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Research Article

The Gross-Pitaevskii Model of Spinor BEC

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The Gross-Pitaevskii model of spinor Bose-Einstein condensates is studied. Using the abstract results obtained for infinite dimensional Hamilton system, we establish the mathematical theory for the model of spinor BEC. Furthermore, three conservative quantities of spinor BEC, that is, the energy, total particle number and magnetization intensity are also proved.

1. Introduction

After the first remarkable experiments concerning the observation of Bose-Einstein condensate (BEC) in dilute gases of alkali atoms such as ^{87}Rb [1], ^{23}Na [2], and ^{7}Li [3] the interest in this phenomenon has revived [4, 5]. On the mathematical side, most of the work has concentrated on the Gross-Pitaevskii (GP) model of BEC, which is usually referred to as nonlinear Schödinger equation (NLSE) (cf. [6–14] and references therein). There are also many pieces of the literature on the spinor BEC ([15–19]). In the spinor BEC case, the constituent bosons have internal degrees of freedom, such as spin, the quantum state, and its properties becomes more complex [20]. What has made the alkali spinor BEC particularly interesting is that optical and magnetic fields can be used to probe and manipulate the system.

In [15], Ho shows that in an optical trap the ground states of spin-1 bosons such as ^{23}Na, ^{39}K, and ^{87}Rb can be either ferromagnetic or polar states, depending on the scattering lengths in different angular momentum channels. In [17], Pu et al. discuss the energy eigenstates, ground and spin mixing dynamics of a spin-1 spinor BEC for a dilute atomic vapor confined in an optical trap. Their results go beyond the mean field picture and are developed within a fully quantized framework. In [19], Zou and Mathis propose a three-step scheme for generating the maximally entangled atomic Greenberger-Horne-Zeilinger (GHZ) states in a spinor BEC by using strong classical laser fields to shift atom level and drive single-atom Raman transition. Their scheme can be directly used to generate the maximally entangled states between atoms with hyperfine spin 0 and 1.

In this paper, we want to establish the mathematical theory of the GP model of spinor BEC in which the internal degrees of freedom of atoms are also under consideration. Furthermore, three conservative quantities of spinor BEC, that is, the energy, total particle number, and magnetization intensity are also proved.

2. Gross-Pitaevskii Model for Spinor BEC

In this section we derive GP equation for spinor BEC, that is, the following equation:

$$\begin{aligned}
i\hbar\frac{\partial\psi_1}{\partial t} &= -\frac{\hbar^2}{2m}\Delta\psi_1 + V(x)\psi_1 + g_n|\Psi|^2\psi_1 + g_s\psi_{-1}^*\psi_0^2 \\
&\quad + g_s\left(|\psi_1|^2 + |\psi_0|^2 - |\psi_{-1}|^2\right)\psi_1, \\
i\hbar\frac{\partial\psi_0}{\partial t} &= -\frac{\hbar^2}{2m}\Delta\psi_0 + V(x)\psi_0 + g_n|\Psi|^2\psi_0 + 2g_s\psi_1\psi_{-1}\psi_0^* \\
&\quad + g_s\left(|\psi_1|^2 + |\psi_{-1}|^2\right)\psi_0, \\
i\hbar\frac{\partial\psi_{-1}}{\partial t} &= -\frac{\hbar^2}{2m}\Delta\psi_{-1} + V(x)\psi_{-1} + g_n|\Psi|^2\psi_{-1} + g_s\psi_1^*\psi_0^2 \\
&\quad + g_s\left(|\psi_{-1}|^2 + |\psi_0|^2 - |\psi_1|^2\right)\psi_{-1}.
\end{aligned} \tag{2.1}$$

We consider the GP model for $F = 1$ spinor BEC. Particles of $F = 1$ have three quantum states: magnetic quantum number $m = 1, 0, -1$. The corresponding wave function of these three quantum states are denote by

$$\Psi = (\psi_1, \psi_0, \psi_{-1}). \tag{2.2}$$

Here the physical meaning of $|\psi_i|^2$ is the density of $m = i$ particles ($i = 1, 0, -1$). The corresponding Hamilton energy functional of $F = 1$ spinor BEC is as follows:

$$E(\Psi, \Psi^*) = \int_\Omega \left[\frac{\hbar^2}{2m}|\nabla\Psi|^2 + V(x)|\Psi|^2 + \frac{1}{2}g_n|\Psi|^4 + \frac{1}{2}g_s|\Psi^* S\Psi|^2\right]dx, \tag{2.3}$$

where m is the boson mass, $\hbar$ is Planck constant, $V(\cdot)$ is the external trapping potential, $|\Psi|^2$ is the density of dilute bosonic atoms, g_n is the interaction constant between atoms, and g_s is the spin exchange interaction constant

$$g_n = \frac{4\pi\hbar^2}{m}\frac{a_0 + 2a_2}{3}, \qquad g_s = \frac{4\pi\hbar^2}{m}\frac{a_2 - a_0}{3}, \tag{2.4}$$

a_0 and a_2 are the scattering length, and $S = S_x\vec{i} + S_y\vec{j} + S_z\vec{k}$ is the spin operator:

$$S_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad S_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad S_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{2.5}$$

