted by u^1_t. In this case, $u^1_t dt$ turns out to be the cash inflow rate per unit of the nett cash inflow over the time period $(t, t + dt)$. A notion related to this is the *adjustment to the cash inflow rate per unit of the nett cash inflow rate for a higher or lower LCR*, $u^2 : \Omega \times [0, T] \to \mathbb{R}^+$, that will in closed loop be made dependent on the LCR. We denote the sum of u^1 and u^2 by the *cash inflow rate* $u^3 : \Omega \times [0, T] \to \mathbb{R}^+$, that is,

$$u^3_t = u^1_t + u^2_t, \quad \forall t. \tag{2.9}$$

Before the GFC, the cash inflow rate increased significantly as a consequence of rising liquidity. The following assumption is made in order to model the LCR in a stochastic framework.

Assumption 2.3 (cash inflow rate). The cash inflow, u^3, is predictable with respect to $\{\mathcal{F}_t\}_{t\geq 0}$.

The cash inflow provides us with a means of controlling LCR dynamics. The dynamics of the *cash outflow per unit of the nett cash outflow*, $e : \Omega \times [0, T] \to \mathbb{R}$, is given by

$$de_t = r^e_t dt + \sigma^e_t dW^e_t, \qquad e(t_0) = e_0, \tag{2.10}$$

where e_t is the cash outflow per unit of the nett cash outflow, $r^e : T \to \mathbb{R}$ is the rate of outflow per unit of the nett cash outflow, the scalar $\sigma^e : T \to \mathbb{R}$ is the volatility in the outflow per nett cash outflow unit, and $W^e : \Omega \times [0, T] \to \mathbb{R}$ is standard Brownian motion.

Next, we take $i : \Omega \times [0, T] \to \mathbb{R}^+$ as the *nett cash outflow increase before cash outflow per monetary unit of the nett cash outflow*, $r^i : T \to \mathbb{R}^+$ is the rate of increase of nett cash outflows before cash outflow per nett cash outflow unit, the scalar $\sigma^i \in \mathbb{R}$ is the volatility in the increase of nett cash outflows before outflow, and $W^i : \Omega \times [0, T] \to \mathbb{R}$ represents standard Brownian motion. Then, we set

$$di_t = r^i_t dt + \sigma^i dW^i_t, \qquad i(t_0) = i_0. \tag{2.11}$$

The stochastic process i_t in (2.11) may typically originate from nett cash flow volatility that may result from changes in market activity, cash supply, and inflation.

2.3. The Liquidity Coverage Ratio

This section discusses ratio analysis and liquidity coverage ratio dynamics.

2.3.1. Ratio Analysis

Ratio analysis is conducted on the bank's balance sheet composition. In this case, the LCR measures a bank's ability to access funding for a 30-day period of acute market stress. In this paper, as in Basel III, we are interested in the LCR that is defined as the sum of interbank assets and securities issued by public entities as a percentage of interbank liabilities. The LCR formula is given by

$$\text{Liquidity Coverage Ratio} = \frac{\text{Stock of High Quality Liquid Assets}}{\text{Nett Cash Outflows over a 30-day Period}}. \tag{2.12}$$

This ratio measures the bank system's liquidity position that allows the assessment of a bank's capacity to ensure the coverage of some of its more immediate liabilities with highly available assets. It also identifies the amount of unencumbered, high-quality liquid assets a bank holds that can be used to offset the nett cash outflows it would encounter under a short-term stress scenario specified by supervisors, including both specific and systemic shocks.

2.3.2. Liquidity Coverage Ratio Dynamics

Using the equations for liquid assets x^1 and nett cash outflow x^2, we have that

$$\begin{aligned}
dx_t^1 &= x_t^1 dh_t + x_t^2 u_t^3 dt - x_t^2 de_t \\
&= \left[r^C(t) x_t^1 + x_t^1 \widetilde{\pi}_t^T \widetilde{r}_t^y + x_t^2 u_t^1 + x_t^2 u_t^2 - x_t^2 r_t^e\right] dt \\
&\quad + \left[x_t^1 \widetilde{\pi}_t^T \widetilde{\Sigma}_t^y dW_t^y - x_t^2 \sigma^e dW_t^e\right], \\
dx_t^2 &= x_t^2 di_t - x_t^2 de_t \\
&= x_t^2 \left[r_t^i dt + \sigma^i dW_t^i\right] - x_t^2 \left[r_t^e dt + \sigma^e dW_t^e\right] \\
&= x_t^2 \left[r_t^i - r_t^e\right] dt + x_t^2 \left[\sigma^i dW_t^i - \sigma^e dW_t^e\right].
\end{aligned} \tag{2.13}$$

The SDEs (2.13) may be rewritten into matrix-vector form in the following way.

Definition 2.4 (stochastic system for the LCR model). Define the stochastic systemfor the LCR model as

$$dx_t = A_t x_t dt + N(x_t) u_t dt + a_t dt + S(x_t, u_t) dW_t, \tag{2.14}$$

with the various terms in this stochastic differential equation being

$$
\begin{aligned}
u_t &= \begin{bmatrix} u_t^2 \\ \widetilde{\pi}_t \end{bmatrix}, \qquad u : \Omega \times [0, T] \longrightarrow \mathbb{R}^{n+1}, \\
A_t &= \begin{bmatrix} r^C(t) & -r_t^e \\ 0 & r_t^i - r_t^e \end{bmatrix}, \\
N(x_t) &= \begin{bmatrix} x_t^2 & x_t^1 \widetilde{r_t^y}^T \\ 0 & 0 \end{bmatrix}, \qquad a_t = \begin{bmatrix} x_t^2 u_t^1 \\ 0 \end{bmatrix}, \\
S(x_t, u_t) &= \begin{bmatrix} x_t^1 \widetilde{\pi}_t^T \widetilde{\Sigma}_t^y & -x_t^2 \sigma^e & 0 \\ 0 & -x_t^2 \sigma^e & x_t^2 \sigma^i \end{bmatrix}, \\
W_t &= \begin{bmatrix} W_t^y \\ W_t^e \\ W_t^i \end{bmatrix},
\end{aligned}
\tag{2.15}
$$

where W_t^y, W_t^e, and W_t^i are mutually (stochastically) independent standard Brownian motions. It is assumed that for all $t \in T$, $\sigma_t^e > 0$, $\sigma_t^i > 0$ and $\widetilde{C}_t > 0$. Often the time argument of the functions σ^e and σ^i is omitted.

We can rewrite (2.14) as follows:

$$
\begin{aligned}
N(x_t)u_t &:= \begin{bmatrix} x_t^2 \\ 0 \end{bmatrix} u_t^2 + \begin{bmatrix} x_t^1 \widetilde{r_t^y}^T \\ 0 \end{bmatrix} \widetilde{\pi}_t \\
&:= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_t u_t^3 + \sum_{m=1}^{n} \begin{bmatrix} x_t^1 \widetilde{r}_t^{y,m} \\ 0 \end{bmatrix} \widetilde{\pi}_t^m \\
&:= B_0 x_t u_t^0 + \sum_{m=1}^{n} \begin{bmatrix} \widetilde{r}_t^{y,m} & 0 \\ 0 & 0 \end{bmatrix} x_t \widetilde{\pi}_t^m \\
&:= \sum_{m=0}^{n} [B_m x_t] u_t^m, \\
S(x_t, u_t) dW_t &= \begin{bmatrix} \left[\widetilde{\pi}_t^T \widetilde{C}_t \widetilde{\pi}_t\right]^{1/2} & 0 \\ 0 & 0 \end{bmatrix} x_t dW_t^1 \\
&\quad + \begin{bmatrix} 0 & -\sigma^e \\ 0 & -\sigma^e \end{bmatrix} x_t dW_t^2 + \begin{bmatrix} 0 & 0 \\ 0 & \sigma^i \end{bmatrix} x_t dW_t^3 \\
&= \sum_{j=1}^{3} \left[M^{jj}(u_t) x_t\right] dW_t^{jj},
\end{aligned}
\tag{2.16}
$$

where B and M are only used for notational purposes to simplify the equations. From the stochastic system given by (2.14) it is clear that $u = (u^2, \widetilde{\pi})$ affects only the stochastic differential equation of x_t^1 but not that of x_t^2. In particular, for (2.14) we have that $\widetilde{\pi}$ affects the variance of x_t^1 and the drift of x_t^1 via the term $x_t^1 \widetilde{r_t^{y}}^T \tilde{\pi}_t$. On the other hand, u^2 affects only the drift of x_t^1. Then (2.14) becomes

$$dx_t = A_t x_t dt + \sum_{j=0}^{n} \left[B^j x_t\right] u_t^j dt + a_t dt + \sum_{j=1}^{3} \left[M^j(u_t) x_t\right] dW_t^{jj}. \tag{2.17}$$

The model can be simplified if attention is restricted to the system with the LCR, as stated earlier, denoted in this section by $x_t = x_t^1 / x_t^2$.

Definition 2.5 (stochastic model for a simplified LCR). Define the simplified LCR system by the SDE

$$\begin{aligned} dx_t = {} & x_t \left[r^C(t) + r_t^e - r_t^i + (\sigma^e)^2 + \left(\sigma^i\right)^2 + \widetilde{r_t^{y}}^T \tilde{\pi}_t\right] dt \\ & + \left[u_t^1 + u_t^2 - r_t^e - (\sigma^e)^2\right] dt \\ & + \left[(\sigma^e)^2 (1 - x_t)^2 + \left(\sigma^i\right)^2 x_t^2 + x_t^2 \tilde{\pi}_t^T \tilde{C}_t \tilde{\pi}_t\right]^{1/2} d\overline{W}_t, \qquad x(t_0) = x_0. \end{aligned} \tag{2.18}$$

Note that in the drift of the SDE (2.18), the term

$$-r_t^e + x_t r_t^e = -r_t^e (x_t - 1), \tag{2.19}$$

appears because it models the effect of the decline of both liquid assets and nett cash outflows. Similarly the term $-(\sigma^e)^2 + x_t (\sigma^e)^2 = (\sigma^e)^2 (x_t - 1)$ appears.

3. Bank Liquidity in a Numerical Quantitative Framework

In this section, we discuss bank liquidity in a numerical quantitative framework. Recently the finance literature has devoted more attention to modeling and assessing liquidity risk in a numerical quantitative framework (see, e.g., [5, 8, 9]).

3.1. Bank Liquidity: Numerical Example 1

In this subsection, we use the data supplied in [6] (see, also, Appendices A.1 and A.2) to assess the liquidity of banks. The dataset originates from a supervisory liquidity report for Dutch banks. It covers a detailed breakdown of liquid assets and liabilities including cash in- and outflows of banks (see, also, [5]).

3.1.1. Data Description: Numerical Example 1

The aforementioned supervisory liquidity report includes on- and off-balance sheet items for about 85 Dutch banks (foreign bank subsidiaries included) with a breakdown per item (average granularity of about 7 items per bank). The report presents month end data available for the period October 2003 to March 2009. In this case, supervisory requirements dictate that actual bank liquidity must exceed required liquidity, at both a one-week and a one-month horizon. Actual liquidity is defined as the stock of liquid assets (weighted for haircuts) and recognized cash inflows (weighted for their liquidity value) during the test period. Required liquidity is defined as the assumed calls on contingent liquidity lines, assumed withdrawals of deposits, drying up of wholesale funding, and liabilities due to derivatives. In this way, the liquidity report comprises a combined stock and cash flow approach, in which respect it is a forward looking concept. The weights, w^i, of the assumed haircuts on liquid assets and run-off rates of liabilities are presented in last two columns of Tables 1 and 2 below. In this regard, the pecking order hypothesis is tested empirically in [5] by classifying the assets and liabilities of the banks in our sample according to the month weights in the liquidity report (as presented in the last column of Tables 1 and 2). In the report, the w^i values are fixed (see, e.g., [6]) and reflect the bank-specific and market-wide situation. The w^i values are based on best practices of values of haircuts on liquid assets and run-off rates of liabilities of the banking industry and credit rating agencies.

The various balance sheet and cash flow items in the prudential report [6] are assumed to reflect the instruments which banks use in liquidity risk management by way of responding to shocks. The instruments are expressed in gross amounts. To enhance the economic interpretation we define coherent groups of instruments and the sum of item amounts per group. The first column of Tables 1 and 2 below provides the group classification. Here, the second columns in these tables describe the particular class of assets and liabilities. For the liquidity test for the full month, a distinction is made between non-scheduled items and scheduled items. By contrast to non-scheduled items, scheduled items are included on the basis of their possible or probable due dates. For the liquidity test for the first week, scheduled items are only included if they are explicitly taken into account in day-to-day liquidity management Treasury operations. In Tables 1 and 2 below, scheduled items are indicated by the letter **S**.

3.1.2. Data Presentation: Numerical Example 1

In this section, we firstly represent data related to assets and then data related to liabilities.