We adopt the following notations:

$$\begin{aligned} \Psi^* S \Psi &= \left(\psi_1^*, \psi_0^*, \psi_{-1}^*\right)\left[S_x\vec{i} + S_y\vec{j} + S_z\vec{k}\right]\begin{pmatrix} \psi_1 \\ \psi_0 \\ \psi_{-1} \end{pmatrix}, \\ |\Psi^* S \Psi|^2 &= |\Psi^* S_x \Psi|^2 + \left|\Psi^* S_y \Psi\right|^2 + |\Psi^* S_z \Psi|^2 \\ &= \frac{1}{2}\left|\psi_1^*\psi_0 + \psi_0^*\left(\psi_1 + \psi_{-1}\right) + \psi_{-1}^*\psi_0\right|^2 \\ &\quad + \frac{1}{2}\left|-\psi_1^*\psi_0 + \psi_0^*\left(\psi_1 - \psi_{-1}\right) + \psi_{-1}^*\psi_0\right|^2 \\ &\quad + \left(\left|\psi_1\right|^2 - \left|\psi_{-1}\right|^2\right)^2. \end{aligned} \tag{2.6}$$

By calculation we can get

$$\begin{aligned} |\Psi^* S \Psi|^2 = {}& \left|\psi_1\right|^4 + \left|\psi_{-1}\right|^4 - 2\left|\psi_1\right|^2\left|\psi_{-1}\right|^2 + 2\left|\psi_0\right|^2\left|\psi_1\right|^2 \\ & + 2\left|\psi_0\right|^2\left|\psi_{-1}\right|^2 + 2\psi_0^2\psi_1^*\psi_{-1}^* + 2\psi_0^{*2}\psi_1\psi_{-1}. \end{aligned} \tag{2.7}$$

By using Lagrange multiplier theorem, from the Hamilton energy functional E (see (2.3) and the total particle number

$$N = \int_\Omega |\Psi|^2 dx, \quad \Psi = \left(\psi_1, \psi_0, \psi_{-1}\right) \tag{2.8}$$

is conservative, and we can obtain the steady state GP equation of spinor BEC as follows:

$$\mu\psi_k = \frac{\delta}{\delta\psi_k^*}E(\Psi, \Psi^*), \quad k = 1, 0, -1, \tag{2.9}$$

where μ is the chemical potential. Furthermore, according to general rules of quantum mechanics from steady state GP equation, we can get the dynamical model as follows:

$$i\hbar\frac{\partial\psi_k}{\partial t} = \frac{\delta}{\delta\psi_k^*}E(\Psi, \Psi^*), \quad k = 1, 0, -1, \tag{2.10}$$

where $i = \sqrt{-1}$. From (2.3) and (2.7), we can obtain the concrete expression of (2.10) as (2.1).

In the spinor BEC g_n and g_s we have following physical meaning:

$$g_n \begin{cases} >0, & \text{corresponding to the repulsive interaction between atoms,} \\ <0, & \text{corresponding to the attractive interaction between atoms,} \end{cases}$$
$$g_s \begin{cases} >0, & \text{corresponding to the antiferromagnetic states,} \\ <0, & \text{corresponding to the ferromagnetic states.} \end{cases} \tag{2.11}$$

3. Equivalent Form of Spinor BEC

Let $\psi_k = \psi_k^1 + i\psi_k^2$. In this section we will show that GP equation (2.1) is equivalent to the following quantum Hamilton systems (see [21]):

$$\begin{aligned} \frac{\partial \psi_k^1}{\partial t} &= \frac{\delta F(\psi^1, \psi^2)}{\delta \psi_k^2}, \\ \frac{\partial \psi_k^2}{\partial t} &= -\frac{\delta F(\psi^1, \psi^2)}{\delta \psi_k^1}, \end{aligned} \qquad k = 1, 0, -1, \tag{3.1}$$

where $\psi^1 = (\psi_1^1, \psi_0^1, \psi_{-1}^1)$, $\psi^2 = (\psi_1^2, \psi_0^2, \psi_{-1}^2)$, and the energy functional defined as

$$F\left(\psi^1, \psi^2\right) = \frac{1}{2\hbar} E(\Psi, \Psi^*), \quad (E \text{ see } (2.3)). \tag{3.2}$$

In fact, on the one hand, splitting real and imaginary parts of (2.1), we can obtain

$$\begin{aligned} \hbar \frac{\partial \psi_1^1}{\partial t} &= \left(-\frac{\hbar^2}{2m} \Delta + V(x) + g_n |\Psi|^2 + g_s \left(\left| \psi_1 \right|^2 + \left| \psi_0 \right|^2 - \left| \psi_{-1} \right|^2 \right) \right) \psi_1^2 \\ &\quad + g_s \left(2\psi_{-1}^1 \psi_0^1 \psi_0^2 - \psi_{-1}^2 \left(\psi_0^1 \right)^2 + \psi_{-1}^2 \left(\psi_0^2 \right)^2 \right), \\ \hbar \frac{\partial \psi_1^2}{\partial t} &= \left(\frac{\hbar^2}{2m} \Delta - V(x) - g_n |\Psi|^2 - g_s \left(\left| \psi_1 \right|^2 + \left| \psi_0 \right|^2 - \left| \psi_{-1} \right|^2 \right) \right) \psi_1^1 \\ &\quad - g_s \left(2\psi_{-1}^2 \psi_0^1 \psi_0^2 + \psi_{-1}^1 \left(\psi_0^1 \right)^2 - \psi_{-1}^1 \left(\psi_0^2 \right)^2 \right), \\ \hbar \frac{\partial \psi_0^1}{\partial t} &= \left(-\frac{\hbar^2}{2m} \Delta + V(x) + g_n |\Psi|^2 + g_s \left(\left| \psi_1 \right|^2 + \left| \psi_{-1} \right|^2 \right) \right) \psi_0^2 \\ &\quad + 2g_s \left(\psi_1^1 \psi_{-1}^2 \psi_0^1 + \psi_{-1}^1 \psi_1^2 \psi_0^1 - \psi_1^1 \psi_{-1}^1 \psi_0^2 + \psi_1^2 \psi_{-1}^2 \psi_0^2 \right), \end{aligned}$$