3.1.3. Data Analysis: Numerical Example 1

From Tables 1 and 2, we have seen that the behavior of banks can be described by rather simple indicators constructed from firm-specific balance sheet data. Although they are descriptive in nature, the measures identify trends in banks behavior that convey forward looking information on market-wide developments. A key insight from the analysis is that while banks usually follow a pecking order in their balance sheet adjustments (by making larger adjustments to the most liquid balance sheet items compared to less liquid items), during the crisis banks were more inclined to a static response. This suggests that they have less room to follow a pecking order in their liquidity risk management in stressed circumstances. It implies that banks responses in crises may have more material effects on the economy, since a static response rule means that banks are more likely to adjust their (less liquid) retail lending and deposits than under normal market conditions. A sufficient stock of liquid buffers could prevent that banks are forced to such detrimental static responses, which lends support to the initiatives of the Basel Committee to tighten liquidity regulation for banks (see, e.g., [1]).

Table 1: Assets for liquidity testing.

Group		Assets	S	Week	Month
		Cash in the form of Banknotes/Coins		100	100
		Receivables from CBs (including ECB)			
1	1	Demand deposits		100	100
1	2	Amounts receivable	**S**	100	100
1	3	Receivables i.r.o reverse repos	**S**	100	100
1	4	Receivables i.t.f.o securities or Tier 2 eligible assets	**S**	d^*	d^*
		Collection documents			
	1	Available on demand		100	100
	2	Receivable	**S**	100	100
		Readily marketable debt instruments/CB eligible receivables			
		Issued by public authorities			
2	1	ECB tier 1 and tier 2 eligible assets		95**	95**
2	2	ECB tier 2 eligible assets deposited		85**	85**
2	3	ECB tier 2 eligible assets not deposited		85	85
2	4	Other readily marketable debt instruments		95	95
		Zone A			
2	5	Other readily marketable debt instruments		70	70
		Zone B			
		Issued by credit institutions			
2	1	ECB tier 1 eligible assets		90**	90**
2	2	ECB tier 2 eligible assets deposited		80**	80**
2	3	Other debt instruments qualifying under the capital adequacy directive (CAD)		90	90
2	4	Other Liquid Debt Instruments		70	70
		Issued by other institutions			
2	1	ECB tier 1 eligible assets		90**	90**
2	2	ECB tier 2 eligible assets deposited		80**	80**
2	3	Other debt instruments qualifying under the capital adequacy directive (CAD)		90	90
2	4	Other liquid debt instruments		70	70
		Amounts receivables			
		Branches and banking subsidiaries not included in the report			
3	1	Demand deposits		50	100
3	2	Amounts receivable i.r.o securities transactions	**S**	100	100
3	3	Other amounts receivables	**S**	100	90
		Other credit institutions			
3	1	Demand deposits		50	100
3	2	Amounts receivable i.r.o securities transactions	**S**	100	100
3	3	Other amounts receivables	**S**	100	90
		Public authorities			
3	1	Demand deposits		50	100
3	2	Amounts receivable i.r.o securities transactions	**S**	100	100
3	3	Other amounts receivables	**S**	100	90

Table 1: Continued.

Group		Assets	S	Week	Month
		Other professional money market players			
3	1	Demand deposits		50	100
3	2	Amounts receivable i.r.o securities transactions	**S**	100	100
3	3	Other amounts receivables	**S**	100	90
		Other counterparties			
	1	Demand deposits		0	0
	2	Amounts receivable i.r.o securities transactions	**S**	100	90
4	3	Other amounts receivables including premature redemptions	**S**	50	40
		Receivables i.r.o REPO and reverse REPO transactions			
		Reverse repo transactions (other than with CBs)			
5	1	Receivables i.r.o Bonds	**S**	100	100
5	2	Receivables i.r.o Shares	**S**	100	100
		Repo Transactions (Other Than with CBs)			
5	1	Receivables i.r.o bonds	**S**	$90/d^*/^{**}$	$90/d^*/^{**}$
5	2	Receivables i.r.o shares	**S**	70	70
		Securities lending/borrowing transactions			
5	1	Securities stock on account of securities		100	100
		Lending/borrowing transactions			
5	2	Securities receivable on account of securities		100	100
		Lending/borrowing transactions			
		Other securities and gold			
6	1	Other liquid shares		70	70
6	2	Unmarketable shares		0	0
2	3	Unmarketable bonds	**S**	100	100
	4	Gold		90	90
		Official standby facilities			
14	1	Official standby facilities received		100	100
14		Receivables i.r.o derivatives	**S**	***	***

*: Less applicable discount.
**: Either at stated percentage or at percentages applicable for ecb/escb collateral purposes.
***: Calculated amount for the period concerned.
$90/d^*/^{**}$: 90% OR less applicable discount (provided the method is consistently applied).

The measures for size and the number of extreme balance sheet adjustments gauge the time dimension of macroprudential risk, and indicators of the dependency and concentration of reactions capture the cross-sectoral dimension. The measures are robust to different specifications and distributions of the data. Applied to Dutch banks, the measures show that the number, size, and similarity of responses substantially changed during the crisis, in particular on certain market segments. They also indicate that the nature of banks behavior is asymmetric, being more intense in busts than in booms. Furthermore, during the crisis the deleveraging of large banks started earlier was more intense and more advanced than the deleveraging of smaller banks.

Given these findings, the indicators are useful for macroprudential analysis, for instance with regard to monitoring frameworks. Our analysis underscores the relevance

Table 2: Liabilities for liquidity testing.

Group		Liabilities	S	Week	Month
		Moneys borrowed from CBs			
7	1	Overdrafts payable within one week		100	100
7	2	Other amounts owed	**S**	100	100
		Debt instruments issued by the bank itself			
8	1	Issued debt securities	**S**	100	100
8	2	Subordinate liabilities	**S**	100	100
		Deposits and fixed term loans			
		Branches and banking subsidiaries not included in the report			
9	1	Amounts owed i.r.o securities transactions	**S**	100	100
9	2	Deposits and other funding—fixed maturity—plus interest payable	**S**	100	90
		Other counterparties			
	1	Amounts owed i.r.o securities transactions	**S**	100	100
10	2	Deposits and other funding—fixed maturity—plus interest payable	**S**	100	90
10		Fixed-term savings deposits	**S**	20	20
		Liabilities i.r.o repo and reverse repo transactions			
		Repo transactions (other than with CBs)			
11	1	Amounts owed i.r.o bonds	**S**	100	100
11	2	Amounts owed i.r.o shares	**S**	100	100
		Securities lending/borrowing transactions			
11	1	Negative securities stock on account of securities lending/borrowing transactions		100	100
11	2	Securities to be delivered on account of securities lending/borrowing transactions	**S**	100	100
		Credit balances and other moneys borrowed with an indefinite effective term			
		Branches and banking subsidiaries not included in the report			
12	1	Current account balances and other demand deposits		50	100
		Other credit institutions			
12	1	Balances on vostro accounts of banks		50	50
12	2	Other demand deposits		50	100
		Other professional money market players			
12	1	Demand deposits		50	100
		Savings accounts			
13	1	Savings accounts without a fixed term		2.5	10
		Other			
13	1	Demand deposits and other liabilities		5	20
13	2	Other amounts due and to be accounted for including the balance of forward transactions and amounts due i.r.o. social and provident funds		5	20

Table 2: Continued.

Group		Liabilities	S	Week	Month
		Official standby facilities			
14	1	Official standby facilities granted		100	100
		Liabilities i.r.o. derivatives			
14	1	Known liabilities i.r.o derivatives	S	***	***
14	1	Unknown liabilities i.r.o derivatives		***	***
		Other contingent liabilities and irrevocable credit facilities			
14	1	Unused irrevocable credit facilities, including underwriting of issues		2.5	10
14	2	Bills accepted	S	100	100
14	3	Credit-substitute guarantees		2.5	10
14	4	Non-credit-substitute guarantees		2.5	10
14	5	Other off-balance sheet liabilities		1.25	5

of using several indicators of liquidity risk at the same time, given the different leads and lags of the measures with systemic risk. The empirical results also provide useful information for financial stability models. A better understanding of banks behavior helps to improve the microfoundations of such models, especially with regard to the behavioral assumptions of heterogeneous institutions. Finally, the empirical findings in our study are relevant to understand the role of banks in monetary transmission and to assess the potential demand for CB finance in stress situations. The measures explain developments of financial intermediation channels (wholesale and retail, unsecured, secured, etc.) along the cross-sectional and time dimensions. They also shed more light on the size and number of banks that rely on CB financing.

3.2. Bank Liquidity: Numerical Example 2

In this section, we provide a simulation of the LCR dynamics given in (2.18).

3.2.1. Simulation: Numerical Example 2

In this subsection, we provide parameters and values for a numerical simulation. The parameters and their corresponding values for the simulation are shown in Table 3.

3.2.2. LCR Dynamics: Numerical Example 2

In Figure 1, we provide the LCR dynamics in the form of a trajectory derived from (2.18).

3.2.3. Properties of the LCR Trajectory: Numerical Example 2

Figure 1 shows the simulated trajectory for the LCR of low liquidity assets. Here different values of banking parameters are collected in Table 3. The number of jumps of the trajectory was limited to 1001, with the initial values for l fixed at 20.

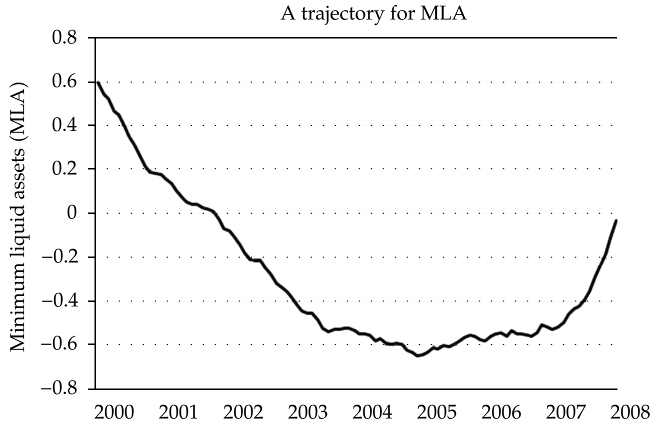

Figure 1: Trajectory of the LCR for low liquidity assets.

Table 3: Choices of liquidity coverage ratio parameters.

Parameter	Value	Parameter	Value	Parameter	Value
C	1 000	r^C	0.06	r^e	0.07
r^i	0.02	σ^e	1.7	σ^i	1.9
$\widetilde{r}^y$	0.05	$\widetilde{\pi}$	0.4	u^1	0.03
u^2	0.01	$\widetilde{C}$	750	$\overline{W}$	0.01

As we know, banks manage their liquidity by offsetting liabilities via assets. It is actually the diversification of the bank's assets and liabilities that expose them to liquidity shocks. Here, we use ratio analysis (in the form of the LCR) to manage liquidity risk relating various components in the bank's balance sheets. In Figure 1, we observe that between $t = 2000$ and $t = 2005$, there was a significant decrease in the trajectory which shows that either liquid assets declined or nett cash outflows increased.

There was also an increase between $t = 2005$ and $t = 2007$ which suggests that either liquid assets increased or nett cash outflows decreased. There was an even sharper increase subsequent to $t = 2007$ which comes as somewhat of a surprise. In order to mitigate the aforementioned increase in liquidity risk, banks can use several facilities such as repurchase agreements to secure more funding. However, a significant increase was recorded between $t = 2005$ and $t = 2008$, with the trend showing that banks have more liquid assets on their books. If $l > 0$, it demonstrated that the banks has kept a high volume of liquid assets which might be stemming from quality liquidity risk management. In order for banks to improve liquidity they may use debt securities that allow savings from nonfinancial private sectors, a good network of branches and other competitive strategies.

The LCR has some limitations regarding the characterization of the banks liquidity position. Therefore, other ratios could be used for a more complete analysis taking into account the structure of the short-term assets and liabilities of residual maturities.

4. Bank Liquidity in a Theoretical Quantitative Framework

In this section, we investigate bank liquidity in a theoretical quantitative framework. In particular, we characterize a liquidity provisioning strategy and discuss residual aggregate risk in order to eventually determine the appropriate value of the price process. In order to model uncertainty, in the sequel, we consider the filter probability space $(\Omega, \mathbb{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$, $T \in \mathbb{R}$ described in Assumption 2.1.