$$\begin{aligned}
\hbar\frac{\partial\psi_0^2}{\partial t} &= \left(\frac{\hbar^2}{2m}\Delta - V(x) - g_n|\Psi|^2 - g_s\left(|\psi_1|^2 + |\psi_{-1}|^2\right)\right)\psi_0^1 \\
&\quad - 2g_s\left(\psi_1^1\psi_{-1}^1\psi_0^1 - \psi_{-1}^2\psi_1^2\psi_0^1 + \psi_1^1\psi_{-1}^2\psi_0^2 + \psi_1^2\psi_{-1}^1\psi_0^2\right), \\
\hbar\frac{\partial\psi_{-1}^1}{\partial t} &= \left(-\frac{\hbar^2}{2m}\Delta + V(x) + g_n|\Psi|^2 + g_s\left(|\psi_{-1}|^2 + |\psi_0|^2 - |\psi_1|^2\right)\right)\psi_{-1}^2 \\
&\quad + g_s\left(2\psi_1^1\psi_0^1\psi_0^2 - \psi_1^2\left(\psi_0^1\right)^2 + \psi_1^2\left(\psi_0^2\right)^2\right), \\
\hbar\frac{\partial\psi_{-1}^2}{\partial t} &= \left(\frac{\hbar^2}{2m}\Delta - V(x) - g_n|\Psi|^2 - g_s\left(|\psi_{-1}|^2 + |\psi_0|^2 - |\psi_1|^2\right)\right)\psi_{-1}^1 \\
&\quad - g_s\left(2\psi_1^2\psi_0^1\psi_0^2 + \psi_1^1\left(\psi_0^1\right)^2 - \psi_1^1\left(\psi_0^2\right)^2\right).
\end{aligned} \tag{3.3}$$

On the other hand, it is easy to check that

$$\begin{aligned}
\frac{\delta F(\psi^1,\psi^2)}{\delta\psi_1^2} &= \frac{1}{\hbar}\Bigg[\left(-\frac{\hbar^2}{2m}\Delta + V(x) + g_n|\Psi|^2 + g_s\left(|\psi_1|^2 + |\psi_0|^2 - |\psi_{-1}|^2\right)\right)\psi_1^2 \\
&\quad + g_s\left(2\psi_{-1}^1\psi_0^1\psi_0^2 - \psi_{-1}^2\left(\psi_0^1\right)^2 + \psi_{-1}^2\left(\psi_0^2\right)^2\right)\Bigg], \\
\frac{\delta F(\psi^1,\psi^2)}{\delta\psi_1^1} &= \frac{1}{\hbar}\Bigg[\left(-\frac{\hbar^2}{2m}\Delta + V(x) + g_n|\Psi|^2 + g_s\left(|\psi_1|^2 + |\psi_0|^2 - |\psi_{-1}|^2\right)\right)\psi_1^1 \\
&\quad + g_s\left(2\psi_{-1}^2\psi_0^1\psi_0^2 + \psi_{-1}^1\left(\psi_0^1\right)^2 - \psi_{-1}^1\left(\psi_0^2\right)^2\right)\Bigg], \\
\frac{\delta F(\psi^1,\psi^2)}{\delta\psi_0^2} &= \frac{1}{\hbar}\Bigg[\left(-\frac{\hbar^2}{2m}\Delta + V(x) + g_n|\Psi|^2 + g_s\left(|\psi_1|^2 + |\psi_{-1}|^2\right)\right)\psi_0^2 \\
&\quad + 2g_s\left(\psi_1^1\psi_{-1}^2\psi_0^1 + \psi_{-1}^1\psi_1^2\psi_0^1 - \psi_1^1\psi_{-1}^1\psi_0^2 + \psi_1^2\psi_{-1}^2\psi_0^2\right)\Bigg], \\
\frac{\delta F(\psi^1,\psi^2)}{\delta\psi_0^1} &= \frac{1}{\hbar}\Bigg[\left(-\frac{\hbar^2}{2m}\Delta + V(x) + g_n|\Psi|^2 + g_s\left(|\psi_1|^2 + |\psi_{-1}|^2\right)\right)\psi_0^1 \\
&\quad + 2g_s\left(\psi_1^1\psi_{-1}^1\psi_0^1 - \psi_{-1}^2\psi_1^2\psi_0^1 + \psi_1^1\psi_{-1}^2\psi_0^2 + \psi_1^2\psi_{-1}^1\psi_0^2\right)\Bigg],
\end{aligned}$$