4.1. Preliminaries about the Liquidity Provisioning Strategy

Firstly, we consider a dynamic liquidity provisioning strategy for a risky underlying illiquid asset process, $(\Lambda_t)_{0\leq t\leq T}$. For purposes of relating the discussion below to the GFC, we choose Λ to be residential mortgage loans hereafter known simply as *mortgages*. Mortgages were very illiquid (nonmarketable) before and during the GFC. In this case, for liquidity provisioning purposes, the more liquid marketable securities, S—judging by their credit rating before and during the GFC—are used as a substitute for mortgages. This was true during the period before and during the GFC, with mortgage-backed securities being traded more easily than the underlying mortgages. Furthermore, we assume that the bank mainly holds illiquid mortgages and marketable securities (compare with the assets presented in Tables 1 and 2) with cash for investment being injected by outside investors. The liquid marketable securities, S, are not completely correlated with the illiquid mortgages, Λ, creating market incompleteness. Under the probability measure, $\mathbb{P}$, the price of the traded substitute securities and the illiquid underlying mortgages are given by

$$dS_t = S_t\left[\mu^s dt + \sigma^s dW_t^S\right], \qquad d\Lambda_t = \Lambda_t\left[\mu^\Lambda dt + \sigma^\Lambda dW_t^\Lambda\right], \tag{4.1}$$

respectively, where μ and σ are constants. We define the constant market price of risk for securities as

$$\lambda_s = \frac{\mu_s - r}{\sigma}. \tag{4.2}$$

We note that if the market correlation $|\rho|$ between W^S and W^Λ is equal to one, then the securities and mortgages are completely correlated and the market is complete.

Let Θ be a liquidity provisioning strategy for the bank's asset portfolio. The dynamics of its wealth process is given by

$$d\Pi_t = n_t^S dS_t + \left(\Pi_t - n_t^S S_t\right) r dt + dC_t, \tag{4.3}$$

where dC_t is an amount of cash infused into the portfolio, n_t^S is the number of shares of securities held in the portfolio at time t, Π_t is the value of the wealth process, and r is the riskless interest rate. The cumulative cost process $C(\Theta)$ associated with the strategy, Θ, is

$$C_t(\Theta) = \grave{\Pi}_t(\Theta) - \int_0^t n_u^S d\grave{S}_u, \quad 0 \leq t \leq T. \tag{4.4}$$

The cost process is the total amount of cash that has been injected from date 0 to date t. We determine a provisioning strategy that generate a payoff $(\Lambda_T - K)^+$ at the maturity T. The quantity $\int_t^T \exp\{-r(s-t)\}dC_s$ is the discounted cash amount that needs to be injected into the portfolio between dates t and T. Since $\int_t^T \exp\{-r(s-t)\}dC_s$ is uncertain, the risk-averse agent will focus on minimizing the associated ex-ante aggregate liquidity risk

$$\mathtt{R}_t(\Theta) = \mathbb{E}^{\mathbb{P}}\left[\left(\int_t^T \exp\{-r(s-t)\}dC_u\right)^2\right], \quad 0 \leq t \leq T. \tag{4.5}$$

It is clear that this concept is related to the conditioned expected square value of future costs. The strategy Θ, $0 \leq t \leq T$ is mean self-financing if its corresponding cost process $C = (C_t)_{0 \leq t = T}$ is a martingale. Furthermore, the strategy Θ is self-financing if and only if

$$\grave{\Pi}_t(\Theta) = \grave{\Pi}_0(\Theta) + \int_0^t n_u^S d\grave{S}_u, \quad 0 \leq t \leq T. \tag{4.6}$$

A strategy $\widehat{\Theta}$ is called an admissible continuation of Θ if $\widehat{\Theta}$ coincides with Θ at all times before t and $\Pi_t(\Theta) = L$, $\mathbb{P}$ a.s. Moreover, a provisioning strategy is called *liquidity risk minimizing* if for any $t \in [0, T]$, Θ minimizes the remaining liquidity risk. In other words, for any admissible continuous $\widehat{\Theta}$ of Θ at t we have

$$\mathtt{R}_t(\Theta) \leq \mathtt{R}_t\left(\widehat{\Theta}\right). \tag{4.7}$$

Criterion given in (4.5) can be formally rewritten as

$$\forall t \min_{(n^S, \Pi)} \mathtt{R}_t, \quad \text{subject to } \Pi_t = (\Lambda_T - K)^+. \tag{4.8}$$

We define the *expected squared error of the cost over the next period* as

$$\mathbb{E}^{\mathbb{P}}\left[(\Delta C_t)^2\right] = \mathbb{E}_t\left[\left(\Pi_{t+\Delta t} - \Pi_t - n_t^S(S_{t+\Delta t} - S_t) - \left(\Pi_t - n_t^S S_t\right)(\exp\{r(t+\Delta t)\} - \exp\{rt\})\right)^2\right]. \tag{4.9}$$

In the next section, we minimize the above quantity at each date, with respect to $(n_0^S, n_\Delta^S, \ldots, n_{t+\Delta t}^S)$ and also discuss the notion of a liquidity provisioning strategy.

4.2. Characterization of the Liquidity Provisioning Strategy

During the GFC, liquidity provisioning strategies involved several interesting elements. Firstly, private provisioning of liquidity was provided via the financial system. Secondly, there was a strong connection between financial fragility and cash-in-the-market pricing. Also, contagion and asymmetric information played a major role in the GFC. Finally, much of the debate on liquidity provisioning has been concerned with the provisioning of liquidity to financial institutions and resulting spillovers to the real economy. The next result characterizes the liquidity provisioning strategy that we study.

Theorem 4.1 (characterization of the provisioning strategy). *The locally liquidity risk minimizing strategy is described by the following.*

(1) *The investment in mortgages is*

$$\check{n}_t^S = \frac{\sigma^\Lambda \Lambda_t}{\sigma^S S_t}\rho C^\Lambda(t,\Lambda_t) = \frac{\sigma^\Lambda \Lambda_t}{\sigma^S S_t}\rho \exp\left\{\left(\mu^\Lambda - r - \rho\sigma^\Lambda \lambda^S\right)(T-t)\right\}\mathrm{N}(d_1,t), \tag{4.10}$$

where λ_s is the Sharpe ratio and $C(t,\Lambda_t)$ is the minimal entropy price

$$\begin{aligned} C(t,\Lambda_t) &= \exp\{-r(T-t)\}\mathbb{E}^{\mathbb{Q}}\left[(\Lambda_T-K)^+\right] \\ &= \exp\left\{\left(\mu^\Lambda - r - \rho\sigma^\Lambda\lambda^S\right)(T-t)\right\}\Lambda_t\mathrm{N}(d_1,t) - K\exp\{-r(T-t)\}\mathrm{N}(d_2,t), \end{aligned} \tag{4.11}$$

where $\mathbb{Q}$ is the minimal martingale measure defined as

$$\begin{gathered} \left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_t = \exp\left\{-\frac{1}{2}\Lambda^{S2}(T-t) - \lambda^S\left(W_T^S - W_t^S\right)\right\}, \\ d_{1,t} = \frac{1}{\sigma^\Lambda\sqrt{T-t}}\left[\ln\frac{\Lambda_t}{K} + \left(\mu^\Lambda - \rho\sigma^\Lambda\lambda^S + \frac{\sigma^{\Lambda 2}}{2}\right)(T-t)\right], \qquad d_{2,t} = d_{1,t} - \sigma^\Lambda\sqrt{T-t}. \end{gathered} \tag{4.12}$$

(2) *The cash investment is*

$$C(t,\Lambda_t) - \check{n}_t^S S_t. \tag{4.13}$$

If the Sharpe ratio, λ_s, of the traded substitute securities is equal to zero, the minimal martingale measure coincides with the original measure $\mathbb{P}$, and the above strategy is globally liquidity risk minimizing.