$$\frac{\delta F(\varphi^1,\varphi^2)}{\delta\varphi^2_{-1}} = \frac{1}{\hbar}\left[\left(-\frac{\hbar^2}{2m}\Delta + V(x) + g_n|\Psi|^2 + g_s\left(\left|\varphi_{-1}\right|^2 + \left|\varphi_0\right|^2 - \left|\varphi_1\right|^2\right)\right)\varphi^2_{-1}\right.$$
$$\left. + g_s\left(2\varphi^1_1\varphi^1_0\varphi^2_0 - \varphi^2_1\left(\varphi^1_0\right)^2 + \varphi^2_1\left(\varphi^2_0\right)^2\right)\right],$$
$$\frac{\delta F(\varphi^1,\varphi^2)}{\delta\varphi^1_{-1}} = \frac{1}{\hbar}\left[\left(-\frac{\hbar^2}{2m}\Delta + V(x) + g_n|\Psi|^2 + g_s\left(\left|\varphi_{-1}\right|^2 + \left|\varphi_0\right|^2 - \left|\varphi_1\right|^2\right)\right)\varphi^1_{-1}\right.$$
$$\left. + g_s\left(2\varphi^2_1\varphi^1_0\varphi^2_0 + \varphi^1_1\left(\varphi^1_0\right)^2 - \varphi^1_1\left(\varphi^2_0\right)^2\right)\right]. \tag{3.4}$$

Consequently, GP equation (2.1) is equivalent to (3.1).

4. Infinite Dimensional Hamilton System

In this section, we consider the following infinite-dimensional Hamilton system

$$\begin{aligned} \frac{du}{dt} &= -D_v F(u,v), \\ \frac{dv}{dt} &= D_u F(u,v), \qquad ((u,v)\in X_1\times X_2) \\ u(0) &= \varphi, \quad v(0) = \psi, \end{aligned} \tag{4.1}$$

where $X \subset X_i \subset H$ $(i = 1,2)$ is dense, X is linear space, X_1, X_2 is reflexive Banach space, H is Hilbert space, $F : X_1 \times X_2 \to R^1$ is C^1 functional, and $DF = (D_u F, D_v F)$ is derived operator.

Remark 4.1. Infinite-dimensional Hamilton system (4.1) not only has some kind of beauty in its own form, but also many equations can be written as (4.1). For example, Schödinger equation, Weyl equations, and Dirac equations can be written as (4.1). Hence, it is worth to study the infinite-dimensional Hamilton system (4.1), also see [21].

Definition 4.2. One says $(u,v) \in X_1 \times X_2$ is a weak solution of Hamilton system (4.1), *provided*

$$\langle u,\tilde{u}\rangle_H + \langle v,\tilde{v}\rangle_H = \int_0^t [\langle D_u F(u,v),\tilde{v}\rangle - \langle D_v F(u,v),\tilde{u}\rangle]dt + \langle\varphi,\tilde{u}\rangle_H + \langle\psi,\tilde{v}\rangle_H, \tag{4.2}$$

for every $\tilde{u} \in X_2, \tilde{v} \in X_1$.

Let $F : X_1 \times X_2 \to R^1$ satisfy

$$||(u,v)||_{X_1\times X_2} \longrightarrow \infty \Longleftrightarrow F(u,v) \longrightarrow \infty \quad (\text{or } -F(u,v) \longrightarrow \infty). \tag{4.3}$$

Then we have the following existence theorem.

Theorem 4.3 (see [21]). *Assume that F satisfies condition* (4.3) *and $DF : X_1 \times X_2 \to (X_1 \times X_2)^*$ is weakly continuous, then for any $(\varphi, \psi) \in X_1 \times X_2$, there exists one global weak solution of equation* (4.1)

$$(u, v) \in L^\infty((0, \infty), X_1 \times X_2). \tag{4.4}$$

Furthermore, $F(u, v)$ is a conservative quantity for weak solution (u, v), that is,

$$F(u(t), v(t)) = F(\varphi, \psi), \quad \forall t > 0. \tag{4.5}$$

Proof. We prove the existence of global solution for (4.1) in $L^\infty((0, \infty), X_1 \times X_2)$ by standard Galerkin method. Choose

$$\{e_k \mid k = 1, 2, \ldots\} \subset X \tag{4.6}$$

as orthonormal basis of space H. Set $X_n, \widetilde{X}_n$ as follows:

$$\begin{aligned} X_n &= \left\{ \sum_{k=1}^{n} \alpha_k e_k \mid \alpha_k \in R^1,\ 1 \le k \le n \right\}, \\ \widetilde{X}_n &= \left\{ \sum_{k=1}^{n} \beta_k(t) e_k \mid \beta_k(\cdot) \in C^1[0, \infty),\ 1 \le k \le n \right\}. \end{aligned} \tag{4.7}$$

Consider the ordinary equations as follows:

$$\begin{aligned} \frac{dx_k(t)}{dt} &= \langle -D_v F(u_n, v_n), e_k \rangle, \\ \frac{dy_k(t)}{dt} &= \langle D_u F(u_n, v_n), e_k \rangle, \qquad (k = 1, \ldots, n), \\ x_k(0) &= \langle \varphi, e_k \rangle_H, \\ y_k(0) &= \langle \psi, e_k \rangle_H, \end{aligned} \tag{4.8}$$

where $u_n = \sum_{k=1}^{n} x_k(t) e_k, v_n = \sum_{k=1}^{n} y_k(t) e_k$.

By the theory of ordinary equations, there exists only one local solution of (4.8):

$$\{x_1(t), y_1(t), \ldots, x_n(t), y_n(t)\}, \quad 0 \le t \le \tau. \tag{4.9}$$

From (4.8) we can obtain the equality

$$\begin{aligned} \langle u_n, \widetilde{u}_n \rangle_H + \langle v_n, \widetilde{v}_n \rangle_H = &\int_0^t [\langle D_u F(u_n, v_n), \widetilde{v}_n \rangle - \langle D_v F(u_n, v_n), \widetilde{u}_n \rangle] dt \\ &+ \langle \varphi, \widetilde{u}_n \rangle_H + \langle \psi, \widetilde{v}_n \rangle_H \end{aligned} \tag{4.10}$$

holds true for any $\widetilde{u}_n, \widetilde{v}_n \in X_n$. Moreover, equality

$$\int_0^t \left[\left\langle \frac{du_n}{dt}, \widetilde{u}_n \right\rangle_H + \left\langle \frac{dv_n}{dt}, \widetilde{v}_n \right\rangle_H \right] dt = \int_0^t [\langle D_u F(u_n, v_n), \widetilde{v}_n \rangle - \langle D_v F(u_n, v_n), \widetilde{u}_n \rangle] dt \tag{4.11}$$

holds true for any $\widetilde{u}_n, \widetilde{v}_n \in \widetilde{X}_n$.

Putting $(\widetilde{u}_n, \widetilde{v}_n) = (-dv_n/dt, du_n/dt)$ in (4.11), we obtain that

$$\begin{aligned} 0 &= \int_0^t \left[\left\langle D_u F(u_n, v_n), \frac{du_n}{dt} \right\rangle + \left\langle D_v F(u_n, v_n), \frac{dv_n}{dt} \right\rangle \right] dt \\ &= \int_0^t \frac{d}{dt} F(u_n, v_n) dt, \end{aligned} \tag{4.12}$$

which implies

$$F(u_n, v_n) = F(\varphi_n, \psi_n), \tag{4.13}$$

where

$$\varphi_n = \sum_{k=1}^n \langle \varphi, e_k \rangle_H e_k, \qquad \psi_n = \sum_{k=1}^n \langle \psi, e_k \rangle_H e_k. \tag{4.14}$$

From (4.3) and (4.10), we deduce that $\{(u_n, v_n)\}_{n=1}^\infty$ is bounded in $L^\infty((0,\infty), X_1 \times X_2)$. Therefore there exists a subsequence; we still write it as $\{(u_n, v_n)\}_{n=1}^\infty$, such that

$$(u_n, v_n) \rightharpoonup (u, v) \quad \text{in } X_1 \times X_2, \text{ a. e. } t \in (0, \infty). \tag{4.15}$$

According to $DF : X_1 \times X_2 \to (X_1 \times X_2)^*$ being weakly continuous and (4.10), (4.15), we know the following equality

$$\langle u, \widetilde{u} \rangle_H + \langle v, \widetilde{v} \rangle_H = \int_0^t [\langle D_u F(u, v), \widetilde{v} \rangle - \langle D_v F(u, v), \widetilde{u} \rangle] dt + \langle \varphi, \widetilde{u} \rangle_H + \langle \psi, \widetilde{v} \rangle_H \tag{4.16}$$

holds true for any $\widetilde{u}, \widetilde{v} \in \bigcup_{n=1}^\infty X_n$. Since $\bigcup_{n=1}^\infty X_n$ is dense in X_1 and X_2, equality (4.16) holds true for all $(\widetilde{u}, \widetilde{v}) \in X_1 \times X_2$, which implies that $(u, v) \in L^\infty((0,\infty), X_1 \times X_2)$ is a global weak solution of (4.1).

Next, we prove $F(u, v)$ is a conservative quantity for weak solution (u, v). From (4.16), for all $h > 0$ we have

$$\begin{aligned} &\langle u(t+h) - u(t), \widetilde{u} \rangle_H + \langle v(t+h) - v(t), \widetilde{v} \rangle_H \\ &\quad = \int_t^{t+h} [\langle D_u F(u(\tau), v(\tau)), \widetilde{v} \rangle - \langle D_v F(u(\tau), v(\tau)), \widetilde{u} \rangle] d\tau. \end{aligned} \tag{4.17}$$

Putting

$$\tilde{u} = -\Delta_h v = -(v(t+h) - v(t)), \qquad \tilde{v} = \Delta_h u = u(t+h) - u(t) \tag{4.18}$$

in (4.17), we obtain that

$$\begin{aligned} 0 &= \frac{1}{h}\int_t^{t+h} [\langle D_u F(u(\tau), v(\tau)), \Delta_h u\rangle + \langle D_v F(u(\tau), v(\tau)), \Delta_h v\rangle] d\tau \\ &= F(u(t+h), v(t+h)) - F(u(t), v(t)). \end{aligned} \tag{4.19}$$

Therefore, $F(u,v)$ is a conservative quantity for weak solution (u,v). The proof is completed. □

Theorem 4.4 (see [21]). *Let X_1, X_2 be Hilbert space and $F : X_1 \times X_2 \to R$ be C^1 functional. Then a C^1 functional $G : X_1 \times X_2 \to R$ is a conservative quantity for the infinite-dimensional Hamilton system* (4.1) *if and only if the following equality*

$$\left\langle \frac{\delta G(u,v)}{\delta u}, \frac{\delta F(u,v)}{\delta v} \right\rangle_{X_1 \times X_2} = \left\langle \frac{\delta G(u,v)}{\delta v}, \frac{\delta F(u,v)}{\delta u} \right\rangle_{X_1 \times X_2} \tag{4.20}$$

holds true for any $(u,v) \in X_1 \times X_2$.