Proof. Let $\check{S}_t \equiv \exp\{-rt\}S_t$ be the discounted value of the traded securities at time t. This process follows a martingale under the martingale measure, $\mathbb{Q}$, since we have

$$d\check{S}_t = \check{S}_t\sigma^S dW_t^{S,\mathbb{Q}}, \tag{4.14}$$

where $dW_t^{S,\mathbb{Q}} \equiv dW_t^S + \lambda^S dt$ is the increment to a $\mathbb{Q}$-Brownian motion. Hence, we can write the Kunita-Watanabe decomposition of the discounted option payoff under $\mathbb{Q}$:

$$\exp\{-rt\}(\Lambda_T-K)^+ = H_0 + \int_0^T \zeta_t d\check{S}_t + V_T^H, \tag{4.15}$$

where V^H is a $\mathbb{P}$-martingale orthogonal to $\grave{S}$ under $\mathbb{Q}$. Lévy's Theorem shows that the process $\mathtt{A}$ defined by

$$d\mathtt{A}_t = \frac{dW_t^{\Lambda} - \rho dW_t^S}{\sqrt{1-\rho^2}} \tag{4.16}$$

is a $\mathbb{P}$-Brownian motion and that it is independent of W^S. Then, by Girsanov's theorem, $(W^{S,\mathbb{Q}}, \mathtt{A})$ also follows two-dimensional $\mathbb{Q}$-Brownian motion. Since V^H is a martingale under $\mathbb{Q}$ and is orthogonal to F, the martingale representation theorem shows that we have $V_t^H = \psi d\mathtt{A}_t$ for some process ψ. In particular, V^H is orthogonal under $\mathbb{P}$ to the martingale part of $\grave{S}$, where the martingale part of $\grave{S}$ under $\mathbb{P}$ is defined as

$$G_t = \int_0^t \sigma^S S_s dW_s^S. \tag{4.17}$$

Next, we suppose that

$$P_t = \exp\{-r(T-t)\}\mathbb{E}^{\mathbb{Q}}\left[(\Lambda_T - K)^+\right]. \tag{4.18}$$

Using (4.15) we obtain

$$P_t = \exp\{rt\}\left[H_0 + \int_0^t \zeta_s d\grave{S}_s + \mathrm{V}_t^H\right]. \tag{4.19}$$

Consider now the non-self-financing strategy with value $\grave{\Pi}_t = P_t$ and the number of securities given by $\grave{n}_t^S = \zeta_t$. Given (4.1) and (4.19), we obtain that $d\grave{C}_t = \exp\{rt\}dV_t^H$. This shows that, $V^H, \grave{C}$ is a $\mathbb{P}$-martingale orthogonal to G. We recall that a strategy (Π, n^S) is *locally risk minimizing* if and only if the associated cost process follows a $\mathbb{P}$-martingale orthogonal to G. Hence the strategy $(\grave{\Pi}, \grave{n}^S)$ is locally risk minimizing.

We now prove an explicitly expression for the random variable P_t, which is called the *minimum entropy price*. The Black-Scholes formula implies that

$$\begin{aligned} P_t &= C(t, \Lambda_t) \\ &= \exp\left\{\left(\mu^{\Lambda} - r - \sigma^{\Lambda}\rho\lambda^S\right)(T-t)\right\}\left[\Lambda_t \mathtt{N}(d_1, t) - K\exp\left\{\left(\mu^{\Lambda} - \rho\sigma^{\Lambda}\lambda^S\right)(T-t)\right\}\mathtt{N}(d_2, t)\right], \end{aligned} \tag{4.20}$$

which can be written as a function $C(t, \Lambda_t)$ of t and Λ_t. Using (4.19), we obtain that

$$\zeta_t = \frac{\sigma^{\Lambda}\Lambda_t}{\sigma^S S_t}\rho C^{\Lambda}(t, \Lambda_t). \tag{4.21}$$

The required expression for $\grave{n}_t^F$ follows immediately. □

Our paper addresses the problem of dynamic bank provisioning for (illiquid) nonmarketable mortgages, Λ, for which substitute (liquid) marketable securities, S, is part of the liquidity provisioning strategy. Due to the presence of cross-hedge liquidity risk we operate in an incomplete market setting. In this regard, we employ a non-self-financing strategy to ensure that uncertainty is reduced and trading is conducted in the Treasuries market. Moreover, the strategy is designed to influence a perfect replication at the cost of continuous cash infusion into the replicating bank portfolio. Since the cash infusion is random, the risk-averse agent would require that the total uncertainty involved over the remaining life of the mortgage be minimized. As a consequence, for $0 \leq t \leq T$, the associated *ex-ante aggregated liquidity risk* is given by

$$\mathrm{R}_t(\Theta) = \mathbb{E}^{\mathbb{P}}\left[\left(\int_t^T \exp\{-r(s-t)\}dC_u\right)^2\right], \quad 0 \leq t \leq T. \tag{4.22}$$