Proof. Let (u,v) be a solution of (4.1). Then we have

$$\begin{aligned} \frac{d}{dt}G(u,v) &= \left\langle \frac{\delta G(u,v)}{\delta u}, \frac{du}{dt} \right\rangle_{X_1 \times X_2} + \left\langle \frac{\delta G(u,v)}{\delta v}, \frac{dv}{dt} \right\rangle_{X_1 \times X_2} \\ &= -\left\langle \frac{\delta G(u,v)}{\delta u}, \frac{\delta F(u,v)}{\delta v} \right\rangle + \left\langle \frac{\delta G(u,v)}{\delta v}, \frac{\delta F(u,v)}{\delta u} \right\rangle, \end{aligned} \tag{4.21}$$

which imply that $(d/dt)G(u,v) = 0$ if and only if equality (4.20) holds true. The proof is completed. □

5. The Existence of Global Solution of Spinor BEC

In this section we consider the Gross-Pitaevskii equation of spinor BEC (2.10) under the Dirichlet boundary condition, to wit the following initial boundary problem:

$$\begin{gathered} i\hbar\frac{\partial \Psi}{\partial t} = \frac{\delta}{\delta \Psi^*} E(\Psi, \Psi^*), \quad x \in \Omega, \\ \Psi|_{\partial\Omega} = 0, \\ \Psi(x,0) = \Psi_0(x), \end{gathered} \tag{5.1}$$

where $\Omega \subset R^n$ $(1 \leq n \leq 3)$ is a domain. When $\Omega = R^n$, then (5.1) become Cauchy problem. By applying Theorem 4.3, we can obtain the following theorem.

Theorem 5.1. *Assume that* $V \in L^2(\Omega)$ *and* $g_n > \max\{0, -2g_s\}$, *then for any* $\Psi_0 \in H^1(\Omega, \mathcal{C}^3)$, *there exists one global weak solution of problem* (5.1)

$$\Psi \in C^0\left([0,\infty), L^2\left(\Omega, \mathcal{C}^3\right)\right) \cap L^\infty\left((0,\infty), H^1\left(\Omega, \mathcal{C}^3\right)\right). \tag{5.2}$$

Remark 5.2. If $g_s = 0$, then (5.1) reduce to the GP equation of BEC. Theorem 5.1 is also consistent with the experiments in repulsive case. In the situation of repulsive interaction, solutions to the GP equation of BEC are well defined for all times [12, 13, 20], which corresponds to the emergence of the BEC.

Proof. Let $H = L^2(\Omega, R^6), H_1 = H^1(\Omega, R^6)$. Firstly, we need to verify condition (4.3) in Theorem 4.3. From Section 2, we know that

$$F\left(\varphi^1, \varphi^2\right) = \frac{1}{2\hbar}\int_\Omega \left[\frac{\hbar^2}{2m}|\nabla\Psi|^2 + V(x)|\Psi|^2 + \frac{1}{2}g_n|\Psi|^4 + \frac{1}{2}g_s|\Psi^* S\Psi|^2\right]dx, \tag{5.3}$$

where $\Psi = (\varphi_1, \varphi_0, \varphi_{-1}),\ \varphi_k = \varphi_k^1 + i\varphi_k^2 (k = 1, 0, -1)$, and

$$\begin{aligned}
|\Psi^* S\Psi|^2 &= |\varphi_1|^4 + |\varphi_{-1}|^4 - 2|\varphi_1|^2|\varphi_{-1}|^2 + 2|\varphi_0|^2|\varphi_1|^2 \\
&\quad + 2|\varphi_0|^2|\varphi_{-1}|^2 + 2\varphi_0^2\varphi_1^*\varphi_{-1}^* + 2\varphi_0^{*2}\varphi_1\varphi_{-1} \\
&\le 2\left(|\varphi_1|^4 + |\varphi_0|^4 + |\varphi_{-1}|^4\right).
\end{aligned} \tag{5.4}$$

Hence, when $g_n > \max\{0, -2g_s\}$, we have

$$\int_\Omega g_n|\Psi|^4 + g_s|\Psi^* S\Psi|^2 dx \ge \lambda \int_\Omega |\Psi|^4 dx, \tag{5.5}$$

where $\lambda = g_n - \max\{0, -2g_s\} > 0$. Therefore, we deduce

$$F\left(\varphi^1, \varphi^2\right) \longrightarrow \infty \Longleftrightarrow \left\|\left(\varphi^1, \varphi^2\right)\right\|_{H_1} \longrightarrow \infty, \tag{5.6}$$

which implies that condition (4.3) holds true.

Next we need to verify the continuous condition in Theorem 4.3. Let operator $DF: H_1 \to H_1$ be defined by

$$\begin{aligned}
\left\langle DF(\Psi), \widetilde{\Psi}\right\rangle = \frac{1}{\hbar}\int_\Omega \Bigg[&\frac{\hbar^2}{2m}\nabla\Psi \cdot \nabla\widetilde{\Psi} + V(x)\Psi \cdot \widetilde{\Psi} \\
&+ g_n|\Psi|^2\Psi \cdot \widetilde{\Psi} + g_s(\Psi^* S\Psi)\widetilde{\Psi}^* S\widetilde{\Psi}\Bigg]dx.
\end{aligned} \tag{5.7}$$

For any $\widetilde{\Psi} \in C_0^\infty(\Omega, R^6)$ and $\Psi_n \rightharpoonup \Phi$ in H_1, we have

$$\lim_{n\to\infty} \left\langle DF(\Psi_n), \widetilde{\Psi} \right\rangle = \left\langle DF(\Phi), \widetilde{\Psi} \right\rangle. \tag{5.8}$$

Since $C_0^\infty(\Omega, R^6)$ is dense in H_1, equality (5.8) holds true for all $\widetilde{\Psi} \in H_1$, which implies that $DF : H_1 \to H_1$ is weakly continuous.

Therefore, according to Theorem 4.3, there exists a global weak solution of (5.1). The proof is completed. □

6. The Conservative Quantities of Spinor BEC

In this section we will discuss the conservative quantities of spinor BEC. Let E be defined as (2.3), N, M as follows:

$$N = \int_\Omega \left[|\psi_1|^2 + |\psi_0|^2 + |\psi_{-1}|^2 \right] dx, \qquad M = \int_\Omega \left[|\psi_1|^2 - |\psi_{-1}|^2 \right] dx. \tag{6.1}$$

Then by using the same method as the proof of Theorem 4.4, we will prove the following theorem.

Theorem 6.1. *Hamilton energy E, the total particle number N, and magnetization intensity M are conservative quantities for problem* (5.1).

Proof. Firstly, from (3.1) and (3.2) we can get

$$\begin{aligned}
\frac{1}{2\hbar}\frac{dE(\Psi,\Psi^*)}{dt} &= \frac{dF(\psi^1,\psi^2)}{dt} = \left\langle \frac{\delta F(\psi^1,\psi^2)}{\delta\psi^1}, \frac{\partial\psi^1}{\partial t} \right\rangle + \left\langle \frac{\delta F(\psi^1,\psi^2)}{\delta\psi^2}, \frac{\partial\psi^2}{\partial t} \right\rangle \\
&= \sum_{k=1,0,-1} \left[\left\langle \frac{\delta F(\psi^1,\psi^2)}{\delta\psi_k^1}, \frac{\partial\psi_k^1}{\partial t} \right\rangle + \left\langle \frac{\delta F(\psi^1,\psi^2)}{\delta\psi_k^2}, \frac{\partial\psi_k^2}{\partial t} \right\rangle \right] \\
&= \sum_{k=1,0,-1} \left[\left\langle -\frac{\partial\psi_k^2}{\partial t}, \frac{\partial\psi_k^1}{\partial t} \right\rangle + \left\langle \frac{\partial\psi_k^1}{\partial t}, \frac{\partial\psi_k^2}{\partial t} \right\rangle \right] \\
&= 0,
\end{aligned} \tag{6.2}$$

which imply that the energy E is a conservative quantity for problem (5.1).

Secondly, by using (3.3) we can get the following equalities:

$$\begin{aligned}
\frac{dN}{dt} &= \frac{d}{dt}\int_\Omega \sum_{k=1,0,-1} \left[\left|\psi_k^1\right|^2 + \left|\psi_k^2\right|^2 \right] dx \\
&= 2\int_\Omega \sum_{k=1,0,-1} \left[\psi_k^1 \frac{\partial\psi_k^1}{\partial t} + \psi_k^2 \frac{\partial\psi_k^2}{\partial t} \right] dx = 0,
\end{aligned} \tag{6.3}$$

which imply that the total particle number N is a conservative quantity for problem (5.1).

At last, we show M is a conservative quantity for problem (5.1). Let $X_1 = X_2 = H_1(\Omega, R^3)$, then

$$\begin{aligned} &E\colon X_1 \times X_2 \longrightarrow R^1 \quad \text{defined by (2.3)}, \\ &M\colon X_1 \times X_2 \longrightarrow R^1 \quad \text{defined by (6.1)} \end{aligned} \tag{6.4}$$

are both functional. Let $\psi^1 = (\psi_1^1, \psi_0^1, \psi_{-1}^1)$, $\psi^2 = (\psi_1^2, \psi_0^2, \psi_{-1}^2)$. It is easy to check that

$$\begin{aligned} \frac{\delta M}{\delta \psi^1} &= \left(2\psi_1^1, 0, -2\psi_{-1}^1\right), \qquad \frac{\delta M}{\delta \psi^2} = \left(2\psi_1^2, 0, -2\psi_{-1}^2\right), \\ \frac{\delta E}{\delta \psi^1} &= \left(\frac{\delta E}{\delta \psi_1^1}, \frac{\delta E}{\delta \psi_0^1}, \frac{\delta E}{\delta \psi_{-1}^1}\right), \qquad \frac{\delta E}{\delta \psi^2} = \left(\frac{\delta E}{\delta \psi_1^2}, \frac{\delta E}{\delta \psi_0^2}, \frac{\delta E}{\delta \psi_{-1}^2}\right). \end{aligned} \tag{6.5}$$