Now $\int_t^T \exp\{-r(s-t)\}dC_u$ is stochastic, so we will focus on minimizing the risk in (4.5). We apply a local risk minimization criterion which entails that instead of minimizing the uncertainty with respect to the cash infusion, C_t, over the process, the strategy attempts to minimize, at each date, the uncertainty over the next infinitesimal period. Also, the incompleteness entails the existence of infinitely many equivalent martingale measures. In order to determine the appropriate price of the asset value one should choose an appropriate equivalent martingale measure. In this case, the process is $\mathbb{Q}$-Brownian motion so that the discounted price process $\exp\{-rt\}S_t$ follows martingale pricing. The equivalent martingale measure will be determine according to the risk minimization criterion in Theorem 4.1. Let us consider the discounted price to be

$$\exp\{-rt\}(\Lambda_T - K)^+. \tag{4.23}$$

Applying the Kunita-Watanabe decomposition for the discounted price under a measure $\mathbb{Q}$, we get

$$K^{\Pi} = H_0 + \int_0^T \zeta_t d\hat{S}_u + V_T^H, \tag{4.24}$$

where V^H is a $\mathbb{P}$-martingale orthogonal to $\hat{S}$ under the measure $\mathbb{Q}$. Let

$$P_t = \exp\{-r(T-t)\}\mathbb{E}_t^{\mathbb{Q}}\left[(\Lambda_T - K)^+\right] \tag{4.25}$$

which can be rewritten as

$$P_t = \exp\{rt\}\left[H_0 + \int_0^t \zeta_u d\hat{S}_u + L_t^H\right]. \tag{4.26}$$

4.3. Residual Aggregate Liquidity Risk

During the GFC, two types of uncertainty concerning liquidity requirements arose. Firstly, each individual bank was faced with idiosyncratic liquidity risk. At any given time its depositors may have more or less liquidity needs. Uncertainty also arose from the fact that banks face aggregate liquidity risk. In some periods aggregate liquidity demand is high while in others it is low. Thus, aggregate risk exposes all banks to the same shock, by increasing or decreasing the demand for liquidity that they face simultaneously. The ability of banks to hedge themselves against these liquidity risks crucially depend on the functioning, or, more precisely, the completeness of financial markets. The next theorem provides an explicit expression for the aggregate liquidity risk when a locally risk minimizing strategy is utilized in an incomplete market.

Theorem 4.2 (residual aggregate liquidity risk). *The aggregate liquidity risk when a locally risk minimizing strategy at time t is implemented is equal to*

$$\mathrm{R}_t^{rm} = \sigma^{\Lambda 2}\left(1-\rho^2\right)\int_t^T \exp\{-2r(s-t)\}\mathbb{E}^{\mathbb{P}}\left[\Lambda^{S2}C^{\Lambda}(s,\Lambda_s)^2\right]ds. \tag{4.27}$$

This can be approximated by

$$\mathrm{R}_t^{rm} \approx \sigma^{\Lambda 2}(1-\rho)^2 C^{\Lambda}(0,\Lambda_0)^2\Lambda_0^2\frac{1-\exp\{-2r(T-t)\}}{2r}. \tag{4.28}$$

Proof. Let us now assume $\Phi = \Phi^{\star}$ and $\Pi_t = C(t,\Lambda_t)$. Under $\mathbb{Q}$, the wealth process, Π, evolves as

$$d\Pi_t = r\Pi_t dt + \rho\sigma^{\Lambda}\Lambda_t C^{\Lambda}(t,\Lambda_t)dW_t^{S,\mathbb{Q}} + d\grave{C}_t. \tag{4.29}$$

In addition, $(\exp\{-rt\}C(t,\Lambda_t))_t$ follows a $\mathbb{Q}$-martingale, where

$$dC(t,\Lambda_t) = rC(t,\Lambda_t)dt + C^{\Lambda}(t,\Lambda_t)\sigma^{\Lambda}\Lambda_t dW_t^{\Lambda,\mathbb{Q}}, \tag{4.30}$$

$$dW_t^{\Lambda,\mathbb{Q}} = dW_t^{\Lambda} + \rho\lambda^S dt \tag{4.31}$$

defines a $\mathbb{Q}$-Brownian motion. One can write it as

$$dW_t^{\Lambda,\mathbb{Q}} = dW_t^{\Lambda} - \rho dW_t^S + \rho dW_t^{S,\mathbb{Q}} = \sqrt{1-\rho^2}dW_t^2 + \rho dW_t^{\Lambda,\mathbb{Q}}. \tag{4.32}$$

Comparing (4.29) and (4.30) we obtain that

$$\exp\{-rt\}dC_t = \exp\{-rt\}C^{\Lambda}(t,\Lambda_t)\sigma^{\Lambda}\Lambda_t\sqrt{1-\rho^2}dW_t^2, \tag{4.33}$$

hence (4.27).

In what follows, we let δ_t be the delta of the mortgage process at time t that is computed from the minimal entropy price so that $\delta_t = C^{\Lambda}(t, \Lambda_t)$. We must now compute $\mathbb{E}^{\mathbb{P}}[\delta_t^2 \Lambda_t^2]$ for all t in $[0, T]$. If $(\delta_t^2, \Lambda_t^2)_{t>0}$ were a martingale, the task would be easy since we would have $\mathbb{E}^{\mathbb{P}}[\delta_t^2 \Lambda_t^2] = \delta_0^2 \Lambda_0^2$. But $(\delta_t^2 \Lambda_t^2)_{t\geq 0}$ is not a martingale. However, it can be shown that for small $\sigma^{\Lambda 2} T$, the expectation $\mathbb{E}^{\mathbb{P}}[\delta_t^2 \Lambda_t^2]$ is approximated by the constant $\delta_0^2 \Lambda_0^2$. The formal proof follows from the fact that $\mathbb{E}^{\mathbb{P}}[\gamma_t \Lambda_t^2] \approx \gamma_0 \Lambda_0^2$, $\gamma_t = C^{\Lambda\Lambda}(t, \Lambda_t)$, denoting the gamma of the value of the asset. Therefore, we finally have that

$$\sigma^{\Lambda 2}\left(1-\rho^2\right)\int_t^T \exp\{-2rs\}\mathbb{E}^{\mathbb{P}}\left[\delta_s^2 \Lambda^{S2}\right] ds \approx \sigma^{\Lambda 2}\left(1-\rho^2\right)\delta_0^2 \Lambda_0^2 \frac{1-\exp\{-2r(T-t)\}}{2r}. \tag{4.34}$$

□

Applying a non-self-financing strategy and considering $\grave{\Pi}_t = P_t$ and $\grave{n}_t^S = \zeta_t$, we obtain that

$$d\grave{C}_t = \exp\{rt\} dV_t^H. \tag{4.35}$$

This implies that $\grave{C}$ is $\mathbb{P}$-martingale orthogonal to G. In this regard, the strategy (Π, n^S) is locally risk minimizing if and only if the associated cost process $C(\Theta)$ follows a $\mathbb{P}$-martingale orthogonal to G. This means the strategy minimizes at each date the uncertainty over the next infinitesimal period. In applying the risk-minimization strategy there remains some "residual" aggregate liquidity risk stemming from the imperfection of the Brownian motion processes W^S and W^{Λ}. After the bank has implemented the locally risk minimizing strategy at time t, the aggregate liquidity risk is

$$\mathrm{R}_t^{rm} = \sigma^{\Lambda 2}\left(1-\rho^2\right)\int_t^T \exp\{-2r(s-t)\}\mathbb{E}^{\mathbb{P}}\left[\Lambda^2 C^{\Lambda}(S, \Lambda_u)^2\right]. \tag{4.36}$$

For δ_t associated with the value process at time t computed via the minimized entropy price, we now need to compute $\mathbb{E}^{\mathbb{P}}[\delta_t^2, \Lambda_t^2]$ for all t in $[0, T]$. Since $\mathbb{E}^{\mathbb{P}}[\delta_t^2, \Lambda_t^2]$ is approximated by the constant $\delta_0^2 \Lambda_0^2$, then $\mathbb{E}^{\mathbb{P}}[\gamma_t \Lambda_t^2] \approx \gamma_0 \Lambda_t^2$, $\gamma_t = C^{\Lambda}(t, \Lambda_t)$ which is the gamma of the mortgage value. Therefore, the residual liquidity risk at time t is

$$\sigma^{\Lambda 2}(1-\rho)^2 \int_t^T \exp\{-2rs\}\mathbb{E}^{\mathbb{P}}\left[\delta_u^2 \Lambda_u^S\right] du \approx \sigma^{\Lambda 2}(1-\rho)^2 \delta_0^2 \Lambda_0^2 \frac{-\exp\{-2r(T-t)\}}{2r}. \tag{4.37}$$

5. Conclusions and Future Directions

In this paper, we discuss liquidity risk management for banks. We investigate the stochastic dynamics of bank items such as loans, reserves, securities, deposits, borrowing and bank capital (compare with Question 1). In accordance with Basel III, our paper proposes that overall liquidity risk is best analyzed using ratio analysis approaches. Here, liquidity risk is measured via the LCR. In this case, we provide numerical results to highlight some important issues. Our numerical quantitative model shows that a low LCR stems from a low level of liquid assets or high nett cash outflows (compare with Question 2). Moreover, we provide a characterization of liquidity risk provisioning by considering an (illiquid) nonmarketable mortgage as an underlying asset and using (liquid) marketable securities for provisioning. In this case, we use non-self-financing strategy that considers market incompleteness to provision for liquidity risk. Then, we provide a quantitative framework for assessing residual risk stemming from the above strategy (compare with Question 3).

Future research should focus on other features of the GFC that are related to liquidity provisioning. The first involves the decrease in prices of AAA-rated tranches of structured financial products below fundamental values. The second is the effect of the GFC on interbank markets for term funding and collateralized money markets. Thirdly, further investigations should address the fear of contagion should a major institution fail. Finally, the effects on the real economy should be considered. In addition, the stochastic dynamic model we have consider in this paper does not take assets and liabilities with residual maturities into account. Such a model should be developed.

Appendices

A. More about Liquidity Risk

In this section, we provide more information about measures by cash flow, liquidity monitoring approaches, liquidity risk ratings and national approaches to liquidity risk.

A.1. Measures by Cash Flow

Banks use the intensity of the cash flow to predict the level of stress events. In this case, we determine the level of both cash in flows and cash out flows depending on both supply and demand for liquidity in the normal market performance. In this regard, the bank cash flow predicts the level of stress event s. Moreover, the use of proforma is an acceptable standard which determine the uses and sources of funds in the bank. It identifies where the bank funding short fall and liquidity gap lies.

A.2. Liquidity Monitoring Approaches

The BCBS has set international standards for sound management of liquidity risk. In this regard, the monitoring and evaluation of the banks operational activities is an internal control measure. However, the monitoring approach is divided into three levels, that is, the liquid assets approach, the cash flow approach, and a mixture of both. *Liquid asset approach* is mostly appropriate used in the Treasury bond market. In this regard, banks are required to maintain some liquid asset in their balance sheet that could be used during the hard period. Assets such as government securities are appropriate to maintain in the balance sheet because they can easily enable the bank to secure funding through securitization. While *Cash flow matching approach* enable banks to match the cash in flows with the cash out flows from the balance sheet activities.

The monitoring approaches for assessing liquidity risk is divided into three classes, that is, liquid asset approach, the cash flow approach and the combination of both. In the liquid asset approach a bank prescribed to a minimum level of cash or high-quality liquid or marketable assets in relation to the deposits and other sources of funds. While maturity

Table 4

Quantitative indicators
Availability of funds
Diversification of funding sources
Alternative funding sources
Capacity to augment liquidity through asset sales and/or securitization
Volume of wholesale liabilities with embedded options
Vulnerability of a bank to funding difficulities
Support provided by parent company
Qualitative indicators
Effectiveness of a board's policy in response to liquidity risk
Effectiveness process in identifying, measuring, monitoring, and controlling
Interacting of management to changing market conditions
Development of contingency funding plans
Information system management
Comprehensive and effective internal audit coverage

mismatch classify the expected inflows and outflows of funds into time bands of their residual maturity.

A.3. Liquidity Risk Rating

The rating of liquidity risk is categorized into two sets of indicators, that is, the quantitative and qualitative liquidity risk indicators.

Table 4 shows the quantitative and qualitative liquidity risk indicators. In light of the above, the rating for quantitative liquidity risk management is classified into three levels, that is, low, moderate level, and high level of liquidity risk. Therefore, a bank with a full set of all the indicated quantitative indicators has a low level of liquidity risk. Moreover, the rating for qualitative liquidity risk is divided into three levels, that is, strong, satisfactory, and weak quality of management of liquidity risk. In the above, we indicated that rating of liquidity risk is divided into two sets of indicators, namely, the quantitative liquidity risk indicators and qualitative liquidity risk indicators. According to Table 4, a bank with a full set of all the indicated qualitative indicators has a low level of liquidity risk, while a bank with a full set of all indicated qualitative indicators has a higher level of liquidity risk management.

A.4. National Approaches to Liquidity Risk

In this section, we discuss a useful principle which needs to be developed by individual countries to ensure sound management of liquidity risk and appropriate level of liquidity insurance by banks. This principle could be enforced via policies that assess liquidity as an internal measure; stress testing and other scenario analysis which determine the probability of a bank culminating into liquidity crisis; contingency funding to provide reliable sources of funds to cover the short fall; setting limitations such as holding of liquid assets, minimum liquid assets, limits on maturity mismatches, and limits on a particular funding sources; reporting about liquidity risks and sources of liquidity as well as through public disclosure to enable investors to access bank information.

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