From (2.3) and (2.7), we have

$$\begin{aligned} \frac{\delta E}{\delta \psi_1^2} &= \left[-\frac{\hbar^2}{2m}\Delta + V(x) + g_n|\Psi|^2 + g_s\left(\left|\psi_1\right|^2 + \left|\psi_0\right|^2 - \left|\psi_{-1}\right|^2\right)\right]\psi_1^2 \\ &\quad + g_s\left(2\psi_{-1}^1\psi_0^1\psi_0^2 - \psi_{-1}^2\left(\psi_0^1\right)^2 + \psi_{-1}^2\left(\psi_0^2\right)^2\right), \\ \frac{\delta E}{\delta \psi_1^1} &= \left[-\frac{\hbar^2}{2m}\Delta + V(x) + g_n|\Psi|^2 + g_s\left(\left|\psi_1\right|^2 + \left|\psi_0\right|^2 - \left|\psi_{-1}\right|^2\right)\right]\psi_1^1 \\ &\quad + g_s\left(2\psi_{-1}^2\psi_0^1\psi_0^2 + \psi_{-1}^1\left(\psi_0^1\right)^2 - \psi_{-1}^1\left(\psi_0^2\right)^2\right), \\ \frac{\delta E}{\delta \psi_{-1}^2} &= \left[-\frac{\hbar^2}{2m}\Delta + V(x) + g_n|\Psi|^2 + g_s\left(\left|\psi_{-1}\right|^2 + \left|\psi_0\right|^2 - \left|\psi_1\right|^2\right)\right]\psi_{-1}^2 \\ &\quad + g_s\left(2\psi_1^1\psi_0^1\psi_0^2 - \psi_1^2\left(\psi_0^1\right)^2 + \psi_1^2\left(\psi_0^2\right)^2\right), \\ \frac{\delta E}{\delta \psi_{-1}^1} &= \left[-\frac{\hbar^2}{2m}\Delta + V(x) + g_n|\Psi|^2 + g_s\left(\left|\psi_{-1}\right|^2 + \left|\psi_0\right|^2 - \left|\psi_1\right|^2\right)\right]\psi_{-1}^1 \\ &\quad + g_s\left(2\psi_1^2\psi_0^1\psi_0^2 + \psi_1^1\left(\psi_0^1\right)^2 - \psi_1^1\left(\psi_0^2\right)^2\right). \end{aligned} \tag{6.6}$$

Combining (6.5), (6.6) with (3.1), (3.2), we can get following equalities:

$$\begin{aligned} \frac{dM}{dt} &= \left\langle \frac{\delta M(\psi^1, \psi^2)}{\delta \psi^1}, \frac{\partial \psi^1}{\partial t} \right\rangle_{X_1 \times X_2} + \left\langle \frac{\delta M(\psi^1, \psi^2)}{\delta \psi^2}, \frac{\partial \psi^2}{\partial t} \right\rangle_{X_1 \times X_2} \\ &= \left\langle \frac{\delta M(\psi^1, \psi^2)}{\delta \psi^1}, \frac{\delta F}{\delta \psi^2} \right\rangle_{X_1 \times X_2} - \left\langle \frac{\delta M(\psi^1, \psi^2)}{\delta \psi^2}, \frac{\delta F}{\delta \psi^1} \right\rangle_{X_1 \times X_2} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\hbar}\int_{\Omega} 2\left[\psi_1^1 \frac{\partial E}{\partial \psi_1^2} - \psi_{-1}^1 \frac{\partial E}{\partial \psi_{-1}^2} - \psi_1^2 \frac{\partial E}{\partial \psi_1^1} + \psi_{-1}^2 \frac{\partial E}{\partial \psi_{-1}^1}\right] dx \\
&= \frac{g_s}{\hbar}\int_{\Omega}\Bigg[\psi_1^1\left(2\psi_{-1}^1\psi_0^1\psi_0^2 - \psi_{-1}^2\left(\left(\psi_0^1\right)^2 - \left(\psi_0^2\right)^2\right)\right) \\
&\qquad\qquad - \psi_{-1}^1\left(2\psi_1^1\psi_0^1\psi_0^2 - \psi_1^2\left(\left(\psi_0^1\right)^2 - \left(\psi_0^2\right)^2\right)\right) \\
&\qquad\qquad - \psi_1^2\left(2\psi_{-1}^2\psi_0^1\psi_0^2 + \psi_{-1}^1\left(\left(\psi_0^1\right)^2 - \left(\psi_0^2\right)^2\right)\right) \\
&\qquad\qquad + \psi_{-1}^2\left(2\psi_1^2\psi_0^1\psi_0^2 + \psi_1^1\left(\left(\psi_0^1\right)^2 - \left(\psi_0^2\right)^2\right)\right)\Bigg] dx \\
&= 0,
\end{aligned}
\tag{6.7}
$$

which imply that the magnetization intensity M is a conservative quantity for problem (5.1). The proof is completed. □

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